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Hypocoercivité et confinement géometrique

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Introduction (version française)

Cette thèse étudie le caractère bien posé et le comportement asymptotique des solutions de différentes équations cinétiques dans des domaines bornés.

L'objectif de cette introduction est d'établir le cadre de la thèse :

Nous présenterons d'abord les différents niveaux de description mathématique des systèmes de particules, où nous expliquerons l'intérêt physique et mathématique de l'étude de la théorie cinétique des gaz.

Nous donnons ensuite un aperçu de la ligne générale de recherche que nous considérons dans notre travail et nous expliquons les différents problèmes physiques qui motivent les différents problèmes étudiés dans chacun des chapitres de cette thèse.

Ensuite, nous introduisons le cadre général de l'étude des équations cinétiques : leur forme fondamentale, les conditions aux bords appropriées pour l'étude du problème de Cauchy, et les techniques de *hypocoercivité* pour les équations cinétiques *linéarisées* dans des domaines bornés.

Par la suite, nous présenterons plusieurs exemples d'équations cinétiques pertinentes pour cette thèse, en soulignant à la fois leurs propriétés mathématiques et leur rôle dans la modélisation de phénomènes et de possibles applications physiques.

Enfin, nous exposerons nos résultats principaux, suivis d'une discussion sur les lignes de recherche qu'ouvre notre travail, et la structure générale de la thèse.

1 Niveaux de description des systèmes de particules

L'un des principaux objectifs de la physique mathématique est de décrire des systèmes généraux de multiples particules qui composent un *gaz dilué* comme par exemple l'air, les galaxies ou le plasma. Pour accomplir une telle tâche, nous connaissons aujourd'hui trois échelles physiques principalles pour décrire un tel système.

• **Le point de vue microscopique/newtonien** qui considère chaque particule du système comme un point dans l'espace, se déplaçant avec un vecteur vitesse, et écrit les équations newtoniennes tenant en compte les interactions entre chaque particule.

Cependant, cette méthode n'est pas pratique dans la plupart des situations d'intérêt, où N (le nombre de particules) peut être très élevé. En particulier, pour trois corps déjà, nous savons que ce système est chaotique, c'est-à-dire que la dynamique des particules est très sensible à de petites perturbations des conditions initiales (voir par exemple [144, 164]).

It is thus clear that when we want to study gases, with the Avogadro number stating that $N \approx 10^{23}$, the milky way galaxy, with $N \approx 10^{11}$ stars (see Figure 1) , or the core of the sun, with $N \approx 10^{21}$ particles, this approach becomes practically impossible.

Il est donc clair que lorsque l'on veut étudier les gaz, dont le nombre d'Avogadro indique que $N \approx 10^{23}$, la voie lactée, avec $N \approx 10^{11}$ étoiles (voir Figure 1) , ou le noyau du soleil, avec $N \approx 10^{21}$ particules, cette approche devient pratiquement impossible.

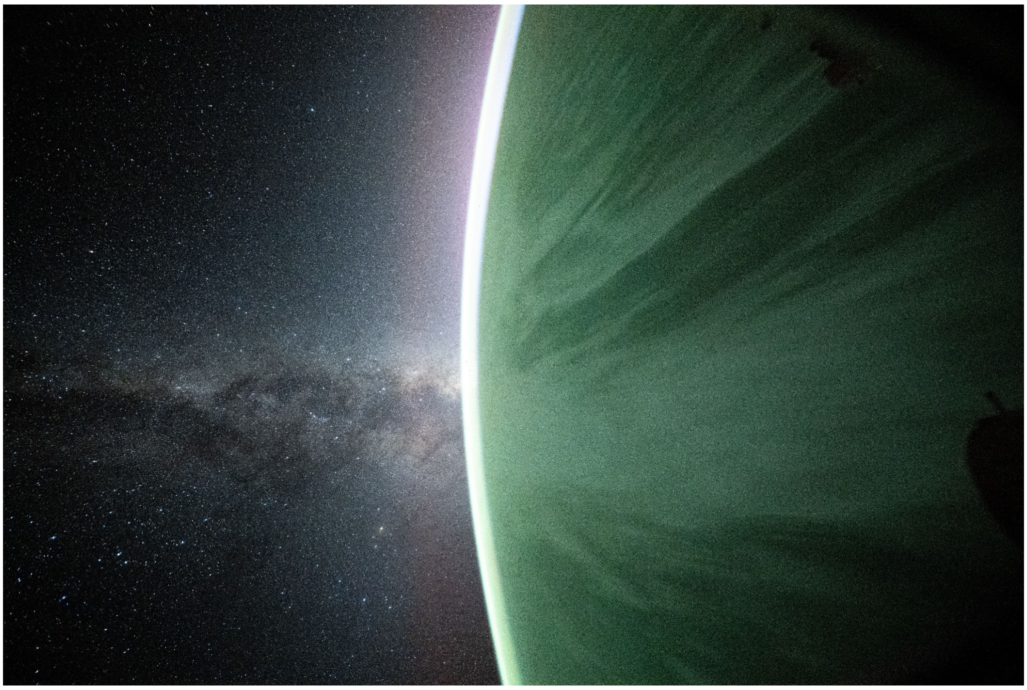

Figure 1: Photo de la Voie lactée apparaissant au-delà de l'horizon terrestre, prise par l'astronaute de la NASA Don Pettit le 29 janvier 2025. *Image credit: NASA/Don Pettit.*

• **Le point de vue macroscopique** où nous prenons un cube dans l'espace, suffisamment petit pour être négligeable par rapport à l'ensemble du système, mais suffisamment grand pour être considéré comme une unité, c'est-à-dire contenant suffisamment de particules pour être traité statistiquement. Nous pourrions alors considérer le comportement des quantités observables de cette unité, telles que la masse, la quantité de mouvement, l'énergie, la pression, la température, le flux de chaleur, entre autres. Cela conduit à la description hydrodynamique du système via les équations d'Euler ou de Navier-Stokes. Pour plus de références à ce sujet, nous renvoyons à [122, Chapter 1],[96, Section2], et aux références qui s'y trouvent.

Ce point de vue présente toutefois un inconvénient majeur : des éléments tels que la viscosité ou la conductivité thermique sont donnés comme des données phénoménologiques ou expérimentales, et ne sont pas liés à des comportements microscopiques.

• **Le point de vue mésoscopique/cinétique** représente une alternative entre les deux descriptions précédentes. Au lieu de chercher à savoir quelles particules font quoi, nous nous intéressons plutôt au *nombre* de particules qui, *en moyenne*, se comportent d'une certaine manière. Cela se traduit mathématiquement par l'étude de la fonction de distribution caractérisant le système, qui mesure les "proportions" des particules dans l'espace de phase de la position et de la vitesse.

De plus, cette fonction de distribution codera à la fois les interactions moléculaires (d'une manière statistique), ainsi que les *quantités macroscopiques (également appelées mesurables ou observables)* telles que la masse, la quantité de mouvement et l'énergie.

La manière formelle d'obtenir une description mathématique d'un tel objet est de prendre la limite, dans la représentation microscopique de l'ensemble de particules du système, lorsque $N \to \infty$ *d'une manière statistique.* Ce type de limite est appelé *limite à nombreuses particules* ou *limite thermodynamique.* Voir par exemple [56, Chapter 2], [123], ou plus récemment [68], pour le cas particulier de l'*équation de Boltzmann*

2 Motivation et cadre de la thèse

Cette thèse s'inscrit dans le cadre de l'étude des systèmes statistiques de particules illustrés par leur description mésoscopique. Nous nous intéressons particulièrement à ceux qui sont en état de *non équilibre thermodynamique*, c'est-à-dire lorsqu'il y a présence de flux de *quantités observables* telles que la masse ou l'énergie.

Il convient de remarquer dans ce contexte que la *deuxième loi de la thermodynamique* implique que les systèmes physiques isolés, bien qu'ils puissent être en non-équilibre à un moment donné du temps, sont tenus d'évoluer vers un état d'équilibre stable. Mathématiquement, cela équivaut au fait qu'un tel état d'équilibre ressemble à une distribution *Maxwellienne* (gaussienne en dehors du monde de la théorie cinétique). Cette hypothèse a été formulée pour la première fois par J. C. Maxwell dans [131] et prouvée plus tard par L. Boltzmann dans [28], avec l'introduction de la première équation cinétique et du *théorème* H, un moyen de quantifier la production d'*entropie* du système et qui implique—au moins qualitativement—un tel comportement à long terme vers des états d'équilibre.

Les résultats remarquables de Boltzmann sont devenus le point de départ de la théorie cinétique des gaz ainsi que de nombreuses études et développements physiques et mathématiques. Nous reviendrons sur ce sujet plus loin dans la Section 6.

Cependant, lorsque notre système de particules n'est pas isolé et qu'il est soumis à des forces non conservatrices telles que la présence de reservoirs thermiques interagissant avec les particules, la deuxième loi de la thermodynamique ne s'applique plus et, même si le système est laissé à évoluer en tant que $t \to \infty$, nous constateront l'existence des flux d'énergie. On dit dans ce cas que l'on est en présence d'un état stationnaire en non-équilibré (NESS), et en particulier qu'il ne peut pas ressembler à une distribution Maxwellienne (voir [162] pour un résultat dans ce sens pour le cas particulier de l'équation *cinétique de Fokker-Planck*).

L'étude de ces systèmes, tant sur le plan physique que mathématique, est aujourd'hui extraordinairement compliquée. Du point de vue physique, la création d'expériences fructueuses est délicate et se heurte souvent aux limites du maintien de températures fixes pour les reservoirs suffisamment longtemps pour faire des observations significatives sur le comportement à long terme des particules, voir par exemple les commentaires dans [94]. Sur le plan mathématique, déjà dans le cadre de l'équilibre de l'équation de Boltzmann et des modèles apparentés, leur étude a nécessité, au cours du siècle dernier, la création de nouveaux domaines, théories et techniques pour donner un sens rigoureux aux solutions et quantifier leur tendance à l'équilibre, en laissant encore aujourd'hui de nombreuses questions ouvertes dans ce domaine.

Ce n'est qu'à la fin du XX$^{\text{th}}$ siècle, avec l'accès à des ordinateurs puissants, que les physiciens ont redonné de l'attention à ces problèmes par le biais de simulations, encourageant ainsi l'étude mathématique de problèmes tels que l'existence d'états stationnaires hors équilibre, leur unicité et stabilité. Pour les travaux mathématiques dans ce cadre, nous nous référons à [7, 8, 10, 42, 44, 82] dans le cadre de *l'équation de Boltzmann* et à [19, 41, 43, 45, 84, 85, 87] et les références qui y figurent pour des résultats connexes dans d'autres modèles cinétiques.

Cette thèse s'inscrit dans ce cadre et est motivée par l'étude des questions de bien posé et de stabilité des états stationnaires hors équilibre pour les équations cinétiques dans des domaines bornés en présence de reservoirs de chaleur sur le bord. Pour être plus précis, nous expliquons maintenant brièvement les principaux axes de notre recherche et les motivations de chacun des chapitres (numérotés) de notre thèse.

• Dans le Chapitre 1, nous étudions une équation de *Fokker-Planck cinétique* linéaire générale en présence d'une température variant sur le bord et nous prouvons l'existence et l'unicité d'un solution stationnaire hors équilibré. De plus, nous caractérisons le comportement à long terme de ses solutions de manière quantitative.

• Dans le Chapitre 2, nous utilisons les résultats du Chapitre 1 pour étudier une équation de *Fokker-Planck cinétique* faiblement non linéaire avec des reservoirs de type *BGK* dans un domaine borné, présentant également des températures variant sur le bord. Nous prouvons l'existence d'une solution stationnaire hors équilibre et sa stabilité, sous la condition de petitesse de la donnée initiale.

• Enfin, nous consacrons le Chapitre 3 à l'étude d'une *équation de Boltzmann* proche de sa limite hydrodynamique. Nous étudions ce problème confiné dans un domaine cylindrique où les bases ont une température fixe et où il y a une réflexion spéculaire dans sa surface latérale. La motivation principale de cette étude est la première étape de l'étude du cadre physiquement pertinent de l'équation de Boltzmann dans un domaine cylindrique avec des températures différentes sur chaque base, et une réflexion spéculaire sur la surface latérale.

Pour une définition précise des conditions aux bords des équations cinétiques mentionnée ci-dessus, ou pour savoir comment représenter la présence d'une température sur le bord, voir Section 4. En outre, pour le lecteur intéressé par des discussions sur les systèmes en équilibre et en non-équilibre thermodynamique d'un point de vue plus orienté vers la physique, voir [94, 117] et [93, Chapitre 9].

3 Cadre général pour l'étude des équations cinétiques

Comme expliqué précédemment, pour une dimension donnée $d \geq 1$, nous nous intéressons à une fonction de distribution $F = F(t, x, v)$ telle que, à chaque instant $t \in \mathbb{R}_+$, elle *represents* une particule à la position $x \in \Omega$, où $\Omega \subset \mathbb{R}^d$ est un domaine, et se déplaçant à une vitesse $v \in \mathbb{R}^d$.

Nous notons que la représentation ci-dessus de F est au sens statistique. Une manière plus mathématiquement précise de décrire F est de la considérer comme une mesure de probabilité dépendante de t (bien que nous puissions également considérer des mesures absolument continues par rapport à la Lebesgue), et telle que pour chaque $t \geq 0$, la quantité $F(t, x, v)\mathrm{d}x\mathrm{d}v$ est la moyenne des particules dans le cube de taille $\mathrm{d}x\mathrm{d}v$ dans l'espace des phases $\Omega \times \mathbb{R}^3$.

En autre, il convient de remarquer que même si le choix le plus physique de la dimension est $d = 3$, il est physiquement significatif et mathématiquement intéressant d'étudier ces problèmes dans des dimensions arbitraires.

La formulation générale d'une équation cinétique se lit alors

$$\partial_t F + \mathbf{T}F = \mathbf{Q}(F) \quad \text{dans } \mathcal{U} := (0, \infty) \times \Omega \times \mathbb{R}^3, \tag{1}$$

où nous avons l'*opérateur de collision* $\mathbf{Q}$, possiblement non-linéaire, et nous avons défini l'opérateur de *transport*

$$\mathbf{T}F := v \cdot \nabla_x F + \nabla_x \phi \cdot \nabla_v F,$$

pour une fonction *potentielle* $\phi : (0, \infty) \times \Omega \to \mathbb{R}$.

En général, la distribution F n'est pas physiquement mesurable mais, pour tout $(t, x) \in \mathbb{R}_+ \times \Omega$ fixes, il est possible d'observer certaines des propriétés des particules que F décrit. Ces propriétés sont appelées *quantités macroscopiques* (QM) et sont liées à la distribution F par les formules suivantes

$$\begin{aligned}
&\text{densité locale} && \rho(t,x) &&:= \int_{\mathbb{R}^3} F(t,x,v)\mathrm{d}v,\\
&\text{vitesse moyenne locale} && u(t,x) &&:= \frac{1}{\rho(t,x)}\int_{\mathbb{R}^3} v\,F(t,x,v)\mathrm{d}v,\\
&\text{température locale} && T(t,x) &&:= \frac{1}{d\,\rho(t,x)}\int_{\mathbb{R}^3} |v-u(t,x)|^2\,F(t,x,v)\mathrm{d}v,\\
&\text{et l'entropie locale} && S(t,x) &&:= -\frac{1}{\rho(t,x)}\int_{\mathbb{R}^3} F(t,x,v)\log F(t,x,v)\mathrm{d}v.
\end{aligned} \tag{QM}$$

En ce qui concerne les choix possibles pour Ω, nous avons quatre possibilités principales :

• D'une part, pour les choix non bornés de Ω, nous rencontrons dans la littérature soit l'espace complet $\mathbb{R}^d$, soit des domaines non bornés avec des frontières, comme par exemple le demi-espace $\mathbb{R}^d_+ := \{(x_1,\dots,x_d) \in \mathbb{R}^d, x_1 > 0\}$.

• D'autre part, si nous voulons travailler dans un ensemble Ω avec une mesure de Lebesgue finie, nous pouvons soit choisir le tore $\mathbb{T}^d$, ce qui signifie que $x \in [0,1]^d$ et nous supposons que F est périodique dans cet ensemble, ou nous prenons Ω comme un domaine arbitraire (suffisamment régulière) borné dans $\mathbb{R}^d$.

En particulier, nous remarquons que lorsque Ω est soit $\mathbb{R}^d_+$ soit un domaine borné, nous devons compléter l'Équation (1) avec des conditions aux bords appropriées.

4 Conditions au bord

Nous supposons maintenant que Ω est un domaine borné dans $\mathbb{R}^d$, et nous supposons qu'il existe une fonction

$$\delta \in W^{1,\infty}(\mathbb{R}^3,\mathbb{R}) \text{ telle que } \Omega = \{x\in\mathbb{R}^3,\ \delta(x)>0\},\ \text{ et } |\delta(x)| = \mathrm{dist}(y,\partial\Omega) \tag{2}$$

sur un voisinage du bord. Ici, la fonction $\mathrm{dist}(\cdot,\partial\Omega)$ représente la fonction de distance à la frontière $\partial\Omega$.

Défini de cette manière, nous observons que $\partial\Omega = \{x\in\mathbb{R}^3,\ \delta(x)=0\}$ et nous avons classiquement que $|\nabla\delta(x)|= 1$. Par conséquent, nous définissons le vecteur normal vers l'extérieur comme suit

$$n_x = n(x) := -\nabla\delta(x) = -\frac{\nabla\delta(x)}{|\nabla\delta(x)|} \quad \text{pour presque tout } x\in\bar{\Omega}. \tag{3}$$

Nous définissons également l'ensemble limite $\Sigma = \partial\Omega\times\mathbb{R}^d$ et nous distinguons les ensembles de vitesses *sortantes* (Σ_+), *entrantes* (Σ_-), et *rasantes* (Σ_0) sur le bord défini par

$$\Sigma_\pm := \{(x,v)\in\Sigma,\ \pm n_x\cdot v>0\}, \quad \text{et} \quad \Sigma_0 := \{(x,v)\in\Sigma,\ n_x\cdot v = 0\}.$$

En autre, nous notons également $\Gamma := (0,\infty)\times\Sigma$ et par conséquent $\Gamma_\pm := (0,\infty)\times\Sigma_\pm$. Nous définissons ensuite γF comme la fonction de trace associée à F sur Γ, et $\gamma_\pm F := \mathbf{1}_{\Gamma_\pm}\gamma F$.

Nous notons qu'en raison de la nature du système que nous décrivons, nous ne sommes pas en mesure de fixer le comportement des particules qui quittent le domaine, c'est-à-dire celles dont les coordonnées $(x,v)\in\Sigma_+$. Ceci est dû au fait que leur dynamique est complètement déterminée par leur “histoire” à l'intérieur de Ω. Par conséquent, pour avoir un problème bien posé, il suffira de donner les conditions au bord que sur $\gamma_- F$, la partie entrante de la trace.

Nous remarquons également qu'une façon très naturelle de définir les particules entrantes est de les relier à celles qui quittent Ω. Physiquement, cela signifie que la particule subit un *reflexion*, d'une sorte ou d'une autre, au bord.

Cependant, le problème de la description de l'interaction entre un gaz et une surface, même s'il a été étudié au moins depuis le XIXe siècle, reste assez difficile, voir par exemple [54, Chapitre III, Section 1] pour une discussion orientée vers la physique à ce sujet.

En particulier, on sait qu'en général, ces collisions peuvent dépendre de la température au bord, de la propreté et de la rugosité du matériau, et même du temps, car la surface est sujette à des changements dus à des processus chimiques entre les particules entrant en collision avec celles présentes sur la frontière.

Néanmoins, sous des hypothèses assez générales, nous pouvons modéliser un tel processus par

$$|n_x \cdot v| \, \gamma_- F(t,x,v) = \int_{\mathbb{R}^d} \gamma_+ F(t,x,u) \, \mathscr{K}(u \to v, x) \, (n_x \cdot u)_+ \, du \qquad \text{sur } \Gamma_-, \tag{4}$$

où $\mathscr{K} = \mathscr{K}(u \to v, x)$ est appelé le *noyau de diffusion* et il décrit l'interaction des particules du gaz avec le bord. Ce noyau peut être supposé posséder les propriétés suivantes :

• *Non négatif:* $\mathscr{K} \geq 0$, puisqu'il représente la distribution de probabilité liée au changement de vitesse après une collision avec le bord.

• *Non-poreux et non-adsorbant/Conservatif* si la surface est supposée réémettre chaque particule, sans tenir compte de sa vitesse, nous supposons que

$$\int_{u \cdot n_x < 0} \mathscr{K}(u \to v, x) du = 1, \qquad \forall x \in \partial\Omega.$$

• *Loi de réciprocité/Equilibre détaillé:* Il existe une distribution Maxwellienne définie comme suit

$$\mathcal{M}_\Theta = \mathcal{M}_\Theta(v) := \frac{1}{(2\pi\Theta)^{d/2}} \exp\left(-\frac{|v|^2}{2\Theta} \right), \tag{5}$$

où $\Theta : \bar{\Omega} \to \mathbb{R}_+^*$ représente une température prescrite au bord, pour laquelle il existe

$$\int_{\mathbb{R}^d} \mathcal{M}_\Theta(t,x,u) \, \mathscr{K}(u \to v, x) \, (n_x \cdot u)_+ \, du = |n_x \cdot v| \mathcal{M}_\Theta(t,x,v).$$

Sous les conditions ci-dessus, nous avons en particulier l'inégalité de Darrozès et Guiraud, voir par exemple [62] ou [54, Chapitre III, Section 4].

La formulation générale (4), pour les conditions au bord, vise à décrire le comportement complexe présent dans les interactions entre les particules d'un gaz et une surface, à travers une formulation générale, et laisse le noyau de diffusion $\mathscr{K}$ comme un opérateur phénoménologique à adapter aux particularités de chaque situation pratique.

Cependant, définies de manière aussi générale, les conditions au bord ne fournissent pas une information suffisante pour établir une théorie de bien-posé pour les équations cinétiques. Nous pouvons donc soit restreindre davantage le comportement de $\mathscr{K}$ en introduisant des hypothèses supplémentaires, soit choisir des cas particuliers de conditions au bord approchant le comportement d'un gaz à une surface.

En optant pour cette dernière solution, nous distinguons dans la littérature les choix suivants de conditions au bord pour compléter l'Équation (1).

• **Condition au bord du type *inflow*.** Pour une fonction donnée $\mathfrak{f} : \Gamma_- \to \mathbb{R}$ nous posons

$$\gamma_- F(t,x,v) = \mathfrak{f}(t,x,v) \quad \text{sur } \Gamma_-.$$

Physiquement, cela revient à injecter dans le domaine des particules suivant la dynamique décrite par $\mathfrak{f}$.

• **Condition au bord du rebond/*bounce-back*.** Nous posons

$$\gamma_- F(t,x,v) = \gamma_+ F(t,x,-v) \quad \text{sur } \Gamma_-.$$

Cette condition au bord n'est pas très physique, mais comme elle établit un lien assez simple entre les particules sortantes et les particules entrantes, elle a été utilisée dans des modèles jouets pour comprendre des problèmes plus compliqués.

• **Réflexion spéculaire.** Nous considérons

$$\gamma_- F(t,x,v) = \mathscr{S}\gamma_+ F(t,x,v) := \gamma_+ F(t,x,\mathcal{V}_x v) \quad \text{sur } \Gamma_-, \tag{6}$$

où $\mathcal{V}_x v := v - 2(n_x \cdot v) n_x$.

Cette condition au bord représente une particule se reflétant sur la surface de la même manière que la lumière se reflète sur un miroir, c'est-à-dire qu'elle suit le principe de *l'angle d'incidence est le même que l'angle de réflexion*, voir la Figure 2 pour une représentation graphique. En effet, pour tout $x \in \mathbb{R}^d$, tel que le vecteur normal n_x est bien défini, et tout $v \in \mathbb{R}^d$ nous observons que

$$\mathcal{V}_x v \cdot n_x = -v \cdot n_x, \quad \text{et} \quad |\mathcal{V}_x v|^2 = |v|^2,$$

ce qui valide le fait que la particule rebondit sur la surface de manière symétrique par rapport à la normale au point de collision, tout en conservant la même vitesse. Il s'agit de l'une des conditions au bord les plus simples, physiquement parlant, puisqu'elle suppose que les collisions sont *parfaites* et qu'il n'y a pas de rugosité ou d'échange d'énergie à la frontière.

De plus, nous remarquons que (6) n'est rien d'autre que (4) lorsque l'on choisit $\mathscr{K} = \delta_D(u - \mathcal{V}_x v)$, où δ_D représente la fonction delta de Dirac.

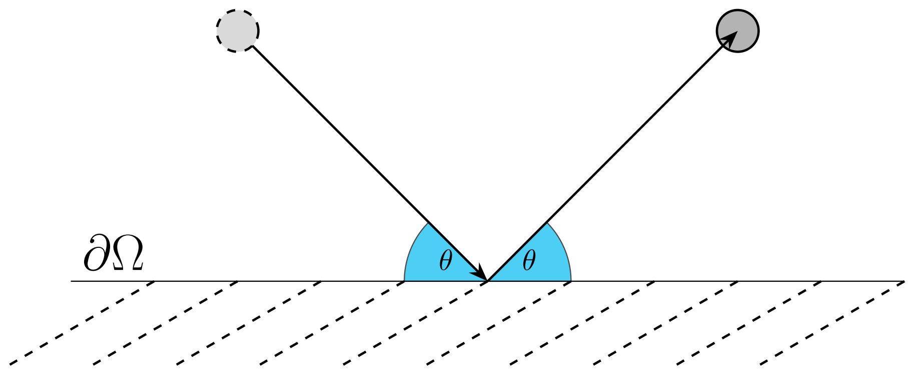

Figure 2: Representation graphique d'une particule subissant une réflexion spéculaire, au bord $\partial\Omega$, avec un angle d'incidence θ.

• **Réflexion diffusive.** Pour une fonction donnée $\Theta : \bar{\Omega} \to \mathbb{R}_+^*$, nous définissons la distribution *Maxwellienne* du bord

$$\mathscr{M}_\Theta := \sqrt{\frac{2\pi}{\Theta}} \mathcal{M}_\Theta, \tag{7}$$

où nous rappelons que $\mathcal{M}_\Theta$ est défini dans (5). Nous prenons alors

$$\gamma_- F(t,x,v) = \mathscr{D}\gamma_+ F(t,x,v) := \mathscr{M}_\Theta(v)\widetilde{\gamma_+ F}(t,x) \quad \text{sur } \Gamma_-, \tag{8}$$

où

$$\widetilde{\gamma_+ F} = \widetilde{\gamma_+ F}(t,x) := \int_{\mathbb{R}^d} \gamma_+ F(t,x,u)\,(n_x \cdot u)_+ \, du,$$

et il convient de remarquer que $\mathcal{M}_\Theta$ a été défini de manière à ce que la condition de normalisation $\widetilde{\mathcal{M}}_\Theta = 1$ soit respectée.

Physiquement, cette condition au bord représente la présence d'une température prescrite Θ à la frontière, qui peut éventuellement varier dans l'espace. L'intuition est que la surface absorbe la particule sortante $(x, v_+) \in \Sigma_+$ et en introduit une nouvelle $(x, v_-) \in \Sigma_-$, où v_- représente une variable indépendante donnée par la distribution gaussienne modifiée $\mathscr{M}_\Theta$.

En outre, il convient de remarquer que, physiquement, la stochasticité inhérente à la condition au bord diffusive peut également être utilisée pour modéliser la *rugosité* de la frontière

De plus, nous remarquons que la condition au bord diffusive (8) est un cas particulier de (4) lorsque l'on choisit $\mathscr{K} = |n_x \cdot v| \mathscr{M}_\Theta(x,v)$.

• **Réflexion Maxwellienne.** J. C. Maxwell a introduit cette condition au bord dans [132, Appendix], où il la décrit comme un moyen approprié de traiter l'interaction entre un gaz et une surface.

La proposition de Maxwell consistait à diviser l'opérateur de réflexion en une partie locale (spéculaire) et une partie diffuse, combinées, de manière convexe, par un *coefficient d'accommodation*.

En effet, nous prenons $\iota : \partial\Omega \to [0,1]$, et nous considérons

$$\gamma_- F(t,x,v) = \mathscr{R}\gamma_+ F(t,x,v) \quad \text{sur } \Gamma_-, \tag{9}$$

où nous avons défini l'opérateur de réflexion de Maxwell

$$\mathscr{R}\gamma_+ F(t,x,v) := (1 - \iota(x))\mathscr{S}\gamma_+ F(t,x,v) + \iota(x)\mathscr{D}\gamma_+ F(t,x,v). \tag{10}$$

Nous rappelons que les opérateurs $\mathscr{S}$ et $\mathscr{D}$ sont donnés par les opérateurs de réflexion spéculaire et diffusive respectivement, et nous notons que ι est le coefficient d'accommodation, qui peut dépendre de la variable spatiale.

Naturellement, nous observons que (8) est un cas particulier de (4) en choisissant

$$\mathscr{K}(u \to v, x) = (1 - \iota(x))\delta_D(u - \mathcal{V}_x v) + \iota(x)|n_x \cdot v| \mathscr{M}_\Theta(x,v).$$

• **Condition au bord de Cercignani-Lampis.** Plus récemment, C. Cercignani et M. Lampis ont introduit dans [52] un opérateur visant à être plus précis physiquement pour modéliser l'interaction entre des particules de gaz et une surface, en généralisant le modèle de Maxwell.

Cette condition au bord s'écrit alors comme (4) avec

$$\begin{aligned} \mathscr{K}(u \to v, x) := \frac{1}{\Theta(x) r_\perp} \frac{1}{(2\pi\Theta(x) r_\parallel (2 - r_\parallel))^{(d-1)/2}} \exp\left(-\frac{|v_\perp|^2}{2\Theta(x) r_\perp} - \frac{(1 - r_\perp)|u_\perp|^2}{2\Theta(x) r_\perp} \right) \\ \times I_0 \left(\frac{(1 - r_\perp)^{1/2} u_\perp \cdot v_\perp}{\Theta(x) r_\perp} \right) \exp\left(-\frac{|v_\parallel - (1 - r_\parallel) u_\parallel|^2}{2\Theta(x) r_\parallel (2 - r_\parallel)} \right), \end{aligned}$$

pour une température de paroi donnée $\Theta : \bar{\Omega} \to \mathbb{R}_+^*$, les notations

$$v_\perp := (v \cdot n_x)n_x, \quad v_\parallel = v - v_\perp, \quad u_\perp := (u \cdot n_x)n_x, \quad u_\parallel = u - u_\perp,$$

avec les coefficients d'accommodation normale et tangentielle $r_\perp \in (0,1]$ et $r_\parallel \in (0,2)$ respectivement, et où I_0 est la fonction de Besel modifiée

$$I_0(y) := \frac{1}{\pi}\int_0^\pi e^{y\cos(w)}\mathrm{d}w.$$

Il convient de noter que d'autres conditions au bord - sans doute plus précises physiquement que celles examinées précédemment - peuvent être envisagées pour modéliser la dynamique entre les particules du gaz et une surface solide, voir par exemple [53, Sections 3.6 et 3.7]. Cependant, ces modèles impliquent souvent des expressions mathématiques complexes pour atteindre un tel degré de précision, ce qui les rend peu pratiques pour leur analyse et pendant les simulations informatiques. De plus, indépendamment de la complexité, et à notre connaissance, nous manquons aujourd'hui de conditions au bord non phénoménologiques pour les équations cinétiques, c'est-à-dire qui ne reposent pas sur des paramètres sélectionnés empiriquement tels que les coefficients d'accommodation.

Comme expliqué dans [52] (voir aussi [53, Sections 3.8]), à part les expériences de faisceaux de gaz sur les phénomènes de réflexion moléculaire, il n'y a actuellement pas assez de données expérimentales pour déterminer quel modèle est le plus approprié pour compléter les équations cinétiques dans les domaines bornés.

Par conséquent, la plupart des articles de recherche considèrent aujourd'hui les conditions au bord de Maxwell comme un modèle pratique et fiable pour les réflexions de gaz aux frontières. En particulier, au cours de cette thèse, nous considérons principalement les conditions au bord de type Maxwell.

5 Théorie d'hypocoercivité

L'étude mathématique de l'Équation (1) dépend fortement de la forme explicite de l'opérateur $\mathbf{Q}$. Dans la plupart des cas d'intérêt, la structure du problème—jusqu'à la théorie connue—n'est pas suffisante pour construire des solutions faibles, nous ne pouvons qu'espérer construire des solutions renormalisées, qui sont encore plus faibles (voir [3, 69, 75, 76, 78, 127, 137, 138]).

Cependant, pour les systèmes en équilibre thermodynamique, nous pouvons généralement inspecter des états d'équilibre explicites (souvent Maxwelliennes). Dans ce cas, nous pouvons étudier les solutions perturbatives de ce problème au voisinage de l'équilibre et la stabilité de cette solution stationnaire. En effet, si nous supposons que $\mathbf{M}$ est un tel état d'équilibre, nous prenons $F = \mathbf{M} + f$ et nous observons que f satisfait, a priori, l'équation suivante

$$\partial_t f + \mathbf{T}f = \mathbf{C}f + \mathbf{Q}(f),$$

où nous avons défini $\mathbf{C}f := \mathbf{Q}(f + \mathbf{M}) - \mathbf{Q}(f) - \mathbf{T}\mathbf{M}$, et nous remarquons que

$$\mathbf{C}f := \begin{cases} 0 & \text{si } \mathbf{Q} \text{ est un opérateur linéaire,} \\ \mathbf{Q}(\mathbf{M}, f) + \mathbf{Q}(f, \mathbf{M}) & \text{si } \mathbf{Q}(f) = \mathbf{Q}(f,f) \text{ est un opérateur bilinéaire.} \end{cases}$$

Cela motive la définition de

$$\mathbf{L}f := \begin{cases} \mathbf{Q}(f) & \text{si } \mathbf{Q} \text{ est un opérateur linéaire,} \\ \mathbf{C}f & \text{si } \mathbf{Q} \text{ est un opérateur bilinéaire,} \end{cases}$$

et nous l'appelons *opérateur de collision linéarisé.*

Afin d'étudier le caractère bien-posé des solutions perturbées et la stabilité de la solutions stationnaire $\mathbf{M}$, nous analysons d'abord l'équation *linéarisée.*

$$\partial_t f + \mathbf{T} f = \mathbf{L} f, \tag{11}$$

ce qui implique de traiter avec l'opérateur conservatif de transport $\mathbf{T}$ et l'opérateur de collision linéarisé $\mathbf{L}$ qui est typiquement dissipatif—dans un certain espace de Hilbert $\mathbf{H}$—mais non coercitif, dans le sens où il n'admet pas de trou spectral, et possède à la place un énorme noyau.

Le but de la *théorie d'hypocoercivité* est alors d'établir des techniques pour traiter ce type de situations (voir [21, 79, 166]) et nous remarquons qu'un tel cadre est similaire au type de problèmes rencontrés dans la théorie de l'*hypoellipticité*, cf. [116]. Son objectif principal est de construire, au moins au niveau des estimations a priori, une estimation explicite de la décroissance des solutions de l'Équation (11) dans $\mathbf{H}$. Cela se fait typiquement en construisant une nouvelle norme, équivalente à celle de $\mathbf{H}$, et sous laquelle l'opérateur $-\mathbf{T} + \mathbf{L}$ sera coercitif.

Pour des résultats explicites dans le cas de domaines spatiaux bornés avec des conditions de Maxwell (isothermes), nous nous référons à la théorie de l'hypocoercivité développée par A. Bernou, K. Carrapatoso, S. Mischler et I. Tristani dans l'article [21]. Pour les situations concernant l'espace complet ou le tore, le lecteur peut consulter l'article [79] de J. Dolbeault, C. Mouhot et C. Schmeiser, ou le mémoire de C. Villani dans [166].

6 Exemples d'équations cinétiques

Dans cette section, nous présentons plusieurs équations cinétiques pertinentes pour cette thèse, ainsi que certains de leurs résultats mathématiques connus et leur motivation physique.

6.1 L'équation de Boltzmann

J. Maxwell [131] et L. Boltzmann [28] ont écrit la première équation de la théorie cinétique, connue aujourd'hui sous le nom d'équation de Boltzmann (BE), qui s'écrit

$$\partial_t F + v \cdot \nabla_x F = \mathcal{Q}(F, F) \quad \text{ dans } \mathcal{U}, \tag{12}$$

où $\mathcal{Q}$, appelé l'opérateur de collision de *Boltzmann*, représente les collisions entre les particules à l'intérieur de Ω, et est donné par la forme bilinéaire suivante

$$\mathcal{Q}(G, H) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathcal{B}(v - v_*, \sigma) \, \left[G'_* H' + H'_* G' - G_* H - G H_* \right] \mathrm{d}\sigma \mathrm{d}v_*, \tag{13}$$

où nous avons utilisé les abréviations

$$H = H(v), \quad H_* = H(v_*), \quad H' = H(v'), \quad H'_* = H(v'_*),$$

et nous avons défini les vitesses post-collisionnelles

$$v' := v - ((v - v_*) \cdot \sigma)\sigma, \qquad v'_* := v_* + ((v - v_*) \cdot \sigma)\sigma. \tag{14}$$

Pour expliquer la définition (14), nous devons nous plonger brièvement dans la modélisation des collisions de particules à l'échelle microscopique : si nous considérons deux particules entrant en collision avec des vitesses v, v_*, et que nous supposons des collisions élastiques, ce qui signifie qu'il y a

$$\begin{array}{rclr} v + v_* &=& v' + v'_* & \text{(conservation de la quantité de mouvement)}, \\ |v|^2 + |v_*|^2 &=& |v'|^2 + |v'_*|^2 & \text{(la conservation de l'énergie)}, \end{array} \tag{15}$$

puis nous voulons calculer à quoi ressemblent les vitesses post-collision. Nous remarquons que (15) représente un système de $d + 1$ équations, alors que nous cherchons un vecteur à $2d$–dimension composé des composantes des vecteurs post-collisionnels v' et v'_*. Cela signifie qu'il y a $d - 1$ paramètres libres à déterminer, ce qui motive l'introduction du vecteur $\sigma \in \mathbb{S}^{d-1}$ et la définition (14). Voir la Figure 3 pour une représentation graphique de ce phénomène.

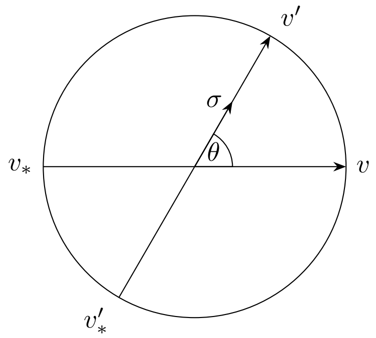

Figure 3: Représentation graphique d'une collision binaire élastique avec des vitesses pré-collisionnelles v, v_*, pour un certain choix du vecteur σ générant les vitesses post-collisionnelles associées v', v'_*. Voir également la représentation graphique de l'angle de déviation θ (voir ci-dessous pour sa définition).

Le *noyau de collision* $\mathcal{B} = \mathcal{B}(v - v_*, \sigma)$, décrit la *configuration* dans laquelle les particules interagissent lors de l'impact et prend la forme de

$$\mathcal{B}(v - v_*, \sigma) = |v - v_*| \Sigma(v - v_*, \sigma),$$

où $\Sigma(v - v_*, \sigma)$ est appelée la *section transversale*. En particulier, nous remarquons que nous pouvons aussi écrire $\mathcal{B} = \mathcal{B}(|v - v_*|, \cos\theta)$ (en abusant de la notation en conservant le symbole $\mathcal{B}$), où

$$\theta = \angle(v - v_*, v' - v'_*) \in [0, \pi], \qquad \text{est appelée l'}\textit{angle de deviation},$$

et de sa définition même, il en résulte que

$$\cos\theta := \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle.$$

Nous notons que nous ne considérons ici que le cas des collisions élastiques entre particules, mais pour les noyaux de collision de Boltzmann relativistes, quantiques ou non élastiques, nous nous référons à [165, Chapitre 1, Section 1.6] et aux références qui s'y trouvent.

Il existe plusieurs choix pour le noyau de collision de l'opérateur de collision de Boltzmann. Tout d'abord, pour un gaz de *sphères dures*, nous avons (éventuellement jusqu'à une constante positive) que

$$\mathcal{B}(v - v_*, \sigma) := |(v - v_*) \cdot \sigma|. \tag{16}$$

Ce modèle décrit la collision de particules comme des boules de billard dans le domaine d−dimensionnel Ω.

Pour les particules chargées suivant des interactions données par—jusqu'à une constante positive—une loi inverse de la forme

$$\phi(r) = \frac{1}{r^{s-1}} \qquad \text{où } r > 0 \text{ représente la distance entre les particules},$$

le noyau de collision prend la forme

$$\mathcal{B}(|v-v_*|,\theta) := |v-v_*|^\gamma b(\cos\theta) \quad \text{avec } \gamma := \frac{s-(2d-1)}{s-1},$$

et où la fonction $b(\cos\theta)$, souvent appelée le *noyau de collision angulaire*, n'est pas explicite dans la plupart des situations. On sait cependant que

$$b(\cos\theta)\sin^{d-2}\theta \underset{\theta\to 0}{\sim} c\,\theta^{-(1+\nu)} \quad \text{avec} \quad \nu = \frac{d-1}{s-1}, \tag{17}$$

pour une constante $c > 0$, voir par exemple [167, Section 4].

Il convient de noter que (17) est valable pour tout $s > 2$. En particulier, dans le cadre tridimensionnel $d = 3$, la valeur limite $s = 2$ correspond à l'interaction de Coulomb, dont on sait qu'elle ne s'inscrit pas dans le cadre de l'équation de Boltzmann, voir par exemple la discussion de [165, Paragraphe 1.7].

Enfin, nous remarquons que, dans la littérature, nous classons les différents noyaux de collisions comme suit :

- $\gamma > 0$ – potentiels durs,
- $\gamma = 0$ – molécules Maxwelliennes,
- $\gamma < 0$ – potentiels faibles, et
- $\gamma < -2$ – potentiels très faibles.

Cette notation, bien qu'utile, peut être trompeuse. Le fait est que les potentiels durs ne sont pas nécessairement *durs* à étudier et que l'équation de Boltzmann présente des propriétés plus connues dans ce cadre, telles que la présence d'un trou spectrale pour le problème linéarisé dans des espaces de Hilbert appropriés et, dans certains cas, également des trous spectrales entropiques. En revanche, pour les potentiels faibles ou très faibles, la théorie de l'équation de Boltzmann présente de nombreuses difficultés mathématiques. Pour plus d'informations sur cette discussion voir [167, Figure 4] et les références qui y figurent.

L'hypothèse de cutoff de Grad introduite par H. Grad dans [98] consiste à imposer que le noyau de collision angulaire b soit intégrable par rapport à la variable angulaire σ, c'est-à-dire que

$$\int_0^\pi b(\cos\theta)\sin^{d-2}\theta\,\mathrm{d}\theta < \infty. \tag{18}$$

Physiquement, cela équivaut à exiger que les particules n'interagissent que sur de courtes distances. D'un point de vue mathématique, cela simplifie considérablement les difficultés mathématiques liées à l'opérateur de collision de Boltzmann et la plupart des résultats connus aujourd'hui sur l'équation de Boltzmann sont valables sous cette hypothèse.

Il convient de noter que pendant longtemps, on a cru que l'hypothèse de cutoff de Grad ne changerait pas les propriétés fondamentales des solutions de l'équation de Boltzmann et que ces solutions seraient équivalentes, dans un certain sens, au phénomène physique réel étudié.

Cependant, comme l'a noté pour la première fois L. Desvillettes dans [73] et comme l'ont étudié une série d'auteurs dont nous mentionnons [2, 67, 100, 128], on sait aujourd'hui que la présence d'interactions à large portée entre les particules fait que l'opérateur de collision $\mathcal{Q}$ se comporte comme un *laplacien fractionnaire*.

En conséquence, les solutions de l'équation de Boltzmann sans cutoff bénéficient d'un gain immédiat de régularité dans la variable vitesse, pour tout temps strictement positif $t > 0$. Ceci est en contraste avec le cas avec cutoff où nous ne pouvons pas attendre de régularité supplémentaire pour les solutions, autre que—au mieux—celle de la donnée initial.

Lois de conservation de l'opérateur de collision de Boltzmann. Il est bien connu que, pour toute fonction régulière $G, H, \varphi : \mathbb{R}^d \in \mathbb{R}$, l'opérateur de collision de Boltzmann satisfait classiquement que

$$\int_{\mathbb{R}^d} \mathcal{Q}(G,H)\varphi = \frac{1}{8}\int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} \mathcal{B}\left(G'_*H' + H'_*G' - G_*H - GH_*\right)\left(\varphi + \varphi_* - \varphi' - \varphi'_*\right), \quad (19)$$

voir par exemple [56, Section 3.1].

En utilisant (19), nous observons que si nous fixons $\mathbb{R}^d \ni v = (v_1, \dots, v_d)$, et en choisissant $\varphi = \varphi(v)$ comme étant soit $1, v_1, \dots, v_d$, soit $|v|^2$, on a

$$\int_{\mathbb{R}^d} \mathcal{Q}(G,H)(v)\,\varphi(v)\,\mathrm{d}v = 0. \quad (20)$$

De plus, (20) n'est rien d'autre que de dire que $\mathcal{Q}$ conserve la masse, la quantité de mouvement et l'énergie localement en $(t,x) \in \mathbb{R}_+ \times \Omega$.

Si nous cherchons maintenant des fonctions (suffisamment régulières) F telles que (20) soit valable pour $G = H = F$, nous constatons que F est nécessairement un Maxwellien de la forme

$$F(t,x,v) = \mathcal{M}_{\rho,u,T} = \mathcal{M}_{\rho,u,T}(v) := \frac{\rho}{(2\pi T)^{d/2}} e^{-\frac{|v-u|^2}{2T}}, \quad (21)$$

où $\rho = \rho(t,x)$ est la densité locale, $u = u(t,x)$ est la vitesse moyenne locale, et $T = T(t,x)$ est la température locale dont nous rappelons qu'elle est donnée par (QM). Pour une preuve de ce résultat, nous renvoyons à [56, Section 3.2], mais voir aussi [31, Théorème 2.1] pour ce résultat sous des hypothèses plus faibles.

Il convient de noter que les fonctions de la forme (21) sont appelées des Maxwelliennes *locales*, tandis que lorsque ρ, u, T sont des constantes, elles sont appelées des Maxwelliennes *globales*.

Lois de conservation pour les solutions de l'équation de Boltzmann. Nous considérons maintenant que Ω est l'espace complet $\mathbb{R}^d$ ou le tore $\mathbb{T}^d$ (mais nous pourrions aussi supposer que Ω est un domaine bordé avec des conditions au bord conservatives), et nous considérons que $F = F(t,x,v)$ est une solution de l'équation de Boltzmann (12), nous observons alors que (20) implique que

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} F\mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} -v\cdot\nabla_x F + \mathcal{Q}(F,F)\mathrm{d}x\mathrm{d}v = 0,\\
\frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} v\,F\mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} v\,(-v\cdot\nabla_x F) + v\mathcal{Q}(F,F)\mathrm{d}x\mathrm{d}v = 0,\\
\frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} |v|^2\,F\mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} |v|^2(-v\cdot\nabla_x F) + |v|^2\mathcal{Q}(F,F)\mathrm{d}x\mathrm{d}v = 0,
\end{aligned}$$

où nous avons utilisé le théorème de divergence ainsi que (20) pour déduire chacun des résultats ci-dessus.

Si nous complémentons alors l'Équation (12) avec une donnée initiale $F_0 : \Omega \times \mathbb{R}^d \to \mathbb{R}$, les calculs ci-dessus impliquent les lois de conservation suivantes

$$\begin{aligned} \int_{\Omega\times\mathbb{R}^d} F_t \,\mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} F_0 \,\mathrm{d}x\mathrm{d}v \quad \text{(masse)}, \\ \int_{\Omega\times\mathbb{R}^d} v\, F_t \,\mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} v\, F_0 \,\mathrm{d}x\mathrm{d}v \quad \text{(quantité de mouvement)}, \\ \int_{\Omega\times\mathbb{R}^d} |v|^2\, F_t \,\mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} |v|^2\, F_0 \,\mathrm{d}x\mathrm{d}v \quad \text{(énergie)}, \end{aligned} \tag{22}$$

pour tout $t \geq 0$.

Entropie et irréversibilité. L'une des principales contributions de L. Boltzmann dans [28] a été l'introduction de la *fonctionnelle–H*

$$H(f) := \int_{\Omega\times\mathbb{R}^d} F(t,x,v) \log F(t,x,v) \,\mathrm{d}x\mathrm{d}v,$$

qui agit comme une fonctionnelle de Lyapunov pour l'équation (12), et qui est physiquement interprétée comme une forme de "quantifier" l'*information* d'un système de particules, ou comme l'opposé de l'*entropie* du système. En effet, nous avons le résultat suivant.

Théorème 1 (Théorème H de Boltzmann)**.** Supposons que Ω soit l'espace complet $\mathbb{R}^d$ ou le tore $\mathbb{T}^d$, et que $F = F(t,x,v)$ soit une solution de l'équation de Boltzmann (12). Nous avons que

$$\frac{d}{dt} H(F) = -D(F) \leq 0,$$

où

$$D(F) := -\int_{\Omega\times\mathbb{R}^d} \mathcal{Q}(F,F) \log F = \frac{1}{4} \int_\Omega \int_{\mathbb{R}^{2d}} \int_{\mathbb{S}^{d-1}} \mathcal{B}\left(F'_* F' - F F_*\right) \log\left(\frac{F' F'_*}{F F_*}\right) \geq 0, \tag{23}$$

est appelée la fonction de *production d'entropie* (ou de *dissipation d'information*).

Remarque 2. La deuxième égalité dans (23) provient de l'utilisation de (19). De plus, l'inégalité $D(F) \geq 0$ est une conséquence du fait que la fonction $(X,Y) \mapsto (X-Y)\log(X/Y)$ est non négative

Remarque 3. Parfois, dans la littérature, la fonctionnelle H est appelée fonctionnelle d'entropie et D est appelée fonctionnelle de dissipation d'entropie. Cependant, dans le cadre de cette thèse, nous avons choisi d'utiliser les désignations présentées ci-dessus, telles que suggérées par C. Villani dans son introduction à [165], et qui sont également souvent utilisées dans la théorie de l'information pour ce type de quantités.

Le Théorème 1 implique (voir par exemple [56, Section 3.2]) que les Maxwelliennes locales sont les seules fonctions où la fonctionnelle H atteint son maximum. Il est intéressant de noter que ces résultats restent valables lorsque Ω est un domaine borné (suffisamment régulier) sous des conditions au bord spéculaires. De plus, lorsqu'il y a présence d'une température au bord (par exemple sous des conditions au bord diffusives ou de type Maxwell), il est également possible d'obtenir une version plus faible du théorème H de Boltzmann, voir [53, Chapitre III, Section 9].

Une possible interprétation du comportement non croissant de la fonctionnelle H est que les collisions microscopiques, selon l'hypothèse du chaos de Boltzmann, produisent de l'entropie à un niveau statistique.

Historiquement, l'introduction de l'Équation (12) et le Théorème H ci-dessus ont fait l'objet de polémiques à l'époque de Boltzmann.

L'une des raisons est que l'hypothèse atomique n'était pas encore totalement acceptée, et les calculs de Boltzmann étaient basés sur l'utilisation de la dynamique Newtonienne pour chacune des particules composant le "gas", et le passage approprié à la limite sous certaines hypothèses.

D'autres polémiques, peut-être plus importantes, ont été soulevées par les paradoxes que les travaux de Boltzmann semblaient évoquer. En effet, J. Loschmidt a souligné que le comportement non décroissant de la fonctionnelle H impliquait la non-réversibilité de l'équation de Boltzmann, ce qui était en contradiction apparente avec le fait que l'Équation (12) provenait d'une limite d'un système d'équations de Newton réversibles. De plus, les équations de Newton étant un cas particulier de dynamique hamiltonienne, le théorème récursif de Poincaré implique que les solutions doivent s'approcher le plus possible de l'état initial, ce qui est manifestement impossible avec une entropie croissante.

Aujourd'hui, nous savons que l'irréversibilité est une condition naturelle résultant de l'introduction d'une direction dans le temps au cours des collisions microscopiques dans les calculs de la dérivation de l'équation de Boltzmann.

Le 6^e problème de Hilbert et la limite hydrodynamique. Lors du Congrès international des mathématiciens qui s'est tenu à Paris en 1900, D. Hilbert a proposé une série de problèmes qu'il considérait comme d'une importance fondamentale pour les mathématiciens du nouveau siècle (voir [115]). Parmi ceux-ci, il y a l'incitation à développer une théorie mathématiquement rigoureuse pour le processus de limite qui mène d'une vision atomistique de la nature aux lois du mouvement des continuums, à savoir obtenir une description unifiée de la dynamique des gaz, y compris tous les niveaux de description.

Le passage de la dynamique microscopique à l'équation mésoscopique de Boltzmann a été obtenu pour la première fois par O. Lanford dans [123] pour *de très petites échelles de temps*—dans le sens où aucune collision est permise—et il a été impossible d'étendre cette méthode pour traiter les grandes échelles de temps. Plus récemment, Y. Deng, Z. Hani et X. Ma ont proposé dans [68] un moyen de rendre cette *limite thermodynamique* rigoureuse pour le modèle des sphères dures dans le tore.

La *limite hydrodynamique*, c'est-à-dire le passage des équations mésoscopiques aux équations macroscopiques, est mieux comprise et il existe une vaste littérature pour ce type de problèmes, voir [156] et les références qui s'y trouvent. Comme les motivations de l'étude du Chapitre 3 sont liées au comportement de l'équation de Boltzmann dans le régime proche de la limite hydrodynamique, nous développons brièvement ce sujet. Néanmoins, nous notons que les idées que nous exposerons sont principalement extraites de [156].

Nous considérons la forme *sans dimension* de l'équation de Boltzmann

$$\mathrm{St}\,\partial_t F + v \cdot \nabla_x F = \mathrm{Kn}\,\mathcal{Q}(F,F) \qquad \text{dans } \mathcal{U}, \tag{24}$$

où $\mathrm{Kn} > 0$ est le *nombre de Knudsen* et $\mathrm{St} > 0$ est le *nombre de Strouhal* qui coïncide souvent avec le *nombre de Mach* $\mathrm{Ma} > 0$, bien que dans certaines situations nous puissions avoir $\mathrm{Ma} << \mathrm{St}$. Nous renvoyons à [156, Chapitre 2] pour plus de détails sur la signification physique de ces constantes et leur lien avec les quantités macroscopiques.

La limite hydrodynamique de l'équation de Boltzmann (24) correspond aux situations où $\mathrm{Kn} << 1$. En effet, si l'on considère la *limite d'Euler compressible*, lorsque $\mathrm{Kn} \to 0$, l'opérateur de collision $\mathcal{Q}$ domine la dynamique et les collisions se produisent à une échelle de temps très petite par rapport aux échelles de temps des observables. Avec le théorème H de Boltzmann, cela signifie que la limite thermodynamique est atteinte presque instantanément et qu'une solution F s'approchera de l'équilibre thermodynamique donné par une Maxwellienne local de la forme (21), et où les fonctions ρ, u, et T satisfont l'équation du gaz parfait *perfect gas equation.* En d'autres termes, le nombre de Knudsen régit la transition du régime mésoscopique au régime macroscopique.

Les nombres de Mach ou de Strouhal n'ont pas de règle fixe, et leurs valeurs peuvent affecter le type d'équation hydrodynamique que l'on obtient à la limite. Par exemple, dans le régime où l'on fait $\mathrm{Kn} << 1$, si l'on prend $\mathrm{St} = \mathrm{Ma} \sim \mathrm{Kn}$ on obtient les équations incompressibles de Navier-Stokes, alors qu'en prenant $\mathrm{Kn} << \mathrm{Ma} = \mathrm{St}$ on obtient les équations incompressibles d'Euler (voir par exemple [156, Figure 2.3]).

Si nous supposons maintenant que nous avons l'Équation (24) dans un domaine spatial borné Ω, et que nous la complémentons avec la condition au bord de type Maxwell(9), il est connu (voir [156, Section 2.2.4]) que le comportement de l'équation de fluide correspondante au bord est déterminé par le rapport ι/Ma :

- lorsque $\iota/\mathrm{Ma} \to \infty$ l'équation macroscopique présentera des conditions au bord de freinage (représentées par des conditions au bord de Dirichlet),
- tandis que si $\iota/\mathrm{Ma} \to 0$, nous aurons une *condition au bord de Navier glissant* représentant l'interaction fluide-paroi.

6.2 L'équation BGK

En raison de la difficulté de la structure de l'équation de Boltzmann et des nombreux défis mathématiques qu'elle pose, de nombreux modèles plus simples ont été proposés au cours du siècle dernier pour le terme de collision. La raison en est que si l'équation *modèle* capture suffisamment bien le comportement des particules dans le régime mésoscopique, elle peut être utilisée pour compléter et prédire les expériences réelles.

C'est le cas de l'équation BGK proposée par Bhatnagar, Gross et Krook dans [22], mais, comme l'a fait remarquer C. Cercignani dans [53, Chapitre II, Section 10], elle a été introduite indépendamment par P. Welander dans [168] à peu près à la même époque.

Ce modèle cinétique est le suivant

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}F := \nu \left(\mathcal{M}_{\rho,u,T}(v) - F \right) \qquad \text{dans } \mathcal{U}, \tag{25}$$

où $\nu > 0$ est la *fréquence de collision* (qui peut dépendre de F dans certaines situations), et nous rappelons que la Maxwellienne locale $\mathcal{M}_{\rho,u,T}$ est définie dans (21), avec la densité locale $\rho = \rho(t,x)$, la vitesse moyenne locale $u = u(t,x)$ et la température locale $T = T(t,x)$ donnée par (QM).

Les principales caractéristiques de l'Équation (25) sont que ses solutions ont les mêmes lois de conservation (22) que l'équation de Boltzmann et qu'elle possède une entropie croissante. En effet, pour toute fonction $F = F(t,x,v)$ on a

$$\begin{aligned} \int_{\Omega\times\mathbb{R}^d} \mathcal{B}F \, \log F &= \int_{\Omega\times\mathbb{R}^d} \nu\, \mathcal{B}F \, \log\left(\frac{F}{\mathcal{M}_{\rho,u,T}}\right) + \int_{\Omega\times\mathbb{R}^d} \nu\, \mathcal{B}F \, \log \mathcal{M}_{\rho,u,T} \\ &= \int_{\Omega\times\mathbb{R}^d} \nu \mathcal{M}_{\rho,u,T} \left(1 - \frac{F}{\mathcal{M}_{\rho,u,T}}\right) \log\left(\frac{F}{\mathcal{M}_{\rho,u,T}}\right) \leq 0, \end{aligned} \tag{26}$$

où nous remarquons que la deuxième intégrale de la droite de la première ligne est nulle par un calcul direct et en utilisant les lois de conservation (22). En outre, l'égalité dans (26) est obtenue si et seulement si $F = \mathcal{M}_{\rho,u,T}$.

Pour des résultats sur le caractère bien posé de l'équation BGK (25), voir [151, 152]. De plus, pour ses propriétés dans la limite hydrodynamique, nous renvoyons à [155].

6.3 L'équation cinétique de Fokker-Planck (KFP)

L'équation cinétique de Fokker-Planck est une équation *hypodiffusive* également connue sous le nom d'équation de Kolmogorov ou d'équation ultraparabolique. Sous sa forme linéaire conservative, elle se lit comme ci-dessous

$$\partial_t F + v \cdot \nabla_x F = \Delta_v F + \mathrm{div}_v\,(vF) \qquad \text{dans } \mathcal{U}. \tag{27}$$

Cependant, elle peut être considérée comme une autre variante de l'équation de Boltzmann (voir [165, Section 1.6] et [53, Chapitre II, Section 10]), en considérant sa version non linéaire

$$\partial_t F + v \cdot \nabla_x F = \rho^\alpha\, \mathrm{div}_v\,(T\nabla_v F + F(v-u)). \tag{28}$$

où $\alpha\ dans[0,1]$, et nous rappelons que $\rho = \rho(t,x)$ est la densité locale, $u = u(t,x)$ est la vitesse moyenne locale , et $T = T(t,x)$ est la température locale, toutes données par (QM).

Il convient de remarquer que, lorsque $\alpha = 1$, l'équation (28) présente le même type d'homogénéité quadratique que l'équation de Boltzmann et satisfait les mêmes lois de conservation. Cependant, différentes variantes concernant l'inclusion—ou non—de ρ, u, T (ou de ses équivalents globaux) sont souvent étudiées.

Pour plus d'informations sur la modélisation et les propriétés des équations KFP, nous nous référons à [57]. En ce qui concerne le caractère bien posé et les propriétés de l'Équation linéaire (27), nous renvoyons à [48] et aux références qui s'y trouvent. Enfin, pour la littérature sur l'équation KFP non linéaire (28) nous mentionnons [5, 88] et les références qui s'y trouvent.

7 Principaux résultats de la thèse.

Nous présentons maintenant un résumé des principaux résultats obtenus au cours de cette thèse.

Tout au long de cette section, nous considérerons $f = f(t,x,v)$, une fonction de distribution telle que présentée ci-dessus, dépendant de la variable temps $t \geq 0$, de la variable position $x \in \Omega \subset \mathbb{R}^d$, pour un domaine convenablement borné, et de la variable vitesse $v \in\in \mathbb{R}^d$.

Nous discutons des résultats de chaque chapitre dans une sous-section dédiée de cette section. Chacune de ces sous-sections décrit le problème, les conditions au bord spécifiques et les principales hypothèses, suivies des principaux théorèmes et d'une discussion sur l'état de l'art et sur la manière dont nos résultats généralisent et étendent les travaux existants dans la littérature.

Nous remarquons que, pour la clarté de notre présentation, nous n'énonçons pas ici toutes les hypothèses précises nécessaires à l'obtention de chaque résultat. Celles-ci sont fournies en détail dans les chapitres individuels où chaque théorème est prouvé. De plus, la discussion de chaque résultat sera revue dans son chapitre respectif.

Nous introduisons maintenant la notion de ce que l'on appelle les *fonctions de poids admissibles.* Il s'agit de fonctions de la forme $\omega : \mathbb{R}^d \to \mathbb{R}$ définies par

$$\omega(v) := \langle v\rangle^k e^{\zeta|v|^s},$$

avec soit

$$s = 0,\ \zeta \geq 0,\ \text{et } k > k_*, \quad \text{soit} \quad s \in (0, s_*) \text{ avec } \zeta \in (0,\zeta_*),\ \text{et pour tout } k \geq 0.$$

Les valeurs exactes de $k_* > 0$, $s_* > 0$, et $\zeta_* > 0$ varient pour chaque sous-section. Afin d'éviter une surcharge de détails techniques, elles ne seront pas exposées ici, et nous les spécifierons dans chaque chapitre respectif.

Avant de présenter nos principaux résultats, nous énonçons quelques notations.

• Nous définissons $\mathcal{O} := \Omega \times \mathbb{R}^d$, et la *masse totale* de f comme $\langle\langle f \rangle\rangle_{\mathcal{O}} := \int_{\mathcal{O}} f(x,v)\mathrm{d}x\mathrm{d}v$.

• Pour un espace de mesure donné $(Z, \mathscr{Z}, \mu)$, une fonction de poids $\rho : Z \to (0,\infty)$ et un exposant $p \in [1,\infty]$, nous définissons l'espace de Lebesgue pondéré $L^p_\rho(Z)$ associé à la norme

$$\|g\|_{L^p_\rho(Z)} = \|\rho g\|_{L^p(Z)}.$$

Nous définissons également $M^1_{\omega,0}(\mathcal{O})$ comme l'espace des mesures de Radon g sur $\mathcal{O}$ dont la masse s'évanouit au bord, c'est-à-dire tel que $|g\omega|(\mathcal{O}\backslash\mathcal{O}_\epsilon) \to 0$ comme $\epsilon \to 0$, où, pour tout $\epsilon > 0$,

$$\Omega_\epsilon := \{x \in \Omega \cap B_{\epsilon^{-1}}, \delta(x) > \epsilon\}, \quad \text{et} \quad \mathcal{O}_\epsilon := \Omega_\epsilon \times B_{\epsilon^{-1}}. \tag{29}$$

De plus, nous définissons l'espace des fonctions continues dans Z comme $C(Z)$.

Enfin, nous remarquons que, pour chacune des sous-sections suivantes, nous abusons de la notation en désignant par $\mathscr{R}$ les différentes conditions au bord de Maxwell associées à différents choix de coefficients d'accommodation et de températures de la frontière.

7.1 Résultat constructif de Krein-Rutman pour les équations KFP dans un domaine

Dans le Chapitre 1, nous considérons une dimension $d \geq 1$, et nous étudions l'équation linéaire KFP

$$\begin{cases} \partial_t f &= \mathcal{L}f := -v \cdot \nabla_x f + \Delta_v f + b \cdot \nabla_v f + cf & \text{dans } \mathcal{U} \\ \gamma_- f &= \mathscr{R}\gamma_+ f & \text{sur } \Gamma_- \\ f_{t=0} &= f_0 & \text{dans } \mathcal{O}, \end{cases} \tag{30}$$

où nous avons défini

$$b = b(x,v) \in \mathbb{R}^d, \quad \text{et} \quad c = c(x,v) \in \mathbb{R}, \tag{31}$$

chacune de ces fonctions étant au moins dans $L^\infty_{\mathrm{loc}}(\mathcal{O})$, et satisfaisant les hypothèses présentées dans la Sous-section 1.1.2. Nous supposons que $\Omega \subset \mathbb{R}^d$ est un domaine borné convenablement lisse, et nous considérons la réflexion de type Maxwell sur le bord

$$\gamma_- f = \mathscr{R}\gamma_+ f = \iota_S \mathscr{S}\gamma_+ f + \iota_D \mathscr{D}\gamma_+ f \quad \text{sur } \Gamma_-, \tag{32}$$

où nous rappelons que les opérateurs de réflexion spéculaire ($\mathscr{S}$) et diffusive ($\mathscr{D}$) sont donnés par (6) et (8) respectivement, et nous supposons que la fonction de température de la surface $\Theta : \bar{\Omega} \to \mathbb{R}^*_+$ satisfait

$$\Theta \in W^{1,\infty}(\Omega), \quad \text{et} \quad \Theta_* \leq \Theta(x) \leq \Theta^*, \tag{33}$$

pour certains $0 < \Theta_* \leq \Theta^* < \infty$. En outre, nous désignons par coefficient d'accommodation $\iota := \iota_S + \iota_D$, et nous supposons que $\iota_S, \iota_D, \iota : \partial\Omega \to [0,1]$.

Nous observons qu'en raison des différents choix possibles pour les fonctions b et c, l'équation (30) ne conserve pas nécessairement la masse. Par conséquent, pour étudier le caractère bien posé et le comportement à long terme de ses solutions, nous recherchons un triplet propre (λ_1, f_1, ϕ_1) qui satisfait aux conditions suivantes

$$\lambda_1 \in \mathbb{R}, \quad \mathcal{L} f_1 = \lambda_1 f_1, \quad \gamma_- f_1 = \mathscr{R} \gamma_+ f_1, \quad \mathcal{L}^* \phi_1 = \lambda_1 \phi_1, \quad \gamma_+ \phi_1 = \mathscr{R}^* \gamma_- \phi_1. \quad (34)$$

Nous énonçons maintenant les principaux résultats du Chapitre 1. Tout d'abord, nous avons un résultat général d'existence et d'unicité pour l'équation KFP (30).

Théorème 4 (Existence et unicité)**.** Nous faisons les hypothèses ci-dessus sur Ω, Θ, b et c, en particulier nous supposons que (31), (33) sont valides, et nous supposons en outre que les hypothèses du Théorème 1.1.1 sont valides. Pour toute fonction de poids admissible ω et toute donnée initiale $f_0 \in L^p_\omega(\mathcal{O})$, $p \in [1, \infty]$, ou $f_0 \in M^1_{\omega,0}(\mathcal{O})$, il existe f unique solution globale faible de l'équation cinétique de Fokker-Planck (30). En particulier, pour tout $(x_0, v_0) \in \mathcal{O}$, il existe une solution fondamentale unique associée à la donnée initiale $f_0 := \delta_D(x_0, v_0)$.

Le sens précis de la solution sera donné dans la Proposition 1.3 (voir aussi le Théorème 1.3.5) dans un cadre L^2, dans le Théorème 1.5.2 dans un cadre général L^p, et dans le Théorème 1.5.3 dans un cadre de mesures de Radon. Ce résultat étend le résultat d'existence et d'unicité de [90, Theorem 11.5] énoncé dans un cadre L^2 plus restrictif (voir aussi [1, 66] pour d'autres résultats antérieurs). Le cadre L^2 est principalement basé sur la variante de Lions du théorème de Lax-Milgram [Chap III, §1]MR0153974, tel qu'utilisé dans [1, 66], une théorie des traces développée dans [49, 137**?** , 138] et des estimations de bord dans l'esprit de [6, 19, 138]. L'estimation de la croissance est obtenue en élaborant une fonction de poids modifiée mais équivalente pour laquelle la dissipativité de l'opérateur complet peut être établie. D'autre part, le cadre général de Lebesgue et le cadre des mesures de Radon sont plus impliqués et sont également basés sur le théorème d'ultracontractivité ci-dessous ainsi que sur certains arguments adaptés de l'équation parabolique tels que développés dans [25–27]. Il est utile de mentionner que le caractère bien posé et certaines questions de régularité pour l'équation KFP dans le tore ont été obtenus dans [1]. Pour l'ensemble de l'espace, nous nous référons aux travaux récents [12, 13] et aux références qui s'y trouvent. Enfin, l'équation KFP dans un domaine borné a été étudiée dans [147, 158, 169].

Nous examinons ensuite le premier problème de valeurs propres et le comportement à long terme qui fournit une réponse quantitative à la question des premiers éléments propres.

Théorème 5 (Comportement asymptotique à long terme)**.** Sous les hypothèses du Théorème 4, il existe deux fonctions de poids ω_1, m_1 et un exposant $r > 2$ avec $L^r_{\omega_1} \subset (L^2_{m_1})'$ tel qu'il existe un unique triplet propre $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times L^r_{\omega_1} \times L^2_{m_1}$ satisfaisant le premier problème propre (34) ainsi que les conditions de normalisation

$$\|\phi_1\|_{L^2_{m_1}} = 1, \quad \langle \phi_1, f_1 \rangle = \langle \phi_1, f_1 \rangle_{L^2_{m_1}, (L^2_{m_1})'} = 1.$$

Ces fonctions propres sont des fonctions continues et satisfont également aux conditions suivantes

$$0 < f_1 \lesssim \omega^{-1}, \quad 0 < \phi_1 \lesssim \omega \quad \text{dans} \quad \mathcal{O},$$

pour toute fonction de poids admissible ω. En outre, il existe des constantes constructives $C \geq 1$ et $\lambda_2 < \lambda_1$ telles que pour toute fonction de poids admissible fortement confinante ω, tout exposant $p\ dans [1, \infty]$ et toute donnée initiale $f_0\ dans L^p_\omega(\mathcal{O})$, la solution associée f à l'équation cinétique de Fokker-Planck (**??**) satisfait que

$$\|f(t) - \langle f_0, \phi_1 \rangle f_1 e^{\lambda_1 t}\|_{L^p_\omega} \leq C e^{\lambda_2 t} \|f_0 - \langle f_0, \phi_1 \rangle f_1\|_{L^p_\omega},$$

pour tout $t \geq 0$.

Ce résultat améliore les travaux récents [90, Section 11] (voir en particulier [90, Théorème 11.6], [90, Théorème 11.8] et [90, Théorème 11. 11]) en généralisant légèrement le cadre à une température de paroi dépendant de la position et en fournissant une approche entièrement constructive pour la stabilité exponentielle de la première fonction propre. Nous renvoyons aux travaux antérieurs [18, 104, 125] (partiellement basés sur [114, 124, 153]) où des résultats similaires sont établis pour le même type d'équation dans un domaine borné avec une condition au bord sans influx. Nous soulignons également que dans le cas conservatif, de nombreux travaux ont été réalisés concernant l'hypocoercivité et le taux constructif de convergence vers l'état d'équilibre dans [70, 81, 106, 112, 113, 166] ou plus récemment dans [1, 32, 39, 48, 79, 134]. D'un point de vue technique, ce résultat est une conséquence de la version abstraite du théorème de Krein-Rutman-Doblin-Harris qui sera présentée dans la Section 1.7.1 (et qui est vraiment dans l'esprit du travail récent [90]) avec la propriété d'ultracontractivité énoncée ci-dessous et les estimations de Harnack établies dans [97].

Les résultats ci-dessus concernant le caractère bien posé du problème et le comportement à long terme sont basés sur la propriété d'ultracontractivité suivante.

Théorème 6 (Ultracontractivité)**.** Il existe $\theta, C > 0$ et $\kappa \geq 0$ tels que toute solution f de l'équation KFP (30) satisfait

$$\|f(T,\cdot)\|_{L^\infty_\omega} \leq C \frac{e^{\kappa T}}{T^\theta} \|f_0\|_{L^1_\omega}, \quad \forall T > 0,$$

pour toute fonction de poids admissible fortement confinante ω, définie dans la Sous-section 1.1.2.

Ce résultat améliore et généralise légèrement [48, Theorem 1.1] qui établit un résultat similaire dans le cas conservatif. La preuve est très similaire à celle de [48, Theorem 1.1] bien que certaines étapes soient légèrement simplifiées. La stratégie est basée sur l'argument du gain d'intégrabilité de Nash [145] qui est cependant effectué sur une inégalité intégrale temporelle comme dans le travail de Moser [142], et il est alors plus commode d'utiliser le gain intérieur d'intégrabilité déduit du résultat de régularité de Bouchut [29] en suivant la voie ouverte par [97, Theorem 6] pour prouver une version locale quelque peu similaire. Contrairement à la dernière référence, le gain d'intégrabilité n'est pas formulé localement en x, v et intégré dans le temps mais globalement en x, v et ponctuellement dans le temps comme dans la théorie de l'ultracontractivité de Davies et Simon [63, 64]. Exactement comme dans [48], l'argument clé consiste à exhiber une fonction de poids modifiée appropriée qui est en quelque sorte légèrement plus élaborée que celle utilisée lors de la preuve des estimations de croissance dans le Théorème 4. Dans le Théorème 4 et le Théorème 6, l'hypothèse de bornes sur Ω n'est vraiment nécessaire que dans la preuve de l'unicité de la solution dans le cadre de $L^p_\omega(\Omega)$ et il est probable qu'elle puisse être supprimée. Ici, nous n'essayons pas de généraliser ces résultats au cas d'un domaine non borné, voir cependant [33] pour des résultats partiels dans cette direction.

7.2 Existence et stabilité de solutions stationnaires hors équilibre pour une équation cinétique de Fokker-Planck faiblement non linéaire dans un domaine

Dans le Chapitre 2, nous utilisons les résultats du Chapitre 1 pour étudier une équation de Fokker-Planck faiblement non linéaire avec des thermostats à chaleur BGK dans un domaine spatialement borné avec des conditions au bord de Maxwell conservatives. En particulier, de telles conditions au bord seront dotées d'un coefficient d'accommodation et d'une température de surface dépendant de l'espace.

Nous considérons $\alpha \in (0, 1/2)$, de dimension ≥ 3, et nous étudions l'équation non linéaire suivante

$$\begin{cases} \partial_t f &= -v \cdot \nabla_x f + (\alpha \mathcal{E}_f + (1-\alpha)\tau)\, \Delta_v f + \mathrm{div}_v(vf) + \mathscr{G} f \quad \text{dans} \quad \mathcal{U} \\ \gamma_- f &= \mathscr{R} \gamma_+ f \quad \text{sur } \Gamma_- \\ f_{t=0} &= f_0 \quad \text{dans } \mathcal{O}, \end{cases} \tag{35}$$

où $\tau = \tau(x) : \Omega \to \mathbb{R}$ est tel que

$$\tau_0 \leq \tau(x) \leq \tau_1 \qquad \forall x \in \Omega,$$

pour certaines constantes $\tau_0, \tau_1 > 0$, et nous avons défini la fonctionnelle d'*énergie totale*

$$\mathcal{E} = \mathcal{E}_f := \frac{1}{d} \int_{\Omega \times \mathbb{R}^d} |v|^2 f \mathrm{d}x \mathrm{d}v,$$

et le *reservoir de chaleur* du type BGK défini par

$$\mathscr{G} f = \sum_{n=1}^{\mathcal{N}} \eta_n \mathscr{G}_n f \quad \text{avec} \quad \mathscr{G}_n f = \mathbf{1}_{\Omega_n} \left(\rho_f \mathcal{M}_{T_n} - f \right),$$

pour un certain $\mathcal{N} \in \mathbb{N}$, certains paramètres $\eta_n \geq 0$, $T_n > 0$, les sous-ensembles $\Omega_n \subset \Omega$, et nous rappelons que la densité locale $\rho_f = \rho$ et la Maxwellienne $\mathcal{M}_T$ sont données par (QM) et (5), respectivement.

Nous présentons dans la Sous-section 2.1.3 une discussion détaillée sur l'interprétation physique des différents opérateurs impliqués dans l'équation (35), cf. [43] pour plus de détails sur la modélisation.

Nous rappelons que l'opérateur de réflexion de Maxwell $\mathscr{R}$ est donné par (10), avec un coefficient d'accomodation $\iota \in C(\partial\Omega, [0, 1])$, et la température de paroi $\Theta : \bar{\Omega} \to \mathbb{R}_+^*$ satisfaisant (33). En outre, nous supposons, sans perte de généralité, que $\langle\!\langle f_0 \rangle\!\rangle_{\mathcal{O}} = 1$.

Voir la Figure 4 pour une représentation graphique d'une possible configuration du problème étudié par l'Équation (35).

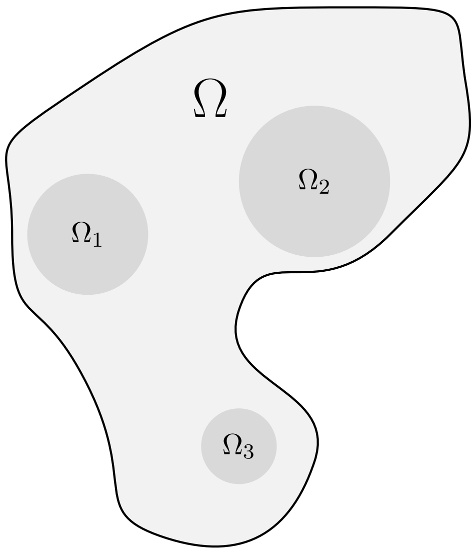

Figure 4: Une configuration possible d'un domaine Ω avec trois zones (en gris) où agissent les thermostats BGK.

Nous obtenons alors les résultats suivants : Nous présentons d'abord un théorème de bien-posé et de stabilité dans le cadre linéaire lorsque $\alpha = 0$.

Théorème 7. On suppose $\alpha = 0$. Il existe $\mathfrak{F}^0 \in L^2(\Omega, H^1(\mathbb{R}^d)) \cap L^\infty(\mathcal{O})$ unique solution stationnaire de l'Équation linéaire (35). De plus, on a $\langle\!\langle \mathfrak{F}^0 \rangle\!\rangle_{\mathcal{O}} = 1$ et, pour toute fonction de poids admissible ς, nous avons que

$$\|\nabla_v \mathfrak{F}^0\|_{L^2_\varsigma(\mathcal{O})} < \infty \qquad \text{et} \qquad \mathfrak{F}^0(x,v) \lesssim (\varsigma(v))^{-1}.$$

En outre, soit ω une fonction poids admissible. Pour toute donnée initiale $f_0 \in L^2_\omega(\mathcal{O})$, il existe une unique solution globale renormalisée $f \in C(\mathbb{R}_+, L^2_\omega(\mathcal{O}))$ de l'équation linéaire (35), et il existe $\lambda > 0$ tel que

$$\|f_t - \mathfrak{F}^0\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-\lambda t}\|f_0 - \mathfrak{F}^0\|_{L^2_\omega(\mathcal{O})} \qquad \forall t \geq 0.$$

Le sens précis des solutions globales fournies par le Théorème 7 est donné par le Théorème 2.3.3 avec le choix $\Lambda = \tau$. Nous remarquons également que le Théorème 2.3.3 n'est rien d'autre qu'une application directe de [49, Theorem 2.11] ainsi que de la théorie des traces issue de [49, Theorem 2.8].

L'existence et l'unicité d'une solution stationnaire pour le problème linéaire, qui est également à comprendre au sens du Théorème 2.3.3, ainsi que sa stabilité, sont obtenues par application directe de la théorie de Krein-Rutmann-Doblin-Harris développée dans [160, Theorem 6.1]. Nous renvoyons également à [45, Theorem 7.1] pour un résultat analogue dans un cadre non conservatif, ainsi qu'à [90] pour l'étude d'un résultat de type Krein-Rutmann-Doblin-Harris dans un cadre théorique plus général.

Nous notons enfin que le Théorème 7 constitue une légère généralisation de [45, Theorems 1.1 and 1.2].

Dans le cadre non-linéaire, nous avons alors l'existence d'un état stationnaire pour $\alpha > 0$ suffisamment petit.

Théorème 8. Il existe $\alpha^\star \in (0, 1/2)$ tel que pour tout $\alpha \in (0, \alpha^\star)$, il existe une fonction positive $\mathfrak{F}^\alpha \in L^2(\Omega, H^1(\mathbb{R}^d)) \cap L^\infty(\mathcal{O})$, solution stationnaire de l'Équation (35). De plus, $\langle\!\langle \mathfrak{F}^\alpha \rangle\!\rangle_{\mathcal{O}} = 1$ et, pour toute fonction de poids admissible ω, nous avons que

$$\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} < \infty, \quad \mathfrak{F}^\alpha(x,v) \lesssim (\omega(v))^{-1}, \quad \text{et} \quad \mathcal{E}_{\mathfrak{F}^\alpha} \leq 2\mathcal{E}_{\mathfrak{F}^0},$$

uniformément en α, et où $\mathfrak{F}^0$ est donné par le Théorème 7.

La conséquence principale du Théorème 8 est l'existence d'un NESS pour l'Équation non-linéaire (35), ainsi que certaines de ses propriétés qualitatives concernant la régularité et le comportement asymptotique en vitesse. Notons que la preuve du Théorème 8 repose sur l'application d'un argument de point fixe dans l'esprit de la preuve de [85, Theorem 1]. De plus, $\mathfrak{F}^\alpha$ est un état stationnaire au sens du Théorème 2.3.3 en prenant $\Lambda = \alpha\mathcal{E}_{\mathfrak{F}^\alpha} + (1-\alpha)\tau$.

Remarquons que le Théorème 8 généralise [43, Theorem 1.2] et que, contrairement à ce travail, nous observons des différences majeures dans le comportement et les propriétés du NESS en l'absence de conditions aux limites périodiques : nous n'avons aucune raison de croire que le NESS sera indépendant de la variable spatiale x, les bornes sur la fonctionnelle d'énergie $\mathcal{E}$ ne peuvent pas être obtenues comme dans [43, Lemma 1.1] (voir la Sous-section 2.1.4), nous ne disposons pas d'assez d'informations pour exclure l'existence d'états stationnaires avec énergie totale non bornée, et nous n'avons pas accès à une représentation explicite du NESS.

Enfin, nous énonçons le résultat de stabilité suivant pour le NESS précédent.

Théorème 9. Nous considérons une fonction de poids admissible ω. Il existe $\alpha^{\star\star} \in (0, \alpha^\star)$ et $\delta > 0$, où $\alpha^\star$ est donné par le Théorème 8, tels que pour tout $\alpha \in (0, \alpha^{\star\star})$ et pour toute donnée initiale $f_0 \in L^2_\omega(\mathcal{O})$ vérifiant

$$\|f_0 - \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \leq \delta,$$

il existe $f \in L^2_\omega(\mathcal{U})$ solution faible globale de l'Équation (35). De plus, il existe $\eta > 0$ tel que

$$\|f_t - \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-\eta t}\|f_0 - \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \qquad \forall t \geq 0.$$

Les solutions globales fournies par le Théorème 9 sont construites en ce sens que la fonction $h := f - \langle\langle f_0 \rangle\rangle_{\mathcal{O}} \mathfrak{F}^\alpha$ satisfait l'Équation (2.8.1), au sens de la Proposition 2.4.

Il convient de souligner que la Proposition 2.4 est essentiellement une application de la variante de Lions du théorème de Lax-Milgram [126, Chap III, §1] telle qu'utilisée dans [49], voir aussi [45, 48, 49, 90] pour des arguments similaires concernant l'existence de solutions d'équations cinétiques. La théorie des traces a été empruntée principalement à [45, 49], mais nous renvoyons également à [90, 137, 138] pour d'autres références sur la théorie des traces pour les équations cinétiques.

En outre, nous soulignons que, pour obtenir les estimations a priori conduisant à la preuve de la Proposition 2.4, nous avons utilisé les fonctions de poids modifiées de [45, 48, 90] afin de contrôler la condition de Maxwell au bord.

De plus, l'estimation de décroissance a été obtenue en définissant une nouvelle norme rappelant [46, Proposition 3.6], [50, Proposition 3.2] et [139, Proposition 4.1]. Il convient de remarquer que nous ne sommes pas en mesure de construire une théorie de *hypocoercivité* dans l'esprit de [21, 48, 79, 166], en raison du manque d'informations supplémentaires sur l'état stationnaire, notamment des bornes de positivité et de régularité.

Nous notons que le Théorème 9 généralise [43, Theorem 1.3] et nous remarquons que les techniques utilisées pour l'obtention de ces résultats diffèrent de celles développées lors de la preuve des théorèmes principaux de [43]. En particulier, nous n'avons pas besoin d'étudier le processus ergodique sous-jacent associé à l'opérateur linéarisé pour obtenir nos résultats.

7.3 L'équation de Boltzmann sur des domaines lisses et cylindriques avec conditions de Maxwell au bord.

Au Chapitre 3 nous étudions le caractère bien posé et le comportement en temps long de l'équation de Boltzmann dans le régime proche de la *limite hydrodynamique*, en présence de conditions de Maxwell isothermes au bord, dans des domaines lisses et cylindriques.

Cette étude est motivée comme une première étape vers l'analyse de l'équation de Boltzmann dans des domaines cylindriques dont chacune des bases présente une condition de bord diffusive associée à des températures différentes. Cette configuration s'inscrit dans le cadre physique de la *thermodynamique hors équilibre*, et soulève des questions mathématiques intéressantes quant à l'existence d'états stationnaires hors équilibre et à leurs propriétés qualitatives, telles que l'unicité et la stabilité.

Nous considérons un petit $\varepsilon > 0$ et nous étudions l'équation de Boltzmann suivante

$$\begin{cases} \varepsilon\partial_t f &= -v\cdot\nabla_x f + \varepsilon^{-1}\mathcal{Q}(f,f) & \text{dans } \mathcal{U} \\ \gamma_- f &= \mathscr{R}\gamma_+ f & \text{sur } \Gamma_- \\ f_{t=0} &= f_0 & \text{dans } \mathcal{O}, \end{cases} \tag{36}$$

où $\langle\langle f_0 \rangle\rangle_{\mathcal{O}} = 1$.

Notons que la présence du petit paramètre $\varepsilon > 0$ dans l'équation reflète le fait que le système est proche de sa *limite hydrodynamique*.

L'*opérateur de collision de Boltzmann* $\mathcal{Q}$ représente les collisions entre particules à l'intérieur de Ω, et est donné par la forme bilinéaire

$$\mathcal{Q}(G,H) := \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\,\left[G(v'_*)H(v') + H(v'_*)G(v') - G(v_*)H(v) - G(v)H(v_*)\right] \mathrm{d}\sigma \mathrm{d}v_*,$$

où nous avons défini

$$v' := v - ((v-v_*)\cdot\sigma)\sigma, \qquad v'_* := v_* + ((v-v_*)\cdot\sigma)\sigma,$$

avec $\sigma \in \mathbb{S}^2$, et le *noyau de collision* $\mathcal{B} = \mathcal{B}(|v-v_*|,\sigma)$, choisi comme celui associé au modèle des *sphères dures* :

$$\mathcal{B}(|v-v_*|,\sigma) := |(v-v_*)\cdot\sigma|.$$

Dans ce cas, l'opérateur de réflexion de Maxwell $\mathscr{R}$, que nous rappelons être donné par (10), est pris dans cette sous-section avec la température de paroi constante $\Theta \equiv 1$.

Nous présentons maintenant les deux types d'*hypothèses géométriques* pour notre domaine Ω, et le choix respectif du coefficient d'accommodation (ι) dans chaque cas.

(H1) Supposons que $\Omega \subset \mathbb{R}^3$ est un domaine C^2 ouvert, et que $\delta \in C^2(\mathbb{R}^3,\mathbb{R}) \cap W^{3,\infty}(\mathbb{R}^3,\mathbb{R})$. De plus, prenons $\iota \in C(\partial\Omega)$ et supposons qu'il existe $\iota_0 \in (0,1]$ tel que pour tout $y \in \partial\Omega$ on ait $\iota(y) \in [\iota_0, 1]$.

(H2) Supposons que $\Omega = (-L,L)\times\Omega_0$, pour un certain $L > 0$ et où $\Omega_0 \subset \mathbb{R}^2$ est la boule de rayon $\mathfrak{R} > 0$ centrée à l'origine. Dans ce cas, nous définissons aussi

$$\Lambda_1 := \{-L\}\times\Omega_0, \quad \Lambda_2 := \{L\}\times\Omega_0, \quad \Lambda_3 := (-L,L)\times\partial\Omega_0,$$

et $\Lambda := \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. En outre, nous imposons des conditions aux limites *mixtes* en prenant $\iota = \mathbf{1}_{\Lambda_1\cup\Lambda_2}$, c'est-à-dire une condition de bord purement diffusive sur les bases du cylindre $(\Lambda_1 \cup \Lambda_2)$, et de la spécularité sur la surface latérale (Λ_3).

Dans ce cadre, nous avons le résultat suivant pour l'équation de Boltzmann.

Théorème 10. Supposons que l'Hypothèse (H1) ou l'Hypothèse (H2) soit vérifiée et soit ω une fonction de poids admissible.

Il existe $\varepsilon_0 > 0$ tel que pour tout $\varepsilon \in (0,\varepsilon_0)$ il existe $\eta(\varepsilon) \in (0,1)$, satisfaisant $\eta(\varepsilon) \to 0$ lorsque $\varepsilon \to 0$, tel que pour toute donnée initiale $f_0 \in L^\infty_\omega(\mathcal{O})$ vérifiant

$$\|f_0 - \mathcal{M}\|_{L^\infty_\omega(\mathcal{O})} \le (\eta(\varepsilon))^2,$$

il existe une fonction $f \in L^\infty_\omega(\mathcal{U})$, solution globale unique de l'équation de Boltzmann (36) au sens des distributions. De plus, il existe une constante constructive $\theta > 0$ telle que

$$\|f_t - \mathcal{M}\|_{L^\infty_\omega(\mathcal{O})} \le e^{-\theta t}\,\eta(\varepsilon) \qquad \forall t \ge 0.$$

Le sens précis de la solution donnée par le Théorème 10 est construit dans le Théorème 3.1.3, après avoir posé $h := f - \mathcal{M}$, effectué le changement de variables de la Sous-section 3.1.3, et étudié l'équation obtenue pour h.

Le cadre de ce chapitre est motivé par, et donc étroitement lié à, celui développé dans [109, 110]. Cependant, dans l'étude de l'équation de Boltzmann, notre résultat constitue une véritable généralisation du leur sur plusieurs aspects essentiels. Premièrement, nous élaborons une théorie $L^2 - L^\infty$ plus fine que celle présentée dans [109, 110] pour le problème linéarisé, nous permettant de dériver des taux explicites et constructifs de décroissance vers l'équilibre, allant au-delà des résultats de bornitude obtenus auparavant. Deuxièmement, nous menons aussi l'analyse dans des domaines cylindriques, introduisant la présence d'irrégularités géométriques dans l'obtention des estimations, généralisant ainsi la méthode d'étirement de [109, 110] au cadre cylindrique. Troisièmement, même si nous n'atteignons pas la gamme complète $[0, 1]$ pour le coefficient d'accommodation dans les domaines lisses, nous permettons à ι d'être une fonction continue dépendant de la position, et dans les domaines cylindriques nous admettons même qu'il soit discontinu.

Ce dernier fait rend le Théorème 3.1.1 également une généralisation de [36], où l'on rappelle que le coefficient d'accommodation devait avoir une borne inférieure imposée de $\sqrt{2/3}$. De plus, nous fournissons aussi des résultats de bien-posé pour des données initiales ayant une large gamme de décroissance à l'infini, y compris polynomiale.

Enfin, notons que, à notre connaissance, il s'agit du premier résultat de bien-posé pour l'équation de Boltzmann dans des domaines cylindriques avec conditions de Maxwell au bord et un coefficient d'accommodation discontinu variant sur l'intervalle $[0, 1]$.

8 Perspectives pour des recherches futures

Nous discutons à présent des axes de recherche que notre travail au cours de cette thèse ouvre pour des investigations futures.

Tout d'abord, nous souhaitons étendre et généraliser les résultats de la Sous-section 7.2 à des cas sans les petits paramètres accompagnant le terme non linéaire, ainsi qu'à d'autres modèles cinétiques de Fokker–Planck non linéaires plus généraux (voir par exemple la discussion de la Sous-section 6.3). L'argument clé pour y parvenir réside, entre autres, dans l'obtention d'inégalités de Harnack renforcées jusqu'à la frontière du domaine spatial, fournissant des conditions de forte positivité locale en espace pour les grandeurs macroscopiques.

Ensuite, nous souhaitons généraliser les résultats concernant l'équation de Boltzmann de la Sous-section 7.3 à des domaines avec conditions de Maxwell non isothermes, ce qui inclura le cas d'un cylindre avec des températures différentes aux bases. Pour ce faire, nous considérerons un cadre perturbatif où la fluctuation de température est petite et nous emploierons, tout en les généralisant, les idées développées dans [82, 83].

Enfin, nous avons l'intention d'utiliser les outils et idées développés au cours de cette thèse pour poursuivre notre investigation d'autres types de problèmes cinétiques dans des domaines bornés avec conditions de Maxwell non isothermes, tels que l'équation de Landau ainsi que les systèmes de Vlasov–Poisson–Boltzmann ou de Vlasov–Poisson–Landau, par exemple.

9 Structure de la thèse

Nous structurons cette thèse de la manière suivante.

La Partie I est consacrée à l'étude des équations de Fokker–Planck linéaires et non linéaires. Elle contient tout d'abord le Chapitre 1, où nous présentons le cadre pour l'équation de Fokker–Planck linéaire (30) et où nous démontrons le Théorème 4, le Théorème 5 ainsi que le Théorème 6. Ce chapitre est suivi du Chapitre 2, consacré à l'étude de l'équation de Fokker–Planck non linéaire (35), dans lequel nous établissons le Théorème 7, le Théorème 8 et le Théorème 9.

Enfin, la Partie II est dédiée à l'équation de Boltzmann (36) dans des domaines réguliers et cylindriques, telle que présentée dans la Sous-section 7.3. Nous y démontrons notamment le Théorème 10.

Pour terminer, nous soulignons que les notations sont définies indépendamment dans chaque chapitre. Ainsi, à l'intérieur d'un même chapitre, les objets et opérateurs sont notés de façon cohérente, mais ces notations peuvent varier d'un chapitre à l'autre.

Introduction (english version)

This thesis studies the well-posedness and long-time behavior of solutions to different *kinetic equations* in bounded spatial domains.

The objective for this introduction is to set the stage of the thesis:

We will first present the different levels of mathematical description for particle systems, where we will explain the physical and mathematical interest on the study of the kinetic theory of gases.

We then give an overview of the general line of research we consider during our work and we explain the several physical problems motivating the different problems studied during each of the chapters of this thesis.

Next, we introduce the general framework on the study of kinetic equations: their fundamental form, the appropriate boundary conditions for the study of their Cauchy problem, and the *hypocoercivity* techniques for *linearized* kinetic equations in bounded domains.

Following this, we will present several examples of kinetic equations relevant to this thesis, emphasizing both their mathematical properties and their role in modeling phenomena and possible physical applications.

At last, we will outline our main results followed by a discussion on future lines of research, and the general structure of the thesis.

1 Levels of description of statistical systems

One of the main objectives of mathematical physics is to describe general many-particle (or many-body) systems composing a *dilute gas* like the air, galaxies or plasma. To accomplish such a task, we know today three main physical levels to describe such a system.

• **The microscopic/Newtonian viewpoint** that takes each particle of the system as a point in space, moving with a velocity vector, and writes the Newtonian equations for the interactions between each particle.

This method is however impractical in most situations of interest, where N (the number of particles) can be very high. In particular, already for three bodies we know that this system is chaotic, i.e the dynamics of the particles are highly sensible to small perturbations of the initial conditions (see for instance [144, 164]).

It is thus clear that when we want to study gases, with the Avogadro number stating that $N \approx 10^{23}$, the milky way galaxy, with $N \approx 10^{11}$ stars (see Figure 5) , or the core of the sun, with $N \approx 10^{21}$ particles, this approach becomes practically impossible.

• **The macroscopic viewpoint** where we take a cube in space, small enough to be negligible in relation to the entire system, but large enough to be considered as a unit, i.e. containing enough particles to be treated statistically. We then could consider the behavior of observable quantities of this unit such as mass, momentum, energy, pressure,

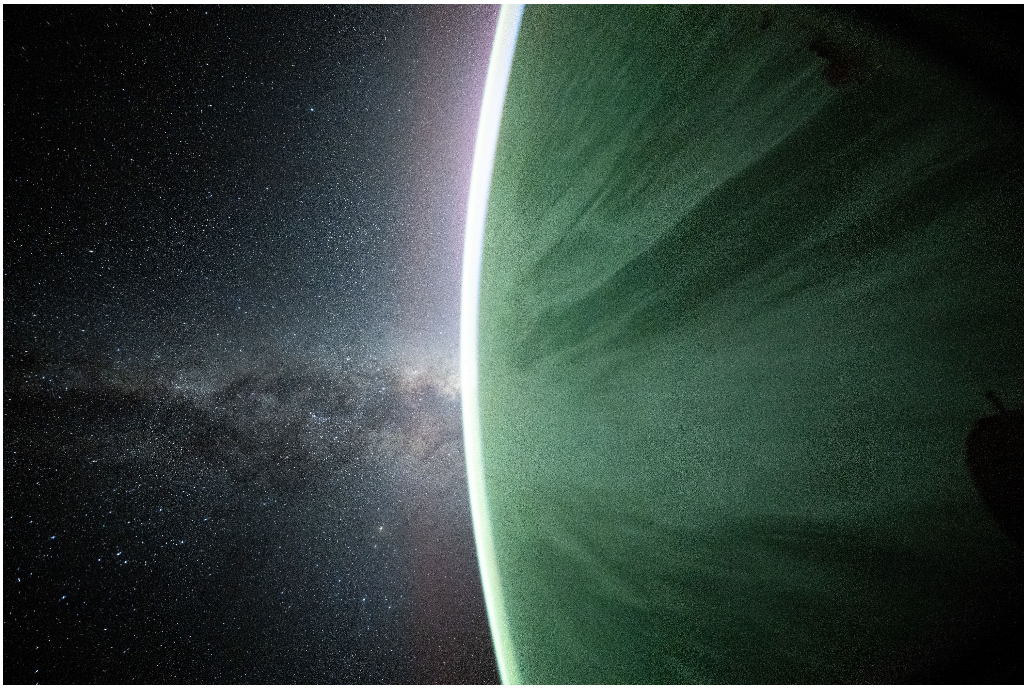

Figure 5: Photo of the Milky Way appearing beyond Earth's horizon taken by NASA astronaut Don Pettit on Jan. 29, 2025. *Image credit: NASA/Don Pettit.*

temperature, heat flux, and others. Doing so leads to the hydrodynamical description of the system via the Euler or Navier-Stokes equations. For more references on this we refer towards [122, Chapter 1],[96, Section2], and the references therein.

This viewpoint has however one major drawback: elements like viscosity or heat conductivity are given as phenomenological or experimental data, and are not related to microscopic behaviors.

• **The mesoscopic/kinetic viewpoint** represents an alternative somewhere in between these two previous descriptions. Instead of looking at which particles does what, we are interested instead in *how many* particles *in average* behave in a certain manner. This translates mathematically on the study of the *distribution function* characterizing the system, and that measures the "proportions" of particles in the phase space of position and velocity.

Moreover, this distribution function will encode at the same time molecular interactions (in a statistical manner), as well as the *macroscopic (also named measurable or observable) quantities* such as mass, momentum and energy.

The formal way of obtaining a mathematical description of such an object is by taking the limit, in the microscopic representation of the many-particle of the system, as $N \to \infty$ *in a statistical way.* This sort of limit is referred to as a *many-particle limit* or *thermodynamical limit.* See for instance [56, Chapter 2], [123], or more recently [68], for the particular case of the *Boltzmann equation.*

2 Motivation and framework of the thesis

This thesis falls within the framework of study of statistical particle systems pictured by their mesoscopic description. We are particularly interested in those within the state of *non thermodynamical equilibrium*, i.e when there is presence of fluxes of *observable quantities* such as mass or energy.

It is worth remarking in this context that the *second law of thermodynamics* implies that isolated physical systems, while they might be in non-equilibrium at some instant of time, are bound to evolve towards an equilibrium steady state. Mathematically, this is equivalent to the fact that such a steady state looks as a *Maxwellian* (gaussian outside the world of kinetic theory) distribution. This was first hypothesized by J. C. Maxwell in [131] and later proved by L. Boltzmann in [28], with the introduction of the first kinetic equation and the *H–theorem*, a way to quantify the production of *entropy* of the system and which implied—at least qualitatively—such long-time behavior towards equilibrium steady states.

Boltzmann's impactful results became the starter point of the kinetic theory of gases and the beginning of many physical and mathematical studies. We revisit this subject later in Section 6.

However, when our system of particles is not isolated, and instead is subject to non-conservative forces like the presence of heat thermostats interacting with particles, the second law of thermodynamics does not applies thus, even if the system is allowed to evolve as $t \to \infty$, we will still witness fluxes of energy. We say in this case that we are in the presence of a non-equilibrium steady state (NESS), and in particular it cannot look as a Maxwellian distribution (see [162] for a result in this direction for the particular case of the *kinetic Fokker-Planck* equation).

The study of such systems, both physically and mathematically, is today extraordinarily complicated. From the physical side, the creation of fruitful experiments is delicate and often encounters limitations on maintaining fixed temperatures for the thermostats long enough to make meaningful observations on the long-time behavior of particles, see for instance the comments from [94]. On the mathematical side, already on the equilibrium setting for Boltzmann's equation and related models, their study has required—over the last century—the creation of whole new fields, theories and techniques to give rigorous sense to solutions and quantify their trend to equilibrium, remaining yet today several open questions on the field.

It wasn't until the end of the XX$^{\text{th}}$ century, with the access to powerful computers that physicist gave back attention to these problems by the means of simulations, thus encouraging the mathematical study of problems such as the existence of non-equilibrium steady states, their uniqueness or lack thereof, and stability. For mathematical works in this setting we refer towards [7, 8, 10, 42, 44, 82] in the framework of *Boltzmann's equation* and towards [19, 41, 43, 45, 84, 85, 87] and the references therein for related results in other kinetic models.

This thesis is inscribed in this framework and its motivated by the study of questions of well-posedness and stability of non-equilibrium steady states for kinetic equations within bounded domains in the presence of heat thermostats. To be more precise, we now briefly explain the main lines of research and motivations for each of the (numbered) chapters of our thesis.

• In Chapter 1 we study a general linear *kinetic Fokker-Planck* equation in the presence of a spatially varying temperature at the boundary and we prove the existence and uniqueness of a non-equilibrium steady state. Moreover, we also characterize the long time behavior of its solutions in a quantitative manner.

• During Chapter 2 we use the results from Chapter 1 to study a weakly non-linear *kinetic Fokker-Planck* equation with *BGK* thermostats within a bounded domain, also presenting spatially varying temperatures at the boundary. We prove the existence of a non-equilibrium steady state and its stability, under a smallness condition for the initial data.

• At last, we dedicate Chapter 3 to study a *Boltzmann equation* near its hydrodynamic limit. We study this problem confined in a cylindrical domain where the bases have a fixed temperature and there is specular reflection in its lateral surface. The main motivation of this study is as a first step on the study of the physically relevant setting of the Boltzmann equation in a cylindrical domain with different temperatures on each bases, and specular reflection on the lateral surface.

For a precise definition of the boundary conditions for kinetic equation mentioned above, or how to represent the presence of a temperature at the boundary, see Section 4. Furthermore, for the reader interested on discussions on equilibrium and non-equilibrium systems from a more physics oriented perspective see [94, 117] and [93, Chapter 9].

3 General framework for kinetic equations

As exposed before, for a given dimension $d \geq 1$, we are interested in a distribution function $F = F(t, x, v)$ such that, at each instant of time $t \in \mathbb{R}_+$, it *represents* a particle at position $x \in \Omega$, where $\Omega \subset \mathbb{R}^d$ is a domain, and moving with velocity $v \in \mathbb{R}^d$.

We note that the above depiction of F is in a *statistical sense.* A more mathematically accurate way to describe F is as a t dependent probability measure (although we can also consider measures absolutely continuous with respect to Lebesgue's), and is such that for each $t \geq 0$, the quantity $F(t, x, v)\mathrm{d}x\mathrm{d}v$ is the average of particles in the cube of size $\mathrm{d}x\mathrm{d}v$ within the phase space $\Omega \times \mathbb{R}^3$.

Furthermore, it is worth remarking that even though the more physical choice of dimension is $d = 3$, it is physically meaningful and mathematically interesting to study these problems in arbitrary dimensions.

The general formulation for a kinetic equation then reads

$$\partial_t F + \mathbf{T}F = \mathbf{Q}(F) \quad \text{in } \mathcal{U} := (0, \infty) \times \Omega \times \mathbb{R}^3, \tag{1}$$

where we have the, possibly non-linear, so called *collision operator* $\mathbf{Q}$, and we have defined the *transport* operator

$$\mathbf{T}F := v \cdot \nabla_x F + \nabla_x \phi \cdot \nabla_v F,$$

for some *potential* function $\phi : (0, \infty) \times \Omega \to \mathbb{R}$.

In general, the distribution F is not physically measurable but, for fixed $(t, x) \in \mathbb{R}_+ \times \Omega$ it is possible to observe certain of the properties of the particles that F describes. Such properties are called *macroscopic quantities* (MQ) and are related to the distribution F by the following formulas

$$\begin{array}{llcl}
\text{local density} & \rho(t,x) & := & \displaystyle\int_{\mathbb{R}^3} F(t,x,v)\mathrm{d}v, \\[2ex]
\text{local bulk velocity} & u(t,x) & := & \displaystyle\frac{1}{\rho(t,x)} \int_{\mathbb{R}^3} v\, F(t,x,v)\mathrm{d}v, \\[2ex]
\text{local temperature} & T(t,x) & := & \displaystyle\frac{1}{d\,\rho(t,x)} \int_{\mathbb{R}^3} |v - u(t,x)|^2\, F(t,x,v)\mathrm{d}v, \\[2ex]
\text{and local entropy} & S(t,x) & := & \displaystyle -\frac{1}{\rho(t,x)} \int_{\mathbb{R}^3} F(t,x,v) \log F(t,x,v)\mathrm{d}v.
\end{array} \tag{MQ}$$

Regarding the possible choices for Ω we have four main possibilities:

• On the one hand, for unbounded choices of Ω we encounter in the literature either the full space $\mathbb{R}^d$, or unbounded domains with boundaries, like for instance the half-space $\mathbb{R}^d_+ := \{(x_1, \dots, x_d) \in \mathbb{R}^d, x_1 > 0\}$.

• On the other hand, if we want to work in a set Ω with finite Lebesgue measure, we can either choose the torus $\mathbb{T}^d$, meaning that $x \in [0,1]^d$ and we assume F to be periodic in this box, or we take Ω to be an arbitrary (smooth enough) bounded domain in $\mathbb{R}^d$.

In particular, we remark that when Ω is either $\mathbb{R}^d_+$ or a bounded domain we need to complement Equation (1) with suitable boundary conditions.

4 Boundary conditions

We assume Ω to be a bounded domain in $\mathbb{R}^d$, and we assume that there exists a function

$$\delta \in W^{1,\infty}(\mathbb{R}^3, \mathbb{R}) \text{ such that } \Omega = \{x \in \mathbb{R}^3,\, \delta(x) > 0\}, \text{ and } |\delta(x)| = \operatorname{dist}(y, \partial\Omega) \tag{2}$$

on a neighborhood of the boundary. Here the function $\operatorname{dist}(\cdot, \partial\Omega)$ represents the distance function to the boundary set $\partial\Omega$.

Defined this way, we observe that $\partial\Omega = \{x \in \mathbb{R}^3,\, \delta(x) = 0\}$ and we classically have that $|\nabla\delta(x)|= 1$. Therefore, we define the normal outward vector as

$$n_x = n(x) := -\nabla\delta(x) = -\frac{\nabla\delta(x)}{|\nabla\delta(x)|} \quad \text{for almost every } x \in \bar{\Omega}. \tag{3}$$

We further define the boundary set $\Sigma = \partial\Omega \times \mathbb{R}^d$ and we distinguish between the sets of *outgoing* (Σ_+), *incoming* (Σ_-), and *grazing* (Σ_0) velocities at the boundary defined by

$$\Sigma_\pm := \{(x,v) \in \Sigma,\, \pm n_x \cdot v > 0\}, \quad \text{and} \quad \Sigma_0 := \{(x,v) \in \Sigma,\, n_x \cdot v = 0\}.$$

Furthermore, we also denote $\Gamma := (0,\infty) \times \Sigma$ and accordingly $\Gamma_\pm := (0,\infty) \times \Sigma_\pm$. We then define γF as the trace function associated with F over Γ, and $\gamma_\pm F := \mathbf{1}_{\Gamma_\pm} \gamma F$.

We note that, due to the nature of the system we are describing, we are not able to fix the behavior of the particles leaving the domain, i.e those with coordinates $(x,v) \in \Sigma_+$. This is due to the fact that their dynamics are completely determined by their "history" within Ω. Therefore the boundary conditions will only be given on $\gamma_- F$, the incoming part of the trace.

We also remark that, a very natural way to define the incoming particles is to relate them with those leaving Ω. Physically, this represents that the particle undergoes a *reflection*, of some kind, at the boundary.

However, the problem of describing the interaction between a gas and a wall, even though is has been investigated at least since the XIX century, remains rather difficult, see [54, Chapter III, Section 1] for a physics oriented discussion on this.

In particular, it is known that in general, such collisions might depend on the temperature at the boundary, the cleanliness and roughness of the material, and even time, as the surface is subject to changes due to chemical processes between the particles colliding with those present at the wall.

Nonetheless, under general assumptions we may model such process by

$$|n_x \cdot v|\, \gamma_- F(t,x,v) = \int_{\mathbb{R}^d} \gamma_+ F(t,x,u)\, \mathscr{K}(u \to v, x)\, (n_x \cdot u)_+ \, du \quad \text{on } \Gamma_-, \tag{4}$$

where $\mathscr{K} = \mathscr{K}(u \to v, x)$ is called the *scattering kernel* and it describes the gasses interaction with the boundary. This kernel can be assumed to have the following properties:

• *Non-negative:* $\mathscr{K} \geq 0$, since it represents the probability distribution related to the change of velocity after a boundary collision.

• *Nonporous and non-adsorbing/Conservative:* if the wall is supposed to re-emit every particle, in disregard with its velocity, we assume that

$$\int_{u \cdot n_x < 0} \mathscr{K}(u \to v, x) du = 1, \qquad \forall x \in \partial\Omega.$$

• *Reciprocity law/Detailed balance:* There is a Maxwellian distribution defined as

$$\mathcal{M}_\Theta = \mathcal{M}_\Theta(v) := \frac{1}{(2\pi\Theta)^{d/2}} \exp\left(-\frac{|v|^2}{2\Theta}\right), \tag{5}$$

where $\Theta : \bar{\Omega} \to \mathbb{R}_+^*$ represents a prescribed temperature at the boundary, for which there holds

$$\int_{\mathbb{R}^d} \mathcal{M}_\Theta(t, x, u)\, \mathscr{K}(u \to v, x)\, (n_x \cdot u)_+\, du = |n_x \cdot v| \mathcal{M}_\Theta(t, x, v).$$

Under the above conditions, we have in particular that there holds the Darrozès and Guiraud inequality, see e.g [62] or [54, Chapter III, Section 4].

The general formulation of (4), for the boundary conditions, aims to describe the complex behavior present in the interactions between the particles of a gas and a wall, through a general formulation of it and leaves the scattering kernel $\mathscr{K}$ as a phenomenological operator to be adapted to the particularities of each practical situation.

However, defined in such a general way, the boundary conditions do not provide good enough information to establish a well-posedness theory for kinetic equations. Therefore, we can either restrict even more the behavior of $\mathscr{K}$ by introducing extra hypothesis, or choose particular cases of boundaries approximating the behavior of a gas at a wall.

Opting for the latter, we distinguish in the literature the following choices of boundary conditions to complement Equation (1).

• **The inflow boundary condition.** For a given $\mathfrak{f} : \Gamma_- \to \mathbb{R}$ we set

$$\gamma_- F(t, x, v) = \mathfrak{f}(t, x, v) \quad \text{ on } \Gamma_-.$$

Physically, this is equivalent to injecting into the domain particles following the dynamics given by $\mathfrak{f}$.

• **Bounce-back boundary condition.** We take

$$\gamma_- F(t, x, v) = \gamma_+ F(t, x, -v) \quad \text{ on } \Gamma_-.$$

This boundary condition is not really very physical but, due to the fact that it relates the outgoing and incoming particles in a rather simple way, it has been used in toy models to understand more complicated problems.

• **Specular reflection boundary condition.** We consider

$$\gamma_- F(t, x, v) = \mathscr{S} \gamma_+ F(t, x, v) := \gamma_+ F(t, x, \mathcal{V}_x v) \quad \text{ on } \Gamma_-, \tag{6}$$

where $\mathcal{V}_x v := v - 2(n_x \cdot v) n_x$.

This boundary condition, represents a particle being reflected on the wall in the same way that light reflects on a mirror, i.e it follows the principle of *the angle of incidence is the same as the angle of reflection*, see Figure 6 for a graphical representation. Indeed, for any $x \in \partial\Omega$, such that the normal vector n_x is well defined, and any $v \in \mathbb{R}^d$ we observe that

$$\mathcal{V}_x v \cdot n_x = -v \cdot n_x, \quad \text{and} \quad |\mathcal{V}_x v|^2 = |v|^2,$$

which validates the fact that the particle bounces off the wall symmetrically with respect to the normal at the point of collision, all while maintaining the same speed. It is one of the simplest—physically speaking—boundary conditions, since it assumes that the collisions are *perfect* and there is no presence of roughness or energy exchange at the boundary.

Moreover, we remark that (6) is nothing but (4) when choosing $\mathscr{K} = \delta_D(u - \mathcal{V}_x v)$, where δ_D stands for the Dirac delta function.

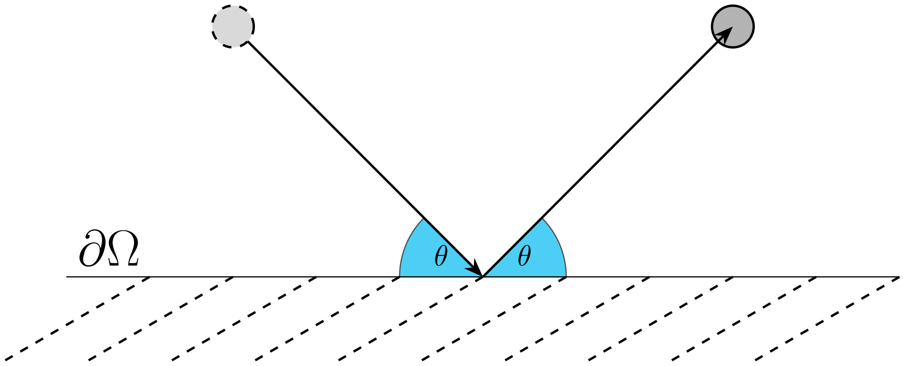

Figure 6: A particle undergoing a specular reflection, at the boundary set $\partial\Omega$, with an angle of incidence θ.

• **Diffusive reflection boundary condition.** For a given function $\Theta : \bar{\Omega} \to \mathbb{R}_+^*$, we define the *Maxwellian* distribution

$$\mathscr{M}_\Theta := \sqrt{\frac{2\pi}{\Theta}} \mathcal{M}_\Theta, \tag{7}$$

where we recall that $\mathcal{M}_\Theta$ is given by (5). We then take

$$\gamma_- F(t,x,v) = \mathscr{D}\gamma_+ F(t,x,v) := \mathscr{M}_\Theta(v) \widetilde{\gamma_+ F}(t,x) \quad \text{on } \Gamma_-, \tag{8}$$

where

$$\widetilde{\gamma_+ F} = \widetilde{\gamma_+ F}(t,x) := \int_{\mathbb{R}^d} \gamma_+ F(t,x,u)\,(n_x \cdot u)_+\,du,$$

and it is worth remarking that $\mathscr{M}_\Theta$ was defined for there to hold the normalization condition $\widetilde{\mathscr{M}_\Theta} = 1$.

Physically, this boundary condition represents the presence of a prescribed temperature Θ at the boundary, which can possibly vary in space. The intuition is that the wall absorbs the outgoing particle $(x, v_+) \in \Sigma_+$ and introduces a new one $(x, v_-) \in \Sigma_-$, where v_- stands for an independent variable given by the modified gaussian distribution $\mathscr{M}_\Theta$.

Moreover, it is also worth remarking that, physically, the inherent stochasticity of the diffusive boundary condition can also be used to model *roughness* at the boundary.

Furthermore, we remark that the diffusive boundary condition (8) is a particular case of (4) when choosing $\mathscr{K} = |n_x \cdot v| \mathscr{M}_\Theta(x,v)$.

• **Maxwell reflection boundary condition.** J. C. Maxwell introduced this boundary condition in [132, Appendix], where he describes it as a suitable way for treating the interaction between a gas and a wall.

Maxwell's proposition was to split the reflection operator into a local (specular) part and a diffuse part, combined, in a convex way, by an *accommodation coefficient*.

Indeed, we take $\iota : \partial\Omega \to [0,1]$, and we consider

$$\gamma_- F(t,x,v) = \mathscr{R}\gamma_+ F(t,x,v) \quad \text{ on } \Gamma_-, \tag{9}$$

where we have defined the Maxwell reflection operator

$$\mathscr{R}\gamma_+ F(t,x,v) := (1-\iota(x))\mathscr{S}\gamma_+ F(t,x,v) + \iota(x)\mathscr{D}\gamma_+ F(t,x,v). \tag{10}$$

We recall that the operators $\mathscr{S}$ and $\mathscr{D}$ are given by the specular and diffusive reflection operators respectively, and we note that ι is the accommodation coefficient, which is possibly dependent on the spatial variable.

Naturally, we observe that (8) is a particular case of (4) by choosing

$$\mathscr{K}(u \to v, x) = (1-\iota(x))\delta_D(u - \mathcal{V}_x v) + \iota(x)|n_x \cdot v|\mathscr{M}_\Theta(x,v).$$

• **Cercignani-Lampis boundary condition.** More recently, C. Cercignani and M. Lampis introduced in [52] an operator aimed to be more physically accurate at modeling the interaction between a gasses particles and a wall, generalizing Maxwell's model.

This boundary condition then writes as (4) with

$$\begin{aligned}\mathscr{K}(u \to v, x) := \frac{1}{\Theta(x) r_\perp} \frac{1}{(2\pi\Theta(x) r_\parallel (2-r_\parallel))^{(d-1)/2}} \exp\left(-\frac{|v_\perp|^2}{2\Theta(x) r_\perp} - \frac{(1-r_\perp)|u_\perp|^2}{2\Theta(x) r_\perp}\right) \\ \times I_0\left(\frac{(1-r_\perp)^{1/2} u_\perp \cdot v_\perp}{\Theta(x) r_\perp}\right) \exp\left(-\frac{|v_\parallel - (1-r_\parallel)u_\parallel|^2}{2\Theta(x) r_\parallel (2-r_\parallel)}\right),\end{aligned}$$

for a given wall temperature $\Theta : \bar{\Omega} \to \mathbb{R}_+^*$, the notations

$$v_\perp := (v \cdot n_x) n_x, \quad v_\parallel = v - v_\perp, \quad u_\perp := (u \cdot n_x) n_x, \quad u_\parallel = u - u_\perp,$$

with the normal and tangential accommodation coefficients $r_\perp \in (0,1]$ and $r_\parallel \in (0,2)$ respectively, and where I_0 is the modified Besel function

$$I_0(y) := \frac{1}{\pi}\int_0^\pi e^{y\cos(w)} \mathrm{d}w.$$

It is worth noting that other boundary conditions—arguably more physically accurate than those previously discussed—can be considered to model the dynamics between gas particles and a solid wall, see for instance [53, Sections 3.6 and 3.7]. However this models often involve complex mathematical expressions to achieve such a degree of precision, which makes them impractical for their analysis and during computational simulations. Moreover, regardless of complexity, and up to our knowledge, we lack today non-phenomenological boundary conditions for kinetic equations, i.e ones that do not rely on empirically selected parameters such as accommodation coefficients.

As explained in [52] (see also [53, Sections 3.8]), aside from gas–beam experiments on molecular reflection phenomena, there is currently insufficient experimental data to determine which model is most suitable to complement kinetic equations in bounded domains.

Therefore, most research papers today consider Maxwell boundary conditions as a practical and reliable model for gas reflections at boundaries. In particular, during this thesis we mainly consider Maxwell boundary conditions at the boundary.

5 Hypocoercivity theory

The mathematical study of Equation (1) heavily depends on the explicit form of the operator $\mathbf{Q}$. In most cases of interest, the structure of the problem—up to the known theory—isn't enough to construct weak solutions, rather we could only hope to construct renormalized solutions, which are even weaker (see [3, 69, 75, 76, 78, 127, 137, 138]).

However, for systems in thermodynamical equilibrium, we usually can inspect explicit (often Maxwellian) steady states. In such cases, we can study perturbative solutions of this problem near the equilibrium and its stability. Indeed, if we assume that $\mathbf{M}$ is such a steady state, we take $F = \mathbf{M} + f$ and we we observe that f satisfies, a priori, the following equation

$$\partial_t f + \mathbf{T}f = \mathbf{C}f + \mathbf{Q}(f),$$

where we have defined $\mathbf{C}f := \mathbf{Q}(f + \mathbf{M}) - \mathbf{Q}(f) - \mathbf{T}\mathbf{M}$, and we remark that

$$\mathbf{C}f := \begin{cases} 0 & \text{if } \mathbf{Q} \text{ is a linear operator,} \\ \mathbf{Q}(\mathbf{M}, f) + \mathbf{Q}(f, \mathbf{M}) & \text{if } \mathbf{Q}(f) = \mathbf{Q}(f, f) \text{ is a bilinear operator.} \end{cases}$$

This motivates the definition for

$$\mathbf{L}f := \begin{cases} \mathbf{Q}(f) & \text{if } \mathbf{Q} \text{ is a linear operator,} \\ \mathbf{C}f & \text{if } \mathbf{Q} \text{ is a bilinear operator,} \end{cases}$$

and we denote it as the *linearized collision operator*.

In order to study then the well-posedness of perturbed solutions and the stability of the steady state $\mathbf{M}$ we first analize the *linearized equation*

$$\partial_t f + \mathbf{T}f = \mathbf{L}f, \tag{11}$$

which involves dealing with the conservative transport operator $\mathbf{T}$ and the linearized collision operator $\mathbf{L}$ that is typically dissipative—in a certain Hilbert space $\mathbf{H}$—but not coercive, in the sense that it does not admit a spectral gap, and instead possess a huge kernel.

The purpose of *hypocoercivity theory* then is to establish techniques for dealing with this type of situations (see [21, 79, 166]) and we remark that such framework is similar to the type of problems encountered in the theory of *hypoellipticity*, cf. [116]. Its main objective is to construct, at least at the level of a priori estimates, an explicit decay estimate for the solutions of Equation (11) in $\mathbf{H}$. This is typically done by constructing a new norm, equivalent to that of $\mathbf{H}$, and under which the operator $-\mathbf{T} + \mathbf{L}$ will be coercive.

For explicit results in the case of bounded spatial domains with (isothermal) Maxwell boundary conditions we refer towards the hypocoercivity theory developed by A. Bernou, K. Carrapatoso, S. Mischler and I. Tristani in [21]. For situations dealing with the full space or the torus the reader may consult the paper [79] by J. Dolbeault, C. Mouhot and C. Schmeiser, or C. Villani's memory in [166].

6 Examples of kinetic equations

In this section, we present several kinetic equations relevant to this thesis, along with some of their known mathematical results and their physical motivation.

6.1 The Boltzmann equation

J. Maxwell [131] and L. Boltzmann [28] wrote down the first equation of kinetic theory, known today as the Boltzmann equation (BE), which writes

$$\partial_t F + v \cdot \nabla_x F = \mathcal{Q}(F,F) \quad \text{in } \mathcal{U}, \tag{12}$$

where $\mathcal{Q}$, named as the *Boltzmann collision operator*, represents the collisions between particles inside Ω, and is given by the bilinear form

$$\mathcal{Q}(G,H) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \mathcal{B}(v-v_*,\sigma)\,\left[G'_*H' + H'_*G' - G_*H - GH_*\right] \mathrm{d}\sigma \mathrm{d}v_*, \tag{13}$$

where we have used the shorthands

$$H = H(v), \quad H_* = H(v_*), \quad H' = H(v'), \quad H'_* = H(v'_*),$$

and we have defined the post-collisional velocities

$$v' := v - ((v-v_*)\cdot\sigma)\sigma, \qquad v'_* := v_* + ((v-v_*)\cdot\sigma)\sigma. \tag{14}$$

To explain the definition (14) we need to briefly dive into the modeling of particle collisions at the microscopic scale: if we consider two colliding particles with velocities v, v_*, and we assume elastic collisions, meaning that there holds

$$\begin{array}{rcll} v + v_* &=& v' + v'_* & \text{(conservation of momentum)},\\ |v|^2+|v_*|^2 &=& |v'|^2+|v'_*|^2 & \text{(conservation of energy)}, \end{array} \tag{15}$$

then we want to compute how the post-collisional velocities look like. We remark that (15) represents a system of $d+1$ equations, while we are looking for a $2d$–dimensional vector composed by the components of the post-collisional vectors v' and v'_*. This means that there are $d-1$ free parameters to be determined, motivating the introduction of the vector $\sigma \in \mathbb{S}^{d-1}$ and the definition (14). See Figure 7 for a graphical representation of this phenomenon.

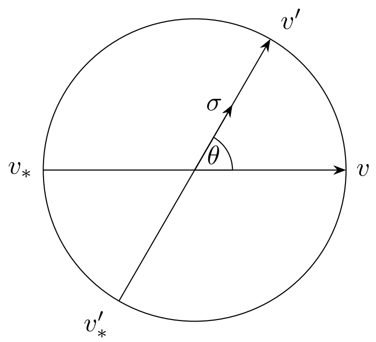

Figure 7: Graphic representation of an elastic binary collision with pre-collisional velocities v, v_*, for a certain choice of the vector σ generating the associated post-collisional velocities v', v'_*. See also the graphical depiction of the *deviation angle* θ (see below for its definition).

The *collision kernel* $\mathcal{B} = \mathcal{B}(v-v_*,\sigma)$, describes the *configuration* in which particles interact during impact and it takes the form of

$$\mathcal{B}(v-v_*,\sigma) = |v-v_*|\Sigma(v-v_*,\sigma),$$

where $\Sigma(v-v_*,\sigma)$ is called the *cross-section.* In particular, we remark that we may also write $\mathcal{B} = \mathcal{B}(|v-v_*|,\cos\theta)$ (abusing notation by maintaining the symbol $\mathcal{B}$), where

$$\theta = \angle(v-v_*, v'-v'_*) \in [0,\pi], \qquad \text{is called the } \textit{deviation angle},$$

and from its very definition there holds

$$\cos\theta := \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle.$$

We note that we consider here only the case of elastic collisions between particles, but for relativistic, quantum, or non-elastic Boltzmann collision kernels we refer to [165, Chapter 1, Section 1.6] and the references therein.

There are several choices for the collision kernel on the Boltzmann collision operator. First, for a gas of *hard spheres* we have (possibly up to a positive constant) that

$$\mathcal{B}(v - v_*, \sigma) := |(v - v_*) \cdot \sigma|. \tag{16}$$

This model describes the collision of particles as billiard balls in the $d-$dimensional domain Ω.

For charged particles following interactions given by—up to a positive constant—an inverse law of the form

$$\phi(r) = \frac{1}{r^{s-1}} \qquad \text{where } r > 0 \text{ represents the distance between the particles,}$$

the collision kernel takes the form

$$\mathcal{B}(|v - v_*|, \theta) := |v - v_*|^\gamma b(\cos\theta) \quad \text{ with } \gamma := \frac{s - (2d-1)}{s-1},$$

and where the function $b(\cos\theta)$, often called the *angular collision kernel*, is not explicit in most situations. It is known however that

$$b(\cos\theta)\sin^{d-2}\theta \underset{\theta\to 0}{\sim} c\,\theta^{-(1+\nu)} \quad \text{ with } \quad \nu = \frac{d-1}{s-1}, \tag{17}$$

for some constant $c > 0$, see for instance [167, Section 4].

It is worth remarking that (17) is valid for any $s > 2$. In particular, in the three dimensional setting $d = 3$, the limit value $s = 2$ corresponds to the Coulomb interaction, which it is known to not fit into the framework of the Boltzmann equation, see for instance the discussion from [165, Paragraph 1.7].

Finally, we remark that in the literature we classify the different collisions kernels as follows:

- $\gamma > 0$ – hard potentials,
- $\gamma = 0$ – Maxwellian mollecules,
- $\gamma < 0$ – soft potentials, and
- $\gamma < -2$ – very soft potentials.

This notation, even though useful, might be misleading. The fact is that hard potentials are not necessarily *hard* to study, and actually the Boltzmann equation presents better known properties in this framework such as presence of a spectral gap for the linearized problem in suitable Hilbert spaces and for some cases also entropic spectral gaps. Whereas for soft or very soft potentials, there are many mathematical difficulties in the theory of the Boltzmann equation. For more on this discussion see [167, Figure 4] and the references therein.

Grad's cutoff assumption introduced by H. Grad in [98] consists in imposing that the angular collision kernel b be integrable with respect to the angular variable σ, that is

$$\int_0^{\pi} b(\cos\theta)\sin^{d-2}\theta\,\mathrm{d}\theta < \infty. \tag{18}$$

Physically, this is equivalent to demand that particles only interact in short ranges. Mathematically, it considerably simplifies the mathematical difficulties in dealing with the Boltzmann collision operator and most known results today on the Boltzmann equation hold under this assumption.

It is worth remarking that for a long time it was believed that the Grad's cutoff assumption wouldn't change the fundamental properties of the solutions of the Boltzmann equation and that such solutions would be equivalent in some sense to the actual physical phenomenon being studied.

However, first noted by L. Desvillettes in [73] and further investigated by a series of authors of whom we mention [2, 67, 100, 128], it is known today that the presence of large range interactions between particles makes the collision operator $\mathcal{Q}$ to behave as a *fractional laplacian.*

As a consequence, the solutions of the Boltzmann equation without cutoff enjoy an immediate gain of regularity in the velocity variable, for any strictly positive time $t > 0$. This is in contrast with the cutoff case where we cannot expect any extra regularity for solutions, than—at best—that of the initial data.

Conservation laws of the Boltzmann collision operator. It is well known that, for any nice enough functions $G, H, \varphi : \mathbb{R}^d \to \mathbb{R}$, the Boltzmann collision operator classically satisfies

$$\int_{\mathbb{R}^d} \mathcal{Q}(G,H)\varphi = \frac{1}{8}\int_{\mathbb{R}^{2d}\times\mathbb{S}^{d-1}} \mathcal{B}\left(G'_*H' + H'_*G' - G_*H - GH_*\right)\left(\varphi + \varphi_* - \varphi' - \varphi'_*\right), \tag{19}$$

see for instance [56, Section 3.1].

Using (19) we observe that if we set $\mathbb{R}^d \ni v = (v_1, \dots, v_d)$, and choosing $\varphi = \varphi(v)$ to be either $1, v_1, \dots, v_d$ or $|v|^2$ there holds

$$\int_{\mathbb{R}^d} \mathcal{Q}(G,H)(v)\,\varphi(v)\,\mathrm{d}v = 0. \tag{20}$$

Furthermore, (20) is nothing but saying that $\mathcal{Q}$ conserves mass, momentum and energy locally in $(t,x) \in \mathbb{R}_+ \times \Omega$.

If we now look for (nice enough) functions F such that (20) holds for $G = H = F$, we have that F is necessarily a Maxwellian of the form

$$F(t,x,v) = \mathcal{M}_{\rho,u,T} = \mathcal{M}_{\rho,u,T}(v) := \frac{\rho}{(2\pi T)^{d/2}} e^{-\frac{|v-u|^2}{2T}}, \tag{21}$$

where $\rho = \rho(t,x)$ is the local density, $u = u(t,x)$ is the local bulk velocity, and $T = T(t,x)$ is the local temperature that we recall are given by (MQ). For a proof of this result we refer to [56, Section 3.2], but see also [31, Theorem 2.1] for this result under weaker hypothesis.

It is worth remarking that functions of the form of (21) are called *local* Maxwellians, whilst when ρ, u, T are constants they are known as *global* Maxwellians.

Conservation laws for the solutions of the Boltzmann equation. We take now Ω to be either the full space $\mathbb{R}^d$ or the torus $\mathbb{T}^d$ (but we could also assume Ω to be a bounded domain with conservative boundary conditions), and we take $F = F(t, x, v)$ to be a solution of the Boltzmann equation (12), we observe then that (20) implies

$$\begin{aligned} \frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} F \mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} -v\cdot\nabla_x F + \mathcal{Q}(F,F)\mathrm{d}x\mathrm{d}v = 0, \\ \frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} v\, F \mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} v\,(-v\cdot\nabla_x F) + v\mathcal{Q}(F,F)\mathrm{d}x\mathrm{d}v = 0, \\ \frac{d}{dt}\int_{\Omega\times\mathbb{R}^d} |v|^2\, F \mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} |v|^2(-v\cdot\nabla_x F) + |v|^2\mathcal{Q}(F,F)\mathrm{d}x\mathrm{d}v = 0, \end{aligned}$$

where we have used the divergence theorem together with (20) to deduce each of the results above.

If we complement then Equation (12) with an initial data $F_0 : \Omega\times\mathbb{R}^d \to \mathbb{R}$, the above computations imply the following conservation laws

$$\begin{aligned} \int_{\Omega\times\mathbb{R}^d} F_t \,\mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} F_0 \,\mathrm{d}x\mathrm{d}v \quad \text{(conservation of total mass)}, \\ \int_{\Omega\times\mathbb{R}^d} v\, F_t \,\mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} v\, F_0 \,\mathrm{d}x\mathrm{d}v \quad \text{(conservation of total momentum)}, \\ \int_{\Omega\times\mathbb{R}^d} |v|^2\, F_t \,\mathrm{d}x\mathrm{d}v &= \int_{\Omega\times\mathbb{R}^d} |v|^2\, F_0 \,\mathrm{d}x\mathrm{d}v \quad \text{(conservation of total energy)}, \end{aligned} \tag{22}$$

for every $t \geq 0$.

Entropy and irreversibility. One of the major contributions of L. Boltzmann in [28] was the introduction of the *H–functional*

$$H(f) := \int_{\Omega\times\mathbb{R}^d} F(t,x,v)\log F(t,x,v)\,\mathrm{d}x\mathrm{d}v,$$

which acts as a Lyapunov functional for Equation (12), and is physically interpreted as a form to "quantify" the *information* of a system of particles, or as the negative of the *entropy* of the system. Indeed, there holds the following result.

Theorem 1 (Boltzmann's H–theorem)**.** *Assume Ω to be either the full space $\mathbb{R}^d$ or the torus $\mathbb{T}^d$, and $F = F(t,x,v)$ to be a solution to the Boltzmann equation* (12)*. There holds*

$$\frac{d}{dt}H(F) = -D(F) \leq 0,$$

where

$$D(F) := -\int_{\Omega\times\mathbb{R}^d} \mathcal{Q}(F,F)\log F = \frac{1}{4}\int_{\Omega}\int_{\mathbb{R}^{2d}}\int_{\mathbb{S}^{d-1}} \mathcal{B}\left(F'_*F' - FF_*\right)\log\left(\frac{F'F'_*}{FF_*}\right) \geq 0, \tag{23}$$

is called the entropy production (or information dissipation) functional.

Remark 0.2. The second equality in (23) comes from using (19). Moreover, the inequality $D(F) \geq 0$ is a consequence from the fact that the function $(X,Y) \mapsto (X-Y)\log(X/Y)$ is nonnegative.

Remark 0.3. Sometimes in the literature, the H–functional is called an entropy functional and D is called an entropy dissipation functional. However, during this thesis, we have opted to use the designations presented above as suggested by C. Villani in his introduction of [165], and which are also often used in information theory for these type of quantities.

Theorem 1 implies (see for instance [56, Section 3.2]) that local Maxwellians are the only functions where the H–functional attain its maximum. It is worth remarking that these results still hold when Ω is a (nice enough) bounded domain under specular boundary conditions. Furthermore, when there is presence of a wall temperature at the boundary (for instance under diffusive or Maxwell boundary conditions) it is also possible to obtain a weaker version of Boltzmann's H–theorem, see [53, Chapter III, Section 9].

A possible interpretation of the non-increasing behavior of the H–functional is that microscopic collisions, under the *Boltzmann chaos hypothesis*, produce entropy at a statistical level.

Historically, the introduction of Equation (12), and the above H–theorem were polemic in Boltzmann's time.

One of the reasons was that the atomic hypothesis was not yet fully accepted, and Boltzmann's computations were based on using the Newtonian dynamics for each of the particles composing the "gas", and carefully passing to the limit under certain hypothesis.

Other polemics, and perhaps more important ones, were raised by the paradoxes that Boltzmann's work seemed to evoke. Indeed, J. Loschmidt's pointed out that the non-decreasing behavior of the H–functional implied the non-reversibility of Boltzmann's equation, which was in apparent contradiction with the the fact that Equation (12) came from a limit of a system of reversible Newton equations. Moreover, since Newton equations are a particular case of Hamiltonian dynamics, Poincaré's recursive theorem implies that the solutions should approach the initial state as much as wanted, however with an increasing entropy this was obviously impossible.

Today we know that the irreversibility is a natural condition coming from the introduction of a direction in time during the microscopic collisions in the computations of the derivation of Boltzmann's equation.

Hilbert's 6^{th} problem and the hydrodynamic limit. During the International Congress of Mathematicians in 1900 held in Paris, D. Hilbert proposed a series of problems he considered were of fundamental importance to be studied by the mathematicians in the new century (see [115]). Within these, there is the incitation to develop a mathematically rigorous theory for the limiting process that lead from an atomistic view of nature to the laws of motion of continua, namely to obtain a unified description of gas dynamics, including all levels of description.

The passage from microscopic dynamics to the mesoscopic Boltzmann equation, was first obtained by O. Lanford in [123] for *very small scales of time*—in the sense that there were no collision allowed—and impossible to extend this method to deal with large times. More recently, Y. Deng, Z. Hani and X. Ma proposed in [68] a way to make this *thermodynamic limit* rigorous for the hard spheres model in the torus.

The *hydrodynamic limit*, i.e passing from mesoscopic to macroscopic equations, is better understood and there is a vast literature for this type of problems, see [156] and the references therein. Since the motivations for the study of Chapter 3 are related to the behavior of the Boltzmann equation in the regime close to the hydrodynamic limit we briefly expand on this subject. Nonetheless, we note that the ideas we will exposed are mainly taken from [156].

We consider the *dimensionless* form of the Boltzmann equation

$$\mathrm{St}\,\partial_t F + v \cdot \nabla_x F = \mathrm{Kn}\,\mathcal{Q}(F,F) \qquad \text{in } \mathcal{U}, \tag{24}$$

where $\mathrm{Kn} > 0$ is the *Knudsen number* and $\mathrm{St} > 0$ is the *Strouhal number* which often coincides with the *Mach number* $\mathrm{Ma} > 0$, although in certain situations we might have

Ma << St. We refer to [156, Chapter 2] for details on the physical meaning of this constants and their connection with macroscopic quantities.

The hydrodynamic limit of the Boltzmann equation (24) corresponds to situations when Kn << 1. Indeed, we have that taking the *compressible Euler limit*, as Kn $\to 0$, the collision operator $\mathcal{Q}$ dominates the dynamics and collisions occur in a very small time scale compared with observables time scales. This together with Boltzmann's H–theorem means that the thermodynamic limit is reached almost instantaneously thus a solution F will approach the thermodynamical equilibrium given by a local Maxwellian of the form of (21), and where the functions ρ, u, and T satisfy the *perfect gas equation*. In other words, the Knudsen number governs the transition from the mesoscopic to the macroscopic regime.

The Mach or Strouhal number have no fixed rule, and their values can affect the type of hydrodynamic equation we obtain at the limit. For instance, in the regime where we make Kn << 1, if we take St = Ma $\sim$ Kn we obtain the incompressible Navier-Stokes equations, whereas by taking Kn << Ma = St we obtain incompressible Euler equations (see for instance [156, Figure 2.3]).

If we now assume that we have Equation (24) in a bounded spatial domain Ω, and complement it with the Maxwell boundary condition (9), it is known (see [156, Section 2.2.4]) that the behavior of the corresponding fluid equation at the boundary is determined by the the ratio ι/Ma:

- when $\iota/\mathrm{Ma} \to \infty$ the macroscopic equation will exhibit braking boundary conditions (represented by Dirichlet boundary conditions),
- whereas if $\iota/\mathrm{Ma} \to 0$ then we will have a *slipping Navier boundary condition* representing the fluid wall interaction.

6.2 The BGK equation

Due to the difficult structure of the Boltzmann equation and the many mathematical challenges it poses, many simpler models have been proposed over the last century for the collision term. The reason being that if the *model equation* captures well enough the behavior of particles in the mesoscopic regime then it might be used to complement and predict real life experiments.

This is the case of the BGK equation proposed by Bhatnagar, Gross and Krook in [22], but, as remarked by C. Cercignani in [53, Chapter II, Section 10], it was independently introduced by P. Welander in [168] around the same time.

This kinetic model reads

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}F := \nu \left(\mathcal{M}_{\rho,u,T}(v) - F \right) \quad \text{in } \mathcal{U}, \tag{25}$$

where $\nu > 0$ is the *collision frequency* (which might depend on F in certain situations), and we recall that the local Maxwellian $\mathcal{M}_{\rho,u,T}$ is defined in (21), with the local density $\rho = \rho(t,x)$, the local bulk velocity $u = u(t,x)$, and the local temperature $T = T(t,x)$ given by (MQ).

The main feature of Equation (25) are that its solutions have the same conservation laws (22) as the Boltzmann equation and it has an increasing entropy. Indeed, for any function $F = F(t,x,v)$ there holds

$$\begin{aligned} \int_{\Omega\times\mathbb{R}^d} \mathcal{B}F \log F &= \int_{\Omega\times\mathbb{R}^d} \nu\, \mathcal{B}F \log\left(\frac{F}{\mathcal{M}_{\rho,u,T}}\right) + \int_{\Omega\times\mathbb{R}^d} \nu\, \mathcal{B}F \log \mathcal{M}_{\rho,u,T} \\ &= \int_{\Omega\times\mathbb{R}^d} \nu \mathcal{M}_{\rho,u,T} \left(1 - \frac{F}{\mathcal{M}_{\rho,u,T}}\right) \log\left(\frac{F}{\mathcal{M}_{\rho,u,T}}\right) \le 0, \end{aligned} \tag{26}$$

where we remark that the second integral of the RHS of the first line is zero by a direct computation and using the conservation laws (22). Furthermore, the equality in (26) is achieved if and only if $F = \mathcal{M}_{\rho,u,T}$.

For well-posedness results on the BGK equation (25) see [151, 152]. Moreover, for its properties on the hydrodynamic limit we refer to [155].

6.3 The kinetic Fokker-Planck (KFP) equation

The kinetic Fokker-Planck equation is a *hypodiffusive* equation that is also known as Kolmogorov or ultraparabolic equation. In its linear conservative form it reads

$$\partial_t F + v \cdot \nabla_x F = \Delta_v F + \mathrm{div}_v\,(vF) \qquad \text{in } \mathcal{U}. \tag{27}$$

However, it can be seen as another variant of the Boltzmann equation (see [165, Section 1.6] and [53, Chapter II, Section 10]), by considering its non-linear version

$$\partial_t F + v \cdot \nabla_x F = \rho^\alpha\,\mathrm{div}_v\,(T\nabla_v F + F(v-u))\,. \tag{28}$$

where $\alpha \in [0,1]$, and we recall that $\rho = \rho(t,x)$ is the local density, $u = u(t,x)$ is the local bulk velocity , and $T = T(t,x)$ is the local temperature, all given by (MQ).

It is worth remarking that, when $\alpha = 1$, Equation (28) has the same type of quadratic homogeneity as the Boltzmann equation and satisfies the same conservation laws. However, different variants with respect to the inclusion—or not—of ρ, u, T (or its global equivalents) are often studied.

For more on the modeling and properties of KFP equations we refer to [57]. Regarding the well-posedness and properties of the linear Equation (27) we refer towards [48] and the references therein. Finally, for literature on the non-linear KFP equation (28) we mention [5, 88] and the references therein.

7 Main results of the thesis

We present now a summary of the main results obtained during this thesis.

Throughout this section we will be considering $f = f(t,x,v)$, a distribution function as presented above, depending on the time variable $t \geq 0$, the position variable $x \in \Omega \subset \mathbb{R}^d$, for a suitably bounded domain, and the velocity variable $v \in \mathbb{R}^d$.

We discuss the results of each chapter in a dedicated subsection within this section. Each of these subsections outlines the problem, the specific boundary conditions, and the main assumptions, followed by the main theorems and a discussion on the state of the art and how our results generalize and extend existing works from the literature.

We remark that, for the clarity of our presentation, we do not state here the precise hypothesis needed to the obtention of each result. These are provided in detail within the individual chapters where each theorem is proven. Moreover, the discussion for each result will be revisited in its respective chapter.

We now introduce the notion of the so-called *admissible weight functions*. These are functions of the form $\omega : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\omega(v) := \langle v \rangle^k e^{\zeta |v|^s},$$

with either

$$s = 0,\ \zeta \geq 0,\ \text{and } k > k_*, \qquad \text{or} \qquad s \in (0, s_*) \text{ with } \zeta \in (0, \zeta_*),\ \text{and any } k \geq 0.$$

The exact values of $k_* > 0$, $s_* > 0$, and $\zeta_* > 0$ vary for each subsection thus, to avoid an overload of technical details, they will not be exposed here, and instead we will specify them in each respective chapter.

Before presenting our main results we state some notations.

• We define $\mathcal{O} := \Omega \times \mathbb{R}^d$, and the *total mass* of f as $\langle\langle f \rangle\rangle_{\mathcal{O}} := \int_{\mathcal{O}} f(x,v)\mathrm{d}x\mathrm{d}v$.

• For a given measure space $(Z, \mathscr{Z}, \mu)$, a weight function $\rho : Z \to (0,\infty)$ and an exponent $p \in [1,\infty]$, we define the weighted Lebesgue space $L^p_\rho(Z)$ associated to the norm

$$\|g\|_{L^p_\rho(Z)} = \|\rho g\|_{L^p(Z)}.$$

We also define $M^1_{\omega,0}(\mathcal{O})$ as the space of Radon measures g on $\mathcal{O}$ with vanishing mass at the boundary, that is such that $|g\omega|(\mathcal{O}\backslash\mathcal{O}_\epsilon) \to 0$ as $\epsilon \to 0$, where, for any $\epsilon > 0$,

$$\Omega_\epsilon := \{x \in \Omega \cap B_{\epsilon^{-1}}, \delta(x) > \epsilon\}, \qquad \text{and} \qquad \mathcal{O}_\epsilon := \Omega_\epsilon \times B_{\epsilon^{-1}}. \tag{29}$$

Furthermore, we define the space of continuous functions in Z as $C(Z)$.

Finally, we remark that, for each of the following subsections, we abuse notation by denoting as $\mathscr{R}$ different Maxwell boundary conditions associated with different choices of accommodation coefficients and wall temperatures.

7.1 Constructive Krein-Rutman result for KFP equations in a domain

In Chapter 1, we consider a dimension $d \geq 1$, and we study the linear KFP equation

$$\begin{cases} \partial_t f &= \mathcal{L} f := -v\cdot\nabla_x f + \Delta_v f + b\cdot\nabla_v f + cf \quad \text{in } \mathcal{U} \\ \gamma_- f &= \mathscr{R}\gamma_+ f \quad \text{on } \Gamma_- \\ f_{t=0} &= f_0 \quad \text{in } \mathcal{O}, \end{cases} \tag{30}$$

where we have defined

$$b = b(x,v) \in \mathbb{R}^d, \qquad \text{and} \qquad c = c(x,v) \in \mathbb{R}, \tag{31}$$

with each of these functions being at least in $L^\infty_{\mathrm{loc}}(\mathcal{O})$, and satisfying the assumptions presented during Subsection 1.1.2. We assume that $\Omega \subset \mathbb{R}^d$ is a suitably smooth bounded domain, and we consider the Maxwell type reflection condition on the boundary

$$\gamma_- f = \mathscr{R}\gamma_+ f = \iota_S \mathscr{S}\gamma_+ f + \iota_D \mathscr{D}\gamma_+ f \quad \text{on} \quad \Gamma_-, \tag{32}$$

where we recall that the specular ($\mathscr{S}$) and diffusive ($\mathscr{D}$) reflection operators are given by (6) and (8) respectively, and we assume that the wall temperature function $\Theta : \bar{\Omega} \to \mathbb{R}^*_+$ satisfies

$$\Theta \in W^{1,\infty}(\Omega), \qquad \text{and} \qquad \Theta_* \leq \Theta(x) \leq \Theta^*, \tag{33}$$

for some $0 < \Theta_* \leq \Theta^* < \infty$. Moreover, we denote as the accommodation coefficient $\iota := \iota_S + \iota_D$, and we assume that $\iota_S, \iota_D, \iota : \partial\Omega \to [0,1]$.

We observe that, due to the general possible choices for the parameters b and c, Equation (30) does not necessarily conserves mass, therefore to study the well-posedness and long-time behavior of its solutions we look for an eigentriplet (λ_1, f_1, ϕ_1) satisfying

$$\lambda_1 \in \mathbb{R}, \quad \mathcal{L} f_1 = \lambda_1 f_1, \quad \gamma_- f_1 = \mathscr{R}\gamma_+ f_1, \quad \mathcal{L}^*\phi_1 = \lambda_1\phi_1, \quad \gamma_+\phi_1 = \mathscr{R}^*\gamma_-\phi_1. \tag{34}$$

We now state the main results of Chapter 1. First, we have a general existence and uniqueness result for the KFP Equation (30).

Theorem 4 (Existence and uniqueness)**.** *We make the above assumptions on Ω, Θ, b and c, in particular we assume* (31), (33) *to hold, and we further assume the assumptions of Theorem 1.1.1 to hold. For any admissible weight function ω and any initial datum $f_0 \in L^p_\omega(\mathcal{O})$, $p \in [1,\infty]$, or $f_0 \in M^1_{\omega,0}(\mathcal{O})$, there exists a unique global weak solution f to the kinetic Fokker-Planck Equation* (30)*. In particular, for any $(x_0, v_0) \in \mathcal{O}$, there exists a unique fundamental solution associated to the initial datum $f_0 := \delta_D(x_0, v_0)$.*

The precise sense of solution will be given in Proposition 1.3 (see also Theorem 1.3.5) in a L^2 framework, in Theorem 1.5.2 in a general L^p framework, and in Theorem 1.5.3 in a Radon measures framework. This result extends the existence and unique result of [90, Theorem 11.5] stated in a more restrictive L^2 framework (see also [1, 66] for further previous results). The L^2 framework is mainly based on Lions' variant of the Lax-Milgram theorem [126, Chap III, §1], as used in [1, 66], a trace theory developed in [49, 90, 137, 138] and boundary estimates in the spirit of [6, 19, 138]. The growth estimate is obtained by cooking up a modified but equivalent weight function for which the dissipativity of the full operator can be established. On the other hand, the general Lebesgue framework and the Radon measures framework are more involved and are also based on the ultracontractivity theorem below as well as some arguments adapted from the parabolic equation as developed in [25–27]. It is worth mentioning that the well-posedness and some regularity issues for the KFP equation set in the torus have been obtained in [1]. For the whole space setting, we refer to the recent works [12, 13] and the references therein. Finally, the KFP equation in a bounded domain has been considered in [147, 158, 169].

We next consider the first eigenvalue problem and the longtime behavior providing a quantitative answer to the first eigenelements issue.

Theorem 5 (Long time asymptotic)**.** *Under the assumptions of Theorem 4, there exist two weight functions ω_1, m_1 and an exponent $r > 2$ with $L^r_{\omega_1} \subset (L^2_{m_1})'$ such that there exists a unique eigentriplet $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times L^r_{\omega_1} \times L^2_{m_1}$ satisfying the first eigenproblem* (34) *together with the normalization conditions*

$$\|\phi_1\|_{L^2_{m_1}} = 1, \quad \langle \phi_1, f_1 \rangle = \langle \phi_1, f_1 \rangle_{L^2_{m_1},(L^2_{m_1})'} = 1.$$

These eigenfunctions are continuous functions and they also satisfy

$$0 < f_1 \lesssim \omega^{-1}, \quad 0 < \phi_1 \lesssim \omega \quad on \quad \mathcal{O},$$

for any admissible weight function ω. Furthermore, there exist some constructive constants $C \geq 1$ and $\lambda_2 < \lambda_1$ such that for any strongly confining admissible weight function ω, any exponent $p \in [1,\infty]$ and any initial datum $f_0 \in L^p_\omega(\mathcal{O})$, the associated solution f to the kinetic Fokker-Planck Equation (30) *satisfies*

$$\|f(t) - \langle f_0, \phi_1 \rangle f_1 e^{\lambda_1 t}\|_{L^p_\omega} \leq C e^{\lambda_2 t} \|f_0 - \langle f_0, \phi_1 \rangle f_1\|_{L^p_\omega},$$

for any $t \geq 0$.

This result improves the recent work [90, Section 11] (see in particular [90, Theorem 11.6], [90, Theorem 11.8] and [90, Theorem 11.11]) by slightly generalizing the framework to a position dependent wall temperature and by providing a fully constructive approach for the exponential stability of the first eigenfunction. We refer to the previous works [18, 104, 125] (partially based on [114, 124, 153]) where similar results are established for the same kind of equation in a bounded domain with no-flow boundary condition. We also emphasize that in the conservative case, many works have been done related to hypocoercivity and constructive rate of convergence to the steady state in [70, 81, 106, 112, 113, 166] or more recently in [1, 32, 39, 48, 79, 134]. From a technical point of view, this result is a consequence of the abstract version of the Krein-Rutman-Doblin-Harris theorem that will be presented in Section 1.7.1 (and which is really in the spirit of the recent work [90]) together with the ultracontractivity property stated below and the Harnack estimates established in [97].

Both the above well-posedness and the longtime behavior results are based on the following ultracontractivity property.

Theorem 6 (Ultracontractivity). *There exist $\theta, C > 0$ and $\kappa \geq 0$ such that any solution f to the KFP Equation* (30) *satisfies*

$$\|f(T,\cdot)\|_{L^\infty_\omega} \leq C\frac{e^{\kappa T}}{T^\theta}\|f_0\|_{L^1_\omega}, \quad \forall T > 0,$$

for any strongly confining admissible weight function ω, defined in Subsection 1.1.2.

This result slightly improves and generalizes [48, Theorem 1.1] which establishes a similar result in the conservative case. The proof is very alike the one of [48, Theorem 1.1] although some steps are slightly simplified. The strategy is based on Nash's gain of integrability argument [145] which is performed however on a time integral inequality as in Moser's work [142], and is then more convenient in order to use the interior gain of integrability deduced from Bouchut's regularity result [29] following the way paved by [97, Theorem 6] for proving a somehow similar local version. Contrary to the last reference, the gain of integrability is not formulated locally in x, v and integrated in time but globally in x, v and pointwisely in time as in the the ultracontractivity theory of Davies and Simon [63, 64]. Exactly as in [48], the key argument consists in exhibiting a suitable twisted weight function which is somehow slightly more elaborated than the one used during the proof of the growth estimates in Theorem 4. In Theorem 4 and Theorem 6, the boundedness assumption on Ω is really only needed in the proof of the uniqueness of the solution in the $L^p_\omega(\Omega)$ framework and it is likely that it can be removed. Here, we do not try to generalize these results to the case of an unbounded domain, see however [33] for partial results in that direction.

7.2 Existence and stability of non-equilibrium steady states of a weakly non-linear kinetic Fokker-Planck equation in a domain

In Chapter 2 we use the results from Chapter 1 to study a weakly non-linear Fokker-Planck equation with BGK heat thermostats in a spatially bounded domain with conservative Maxwell boundary conditions. In particular, such boundary conditions will be equipped with space dependent accommodation coefficient and wall temperature.

We consider $\alpha \in (0, 1/2)$, dimension $d \geq 3$, and we study the following non-linear equation

$$\begin{cases} \partial_t f &= -v\cdot\nabla_x f + (\alpha\mathcal{E}_f + (1-\alpha)\tau)\,\Delta_v f + \mathrm{div}_v(vf) + \mathscr{G}f & \text{in } \ \mathcal{U} \\ \gamma_- f &= \mathscr{R}\gamma_+ f & \text{on } \Gamma_- \\ f_{t=0} &= f_0 & \text{in } \mathcal{O}, \end{cases} \tag{35}$$

where $\tau = \tau(x) : \Omega \to \mathbb{R}$ is such that

$$\tau_0 \leq \tau(x) \leq \tau_1 \qquad \forall x \in \Omega,$$

for some constants $\tau_0, \tau_1 > 0$, and we have defined the *total energy* functional

$$\mathcal{E} = \mathcal{E}_f := \frac{1}{d} \int_{\Omega \times \mathbb{R}^d} |v|^2 f \mathrm{d}x\mathrm{d}v,$$

and the BGK *heat thermostat*

$$\mathscr{G} f = \sum_{n=1}^{\mathcal{N}} \eta_n \mathscr{G}_n f \quad \text{ with } \quad \mathscr{G}_n f = \mathbf{1}_{\Omega_n} \left(\rho_f \mathcal{M}_{T_n} - f \right),$$

for some $\mathcal{N} \in \mathbb{N}$, some parameters $\eta_n \geq 0$, $T_n > 0$, the subsets $\Omega_n \subset \Omega$, and we recall that the local density $\rho_f = \rho$ and the Maxwellian $\mathcal{M}_T$ are given by (MQ) and (5) respectively.

We present in Subsection 2.1.3 a detailed discussion on the physical interpretation of the different operators involved in Equation (35), cf. [43] for further details on the modeling.

We recall that the Maxwell reflection operator $\mathscr{R}$ is given by (10), with an accommodation coefficient $\iota \in C(\partial\Omega, [0,1])$, and the wall temperature $\Theta : \bar{\Omega} \to \mathbb{R}^*$ satisfying (33). Furthermore, we assume, without loss of generality, that $\langle\langle f_0 \rangle\rangle_{\mathcal{O}} = 1$.

See Figure 8 for a graphical representation of a possible configuration of the problem studied by Equation (35).

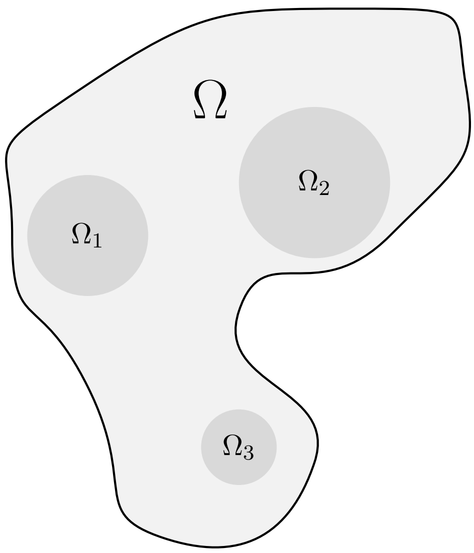

Figure 8: A possible configuration of a domain Ω with three areas (in grey) where the BGK thermostats act.

We then have the following results: We first present a well-posedness and stability theorem in the linear framework when $\alpha = 0$.

Theorem 7. *We assume $\alpha = 0$. There exists $\mathfrak{F}^0 \in L^2(\Omega, H^1(\mathbb{R}^d)) \cap L^\infty(\mathcal{O})$ unique steady solution to the linear Equation (35). Moreover, $\langle\langle \mathfrak{F}^0 \rangle\rangle_{\mathcal{O}} = 1$ and for any admissible weight function ς there holds*

$$\|\nabla_v \mathfrak{F}^0\|_{L^2_\varsigma(\mathcal{O})} < \infty \qquad \textit{and} \qquad \mathfrak{F}^0(x,v) \lesssim (\varsigma(v))^{-1}.$$

Furthermore, let ω be an admissible weight function, for any initial data $f_0 \in L^2_\omega(\mathcal{O})$ there is a unique global renormalized solution $f \in C(\mathbb{R}_+, L^2_\omega(\mathcal{O}))$ to the linear Equation (35) and there is $\lambda > 0$ such that

$$\|f_t - \mathfrak{F}^0\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-\lambda t} \|f_0 - \mathfrak{F}^0\|_{L^2_\omega(\mathcal{O})} \qquad \forall t \geq 0.$$

The precise sense of the global solutions provided in Theorem 7 is given by Theorem 2.3.3 with the choice of $\Lambda = \tau$. We also remark that Theorem 2.3.3 is but a direct application of [49, Theorem 2.11] and the trace theory from [49, Theorem 2.8].

The existence and uniqueness of a stationary solution for the linear problem, which we remark is also in the sense of Theorem 2.3.3, as well as its stability are obtained as a direct application of the Krein-Rutmann-Doblin-Harris theory developed in [160, Theorem 6.1] but we also refer towards [45, Theorem 7.1] for a similar result in a non-conservative setting and to [90] for the study of a general Krein-Rutmann-Doblin-Harris result in a general theoretical framework.

We further note that Theorem 7 is a slight generalization of [45, Theorems 1.1 and 1.2].

In the non-linear framework we have then the existence of a steady state for $\alpha > 0$ sufficiently small.

Theorem 8. *There exists $\alpha^\star \in (0, 1/2)$ such that for every $\alpha \in (0, \alpha^\star)$, there is a positive function $\mathfrak{F}^\alpha \in L^2(\Omega, H^1(\mathbb{R}^d)) \cap L^\infty(\mathcal{O})$, steady solution of Equation* (35)*. Moreover, $\langle\langle \mathfrak{F}^\alpha \rangle\rangle_{\mathcal{O}} = 1$ and for every admissible weight function ω there holds*

$$\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} < \infty, \quad \mathfrak{F}^\alpha(x,v) \lesssim (\omega(v))^{-1}, \quad \text{and} \quad \mathcal{E}_{\mathfrak{F}^\alpha} \leq 2\mathcal{E}_{\mathfrak{F}^0},$$

uniformly in α, and where $\mathfrak{F}^0$ is given by Theorem 7.

The main consequence of Theorem 8 is the existence of a NESS for the non-linear Equation (35), as well as some of its qualitative properties regarding regularity and decay tail in velocity. We remark that the proof of Theorem 8 is based in the application of a fixed point argument in the spirit of the proof of [85, Theorem 1]. Additionally $\mathfrak{F}^\alpha$ is a stationary state in the sense of Theorem 2.3.3 by taking $\Lambda = \alpha\mathcal{E}_{\mathfrak{F}^\alpha} + (1-\alpha)\tau$.

We remark that Theorem 8 generalizes [43, Theorem 1.2] and that, in contrast with this work, we observe major differences on the behavior and properties of the NESS in the absence of periodic boundary conditions: we have no reasons to believe that the NESS will be independent of the spatial variable x, the bounds on the energy functional $\mathcal{E}$ cannot be obtained as during [43, Lemma 1.1] (see Subsection 2.1.4), we lack the information to rule out the existence of steady states with unbounded total energy, and we don't have access to an explicit representation of the NESS.

Finally, we state the following stability result for the previous NESS.

Theorem 9. *We consider an admissible weight function ω. There are $\alpha^{\star\star} \in (0, \alpha^\star)$ and $\delta > 0$, where $\alpha^\star$ is given by Theorem 8, such that for every $\alpha \in (0, \alpha^{\star\star})$ and for any initial datum $f_0 \in L^2_\omega(\mathcal{O})$ satisfying*

$$\|f_0 - \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \leq \delta,$$

there is $f \in L^2_\omega(\mathcal{U})$ global weak solution of Equation (35)*. Moreover, there is $\eta > 0$ for which there holds*

$$\|f_t - \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-\eta t}\|f_0 - \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \qquad \forall t \geq 0.$$

The global solutions provided by Theorem 9 are constructed in the sense that the function $h := f - \langle\langle f_0 \rangle\rangle_{\mathcal{O}} \mathfrak{F}^\alpha$ satisfies Equation (2.8.1), in the sense of Proposition 2.4.

It is worth remarking that Proposition 2.4 is mainly an application of the Lion's variant of the Lax-Milgram theorem [126, Chap III, §1] as used in [49], see also [45, 48, 49, 90] for similar arguments on the existence of solutions of kinetic equations. The trace theory was taken mainly from [45, 49] but we also refer to [90, 137, 138] for further references on the trace theory for kinetic equations.

Furthermore, we emphasize that to obtain the a priori estimates leading to the proof of Proposition 2.4 we have used the modified weight functions from [45, 48, 90] to control the Maxwell boundary condition.

In addition, the decay estimate was obtained by defining a new norm reminiscent of [46, Proposition 3.6], [50, Proposition 3.2] and [139, Proposition 4.1]. It is worth remarking that we are not able to construct a *hypocoercivity* theory in the spirit of [21, 48, 79, 166] due to the lack of extra information on the steady state, namely positivity bounds and regularity.

We note that Theorem 9 generalizes [43, Theorem 1.3] and we remark that the techniques used for the obtention of these results are different from those developed during the proof of the main theorems from [43]. In particular, we do not need to study the underlying ergodic process associated with the linearized operator to obtain our results.

7.3 The Boltzmann equation on smooth and cylindrical domains with Maxwell boundary conditions.

In Chapter 3 we study the well-posedness and long-time behavior of the Boltzmann equation in the regime close to the *hydrodynamic limit* presenting isothermal Maxwell boundary conditions, within smooth and cylindrical domains.

This is motivated as a first step building to study the Boltzmann equation in cylindrical domains where each base presents diffusive boundary condition associated with different temperatures. This setup falls within the physical framework of *non-equilibrium thermodynamics*, and raises mathematically interesting questions regarding the existence of non-equilibrium steady states and their qualitative properties, such as uniqueness and stability.

We consider a small $\varepsilon > 0$ and we study the following Boltzmann equation

$$\left\{\begin{array}{lcll} \varepsilon\partial_t f &=& -v\cdot\nabla_x f + \varepsilon^{-1}\mathcal{Q}(f,f) & \text{in } \ \mathcal{U} \\ \gamma_- f &=& \mathscr{R}\gamma_+ f & \text{on } \Gamma_- \\ f_{t=0} &=& f_0 & \text{in } \mathcal{O}, \end{array}\right. \tag{36}$$

where $\langle\langle f_0 \rangle\rangle_{\mathcal{O}} = 1$.

We remark that the presence of the small parameter $\varepsilon > 0$ in the equation reflects the fact that the system is close to its *hydrodynamic limit*.

The *Boltzmann collision operator* $\mathcal{Q}$ represents the collisions between particles inside Ω, and is given by the bilinear form

$$\mathcal{Q}(G,H) := \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\,\left[G(v'_*)H(v') + H(v'_*)G(v') - G(v_*)H(v) - G(v)H(v_*)\right] \mathrm{d}\sigma \mathrm{d}v_*,$$

where we have defined

$$v' := v - ((v-v_*)\cdot\sigma)\sigma, \qquad v'_* := v_* + ((v-v_*)\cdot\sigma)\sigma,$$

with $\sigma \in \mathbb{S}^2$, and the *collision kernel* $\mathcal{B} = \mathcal{B}(|v-v_*|,\sigma)$, chosen the be the one associated with the *hard spheres* model

$$\mathcal{B}(|v-v_*|,\sigma) := |(v-v_*)\cdot\sigma|.$$

In this case the Maxwell reflection operator $\mathscr{R}$, which we recall is given by (10), is taken during this subsection with the constant wall temperature $\Theta \equiv 1$.

We present now the two types of *geometric assumptions* for our domain Ω, and the respective choice for the accommodation coefficient (ι) in each case.

(H1) Assume $\Omega \subset \mathbb{R}^3$ is an open C^2 domain, and $\delta \in C^2(\mathbb{R}^3, \mathbb{R}) \cap W^{3,\infty}(\mathbb{R}^3, \mathbb{R})$. Moreover, take $\iota \in C(\partial\Omega)$ and assume that there is $\iota_0 \in (0, 1]$ such that for every $y \in \partial\Omega$ there holds $\iota(y) \in [\iota_0, 1]$.

(H2) Assume $\Omega = (-L, L) \times \Omega_0$, for some $L > 0$ and where $\Omega_0 \subset \mathbb{R}^2$ is the 2-dimensional ball of radius $\mathfrak{R} > 0$ centered at the origin. In this case we also define

$$\Lambda_1 := \{-L\} \times \Omega_0, \quad \Lambda_2 := \{L\} \times \Omega_0, \quad \Lambda_3 := (-L, L) \times \partial\Omega_0,$$

and $\Lambda := \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. Furthermore, we impose *mixed* boundary conditions by taking $\iota = \mathbf{1}_{\Lambda_1 \cup \Lambda_2}$, i.e purely diffusive boundary condition on the bases of the cylinder ($\Lambda_1 \cup \Lambda_2$), and specularity on the lateral surface (Λ_3).

In this framework we have the following result for the Boltzmann equation.

Theorem 10. *Consider either Assumption (H1) or Assumption (H2) to hold and let ω be an admissible weight function.*

There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is $\eta(\varepsilon) \in (0, 1)$, satisfying $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for every $f_0 \in L^\infty_\omega(\mathcal{O})$ satisfying

$$\|f_0 - \mathcal{M}\|_{L^\infty_\omega(\mathcal{O})} \leq (\eta(\varepsilon))^2,$$

there exists a function $f \in L^\infty_\omega(\mathcal{U})$ unique global solution to the Boltzmann equation (36) *in the distributional sense. Furthermore, there is a constructive constant $\theta > 0$ such that*

$$\|f_t - \mathcal{M}\|_{L^\infty_\omega(\mathcal{O})} \leq e^{-\theta t}\, \eta(\varepsilon) \qquad \forall t \geq 0.$$

The precise sense of the solution given by Theorem 10 is constructed in Theorem 3.1.3, after taking $h := f - \mathcal{M}$, performing the change of variables from Subsection 3.1.3, and studying the resulting equation for h.

The framework for this chapter is motivated by, and therefore closely related to, the one developed in [109, 110]. However, within the study of the Boltzmann equation, our result constitutes a genuine generalization of theirs in several key aspects. First, we craft a more delicate $L^2 - L^\infty$ theory than the one presented in [109, 110] for the linearized problem, allowing us to derive explicit and constructive decay rates to the equilibrium, going beyond the boundedness results previously obtained. Second, we also carry out the analysis in cylindrical domains, introducing the presence of geometric irregularities to the obtention of the estimates, thereby generalizing the stretching method from [109, 110] for the cylindrical setting. Third, even though we do not achieve the full range of $[0, 1]$ for the accommodation coefficient in smooth domains, we allow ι to be a spatially dependent continuous function, and within cylindrical domains we further have it to be discontinuous.

This last fact makes Theorem 3.1.1 also a generalization of [36], where we recall that the accommodation coefficient had an imposed lower bound of $\sqrt{2/3}$. Furthermore, we also provide well-posedness results for initial data with a wide range of decaying tail at infinity, including polynomial.

Notably, we emphasize that, to the best of our knowledge, this is the first well-posedness result for the Boltzmann equation in cylindrical domains with Maxwell boundary conditions and a discontinuous accommodation coefficient ranging over the interval $[0, 1]$.

8 Perspectives on future research

We discuss now the lines of research that our work during this thesis opens for future investigations.

First, we are interested in extending and generalizing the results from Subsection 7.2 to cases without the smallness parameters accompanying the non-linear term, and to other more general non-linear kinetic Fokker-Planck models (see for instance the discussion from Subsection 6.3). The key argument to achieve this lies, for instance, in the obtention of stronger Harnack inequalities up to the boundary of the spatial domain, providing strong positivity conditions locally in space for the macroscopic quantities.

Secondly, we are interested in generalizing the results on the Boltzmann equation from Subsection 7.3 to domains with non-isothermal Maxwell boundary conditions, which will include the case of a cylinder with different temperatures at the bases. To do this we will consider a perturbative framework where the fluctuation of temperature is small and we will employ and generalize the ideas developed in [82, 83].

At last, we intend to use the tools and ideas developed during this thesis to continue our investigation of other type of kinetic problems in bounded domains with non-isothermal Maxwell boundary conditions, such as the Landau equation and the Vlasov–Poisson–Boltzmann and Vlasov–Poisson–Landau systems for instance.

9 Structure of the thesis

We structure this thesis as follows.

We dedicate Part I to the study of both linear and non-linear Fokker-Planck equations. This contains Chapter 1 where we present the framework for the linear KFP equation (30) and we prove Theorem 4, Theorem 5, and Theorem 6. This is followed by Chapter 2 devoted to study the non linear KFP equation (35) and we prove Theorem 7, Theorem 8, and Theorem 9.

At last, during Part II, we present Chapter 3 where we study the Boltzmann equation (36) in smooth and cylindrical domains, as exposed in Subsection 7.3 and we prove Theorem 10.

Finally, we note that we define notations throughout the thesis independently for each chapter. This means that, for each chapter the notation for objects and operators is consistent, but they may vary from one chapter to another.

Part I

Kinetic Fokker-Planck equations

Chapter 1

Constructive Krein-Rutman result for KFP equations in a domain

We consider a general Kinetic Fokker-Planck (KFP) equation in a domain with Maxwell reflection condition on the boundary, not necessarily with conservation of mass. We establish the well-posedness in many spaces including Radon measures spaces, and in particular the existence and uniqueness of fundamental solutions. We also establish a Krein-Rutman theorem with constructive rate of convergence in an abstract setting that we use for proving that the solutions to the KFP equation converge toward the conveniently normalized first eigenfunction. Both results use the ultracontractivity of the associated semigroup in a fundamental way.

The results presented on this chapter are based on the preprint [45], in collaboration with K. Carrapatoso, P. Gabriel and S. Mischler.

1.1 Introduction

1.1.1 The KFP equation in a domain

In this chapter, we consider the Kinetic Fokker-Planck (KFP) equation (also denominated sometimes as Kolmogorov equation or ultraparabolic equation)

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f + b \cdot \nabla_v f + cf \quad \text{in} \quad \mathcal{U} \tag{1.1.1}$$

on the function $f := f(t,x,v)$ depending on the time variable $t \geq 0$, the position variable $x \in \Omega$, where $\Omega \subset \mathbb{R}^d$ is a suitably smooth bounded domain, $d \geq 1$, and the velocity variable $v \in \mathbb{R}^d$. For $T \in (0,+\infty]$, we use the shorthands $\mathcal{U} := (0,T) \times \mathcal{O}$, $\mathcal{O} := \Omega \times \mathbb{R}^d$. We assume that

$$b = b(x,v) \in \mathbb{R}^d, \quad c = c(x,v) \in \mathbb{R}, \tag{1.1.2}$$

each of these functions being at least in $L^\infty_{\rm loc}(\mathcal{O})$. We complement the above KFP evolution equation with the Maxwell type reflection condition on the boundary

$$\gamma_- f = \mathscr{R}\gamma_+ f = \iota_S \mathscr{S}\gamma_+ f + \iota_D \mathscr{D}\gamma_+ f \quad \text{on} \quad \Gamma_-, \tag{1.1.3}$$

and with an initial condition

$$f(0,x,v) = f_0(x,v) \quad \text{on} \quad \mathcal{O}. \tag{1.1.4}$$

Here Γ_- denotes the incoming part of the boundary, $\mathscr{S}$ denotes the specular reflection operator, $\mathscr{D}$ denotes the diffusive reflection operator, ι_S and ι_D are nonnegative coefficients. More precisely, we assume that $\Omega := \{x \in \mathbb{R}^d;\, \delta(x) > 0\}$ for a $W^{2,\infty}(\mathbb{R}^d)$ function δ such that $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ on a neighborhood of the boundary set $\partial\Omega$ and thus $n_x = n(x) := -\nabla\delta(x)$ coincides with the unit normal outward vector field on $\partial\Omega$. We next define $\Sigma^x_\pm := \{v \in \mathbb{R}^d; \pm\, v \cdot n_x > 0\}$ the sets of outgoing (Σ^x_+) and incoming (Σ^x_-) velocities at point $x \in \partial\Omega$, then the sets

$$\Sigma_\pm := \{(x,v);\, x \in \partial\Omega,\, v \in \Sigma^x_\pm\}, \quad \Gamma_\pm := (0,T) \times \Sigma_\pm,$$

and finally the outgoing and incoming trace functions $\gamma_\pm f := \mathbf{1}_{\Gamma_\pm}\gamma f$. The specular reflection operator $\mathscr{S}$ is defined by

$$(\mathscr{S}g)(x,v) := g(x, \mathcal{V}_x v), \quad \mathcal{V}_x v := v - 2n(x)(n(x)\cdot v), \tag{1.1.5}$$

and the diffusive operator $\mathscr{D}$ is defined by

$$(\mathscr{D}g)(x,v) := \mathscr{M}_x(v)\widetilde{g}(x), \quad \widetilde{g}(x) := \int_{\Sigma^x_+} g(x,w)\, n(x)\cdot w\, dw, \tag{1.1.6}$$

where $\mathscr{M}_x$ stands for the Maxwellian function

$$\mathscr{M}_x(v) := (2\pi\Theta_x)^{-(d-1)/2} \exp(-|v|^2/(2\Theta_x)) > 0, \tag{1.1.7}$$

associated to the wall temperature Θ_x which is assumed to satisfy

$$\Theta_x \in W^{1,\infty}(\Omega), \quad 0 < \Theta_* \le \Theta_x \le \Theta^* < \infty. \tag{1.1.8}$$

It is worth observing that $\mathscr{M}_x$ is conveniently normalized in such a way that $\widetilde{\mathscr{M}_x} = 1$. Denoting the accommodation coefficient $\iota := \iota_S + \iota_D$, we assume

$$\iota_S, \iota_D, \iota : \partial\Omega \to [0,1].$$

Let us introduce some notations and then discuss some particular cases. In view of (1.1.1), we define the interior collisional operator

$$\mathscr{C} f := \Delta_v f + b \cdot \nabla_v f + cf \tag{1.1.9}$$

and next the (full) interior operator

$$\mathscr{L} := \mathscr{T} + \mathscr{C}, \quad \mathscr{T} := -v \cdot \nabla_x. \tag{1.1.10}$$

We name *microscopic or interior mass conservative* case, the case when

$$\mathscr{L}^*1 = \mathscr{C}^*1 = 0, \quad \text{or equivalently} \quad c = \operatorname{div} b,$$

and we name *macroscopic or boundary mass conservative* case, the case when

$$\mathscr{R}^*1 = 1, \quad \text{or equivalently} \quad \iota \equiv 1.$$

Here and below, the operators $\mathscr{C}^*$, $\mathscr{L}^*$ and $\mathscr{R}^*$ denote the (formal) dual operators. It is worth emphasizing that we always have $\mathscr{R}^*1 \le 1$ from the very definition (1.1.3), (1.1.5), (1.1.6) and the assumption $\iota \le 1$, so that mass is never added from the boundary, it is only (possibly partially) returned. The boundary condition (1.1.3) corresponds to the *pure* *specular reflection* boundary condition when $\iota = \iota_S \equiv 1$ and it corresponds to the *pure diffusive* boundary condition when $\iota = \iota_D \equiv 1$. When both mass conservation conditions are fulfilled then equation (1.1.1)–(1.1.3) is mass conservative, meaning that any solution (at least formally) satisfies

$$\int_{\mathcal{O}} f(t,x,v)\, dxdv = \int_{\mathcal{O}} f_0(x,v)\, dxdv, \quad \forall\, t \geq 0.$$

We name *equilibrium or detailed balance condition* case when the maxwellian $\mathscr{M}$ with constant temperature is a stationary state for each operator separately, namely

$$\mathscr{L}\mathscr{M} = \mathscr{C}\mathscr{M} = 0, \quad \mathscr{R}\mathscr{M} = \mathscr{M}.$$

When $\Theta \equiv 1$, that corresponds to the situation when $\iota \equiv 1$ (ι_S and ι_D can be space dependent) and $\mathscr{C}$ is the usual harmonic Fokker-Planck operator with $b(x,v) = v$, $c(x,v) = d$, that is

$$\mathscr{C} f := \Delta f + \mathrm{div}(vf).$$

This very specific but physically motivated situation has been studied in the recent paper [48] where, in particular, a constructive exponential stability result is established.

On the other hand, in the general situation when at least one of the two above conservations fails, we rather look for an eigentriplet (λ_1, f_1, ϕ_1) satisfying

$$\lambda_1 \in \mathbb{R}, \quad \mathscr{L} f_1 = \lambda_1 f_1, \quad \gamma_- f_1 = \mathscr{R}\gamma_+ f_1, \quad \mathscr{L}^* \phi_1 = \lambda_1 \phi_1, \quad \gamma_+ \phi_1 = \mathscr{R}^* \gamma_- \phi_1. \quad (1.1.11)$$

This issue has been tackled recently in [90] with the restriction $\Theta_x = \Theta$ is a constant, where the existence and uniqueness of such eigenelements have been established as well a non-constructive exponential asymptotical stability of the associated eigenfunction $F = e^{\lambda_1 t} f_1$.

We refer to [29, 30, 51, 66, 116, 120, 138, 146, 166, 169] for a general discussion and mathematical analysis of the kinetic Fokker-Planck equation set in the whole space or in a domain and to related problems.

In the present chapter, we carry on the analysis made in [48, 90] by establishing the following results.

(1) We prove the existence and uniqueness of solutions in many weighted Lebesgue spaces by establishing dissipativity estimates on the associated operator and next growth bound on the corresponding semigroup. We also establish the existence and uniqueness of a fundamental solution.

(2) We establish the ultracontractivity of the above semigroup associated to the evolution problem (1.1.1)–(1.1.3)–(1.1.4), that is some immediate gain of stronger Lebesgue integrability and even immediate gain of uniform bound.

(3) We prove a constructive version of a Krein-Rutman-Doblin-Harris theorem providing existence, uniqueness and exponential asymptotic stability with constructive rate of the first eigentriplet for a general class of positive semigroup in an abstract framework.

(4) We show that the KFP model addressed here satisfies the requirement of the above Krein-Rutman-Doblin-Harris theorem and thus give a clear and constructive understanding of the large time behavior of the solutions.

These results generalize or make more accurate some previous similar known results.

1.1.2 Confinement in the velocity variable and admissible weight functions

We introduce additional assumptions on b and c in order that the interior collisional operator $\mathscr{C}$ provides a convenient velocity confinement mechanism. We first assume

$$\liminf_{|v|\to\infty} \inf_{\Omega} b \cdot \frac{v}{|v|} = +\infty, \quad \frac{|b|}{\langle v \rangle}, \operatorname{div}_v b, c = \mathcal{O}\Big(b \cdot \frac{v}{|v|^2}\Big). \tag{1.1.12}$$

These two conditions are fundamental in order that first the interior collisional operator $\mathscr{C}$ and next the full operator are dissipative and even have a discrete spectrum in convenient functional spaces. In order to identify these spaces and to make the discussion simpler, we make more precise the confinement conditions by assuming that there exist $R_0, b_0, b_1 > 0$, $\gamma > 1$ and for any $p \in [1,\infty]$ there exists $k_p^* \geq 0$ such that

$$\forall\, x \in \Omega,\ v \in B_{R_0}^c, \quad b_0 |v|^\gamma \leq b \cdot v \leq b_1 |v|^\gamma, \quad c - \frac{1}{p} \operatorname{div}_v b \leq k_p^*\, b \cdot \frac{v}{|v|^2}. \tag{1.1.13}$$

We now introduce the class of the so-called *admissible* weight functions $\omega : \mathbb{R}^d \to (0,\infty)$ we will work with, which will be either a polynomial weight function

$$\omega = \langle v \rangle^k := (1 + |v|^2)^{k/2}, \quad k > k_* := \max(k_1', k_\infty^*), \tag{1.1.14}$$

$k_1' := k_1 + d/2 + \max(1, \gamma/2 - 1)$, $k_1 := \max(k_1^*, d+2)$, and we set $s := 0$ in that case, or either an exponential weight function

$$\omega = \exp(\zeta \langle v \rangle^s), \tag{1.1.15}$$

with the restrictions

$$\begin{aligned} &s < \min(\gamma, 2),\ \zeta > 0; \quad s = \gamma < 2,\ \zeta \in (0, b_0/2); \\ &s = \gamma = 2,\ \zeta \in (0, \min(1/\Theta^*, b_0)/2); \quad s = 2 < \gamma,\ \zeta \in (0, 1/(2\Theta^*)). \end{aligned} \tag{1.1.16}$$

In order to explain that choice of weight functions, we introduce the function

$$\varpi = \varpi^{\mathscr{C}}_{\varphi,p}(x,v) := 2\left(1 - \frac{1}{p}\right) \frac{|\nabla_v \varphi|^2}{\varphi^2} + \left(\frac{2}{p} - 1\right) \frac{\Delta_v \varphi}{\varphi} - b \cdot \frac{\nabla_v \varphi}{\varphi} + c - \frac{1}{p} \operatorname{div}_v b, \tag{1.1.17}$$

which is the key quantity in order to reveal the velocity confinement mechanism. We may notice that for $\omega := \langle v \rangle^\ell e^{\zeta |v|^s}$, with $\ell \in \mathbb{R}$ and $s, \zeta \geq 0$, and because of the second condition in (1.1.12), we have

$$\varpi^{\mathscr{C}}_{\omega,p} \underset{|v|\to\infty}{\sim} (s\zeta)^2 |v|^{2s-2} - s\zeta b \cdot v |v|^{s-2} \quad \text{if} \quad s > 0, \tag{1.1.18}$$

$$\varpi^{\mathscr{C}}_{\omega,p} \underset{|v|\to\infty}{\sim} c - \frac{1}{p} \operatorname{div}_v b - \ell b \cdot v |v|^{-2} \quad \text{if} \quad s = 0. \tag{1.1.19}$$

As a consequence, whatever is the value $\gamma > 1$, we have $(\varpi^{\mathscr{C}}_{\omega,p})_+ \in L^\infty(\mathcal{O})$ for any admissible weight function, what is the key information in order to establish a growth estimate in the corresponding weighted L^p space. Moreover, we have

$$\limsup_{|v|\to\infty} \sup_{\Omega} \varpi^{\mathscr{C}}_{\omega,p} = -\infty \tag{1.1.20}$$

for any admissible weight function when $\gamma > 2$ and for any exponential weight function with exponent $s \in (2-\gamma, \gamma]$ when $\gamma \in (1,2]$, what gives a key information on the spectrum of $\mathscr{C}$ in the corresponding functional space. We will call *strongly confining* any admissible weight function satisfying (1.1.20). For further references, we also notice that

$$\sup_{\Omega} \mathscr{M}\omega\langle v\rangle \in (L^1 \cap L^\infty)(\mathbb{R}^d), \quad \omega^{-1}\langle v\rangle \in (L^1 \cap L^2)(\mathbb{R}^d), \tag{1.1.21}$$

for any admissible weight function because of the restrictions $k_* \geq d+1$, $s \leq 2$ and $\zeta < 1/(2\Theta^*)$ when $s = 2$. The bound (1.1.21) provides the compatibility of the weight function with the boundary condition.

1.1.3 The main results

In order to state our main results, we need to introduce some functional spaces. For a given measure space $(E, \mathscr{E}, \mu)$, a weight function $\rho : E \to (0, \infty)$ and an exponent $p \in [1, \infty]$, we define the weighted Lebesgue space L^p_ρ associated to the norm

$$\|g\|_{L^p_\rho} = \|\rho g\|_{L^p}.$$

We also define $M^1_{\omega,0}$ as the space of Radon measures g on $\mathcal{O}$ with vanishing mass at the boundary, that is such that $|g\omega|(\mathcal{O}\backslash\mathcal{O}_\varepsilon) \to 0$ as $\varepsilon \to 0$, where, for any $\varepsilon > 0$,

$$\Omega_\varepsilon := \{x \in \Omega \cap B_{\varepsilon^{-1}}, \delta(x) > \varepsilon\}, \quad \mathcal{O}_\varepsilon := \Omega_\varepsilon \times B_{\varepsilon^{-1}}. \tag{1.1.22}$$

We first state a general existence and uniqueness result for the kinetic Fokker-Planck equation (1.1.1), (1.1.3), (1.1.4).

Theorem 1.1.1 (Existence and uniqueness)**.** *We make the above assumptions on* Ω*,* Θ*,* b *and* c*, in particular* (1.1.2)*,* (1.1.8)*,* (1.1.12) *and* (1.1.13) *hold. For any admissible weight function* ω *and any initial datum* $f_0 \in L^p_\omega$*,* $p \in [1, \infty]$*, or* $f_0 \in M^1_{\omega,0}$*, there exists a unique global weak solution* f *to the kinetic Fokker-Planck equation* (1.1.1)*–*(1.1.3)*–*(1.1.4)*. In particular, for any* $(x_0, v_0) \in \mathcal{O}$*, there exists a unique fundamental solution associated to the initial datum* $f_0 := \delta_{(x_0,v_0)}$*.*

The precise sense of solution will be given in Proposition 1.3 (see also Theorem 1.3.5) in a L^2 framework, in Theorem 1.5.2 in a general L^p framework, and in Theorem 1.5.3 in a Radon measures framework. This result extends the existence and unique result of [90, Theorem 11.5] stated in a more restrictive L^2 framework (see also [1, 66] for further previous results). The L^2 framework is mainly based on Lions' variant of the Lax-Milgram theorem [126, Chap III, §1], as used in [1, 66], a trace theory developed in [49, 90, 137, 138] and boundary estimates in the spirit of [6, 19, 138]. The growth estimate is obtained by cooking up a modified but equivalent weight function for which the dissipativity of the full operator can be established. On the other hand, the general Lebesgue framework and the Radon measures framework are more involved and are also based on the ultracontractivity theorem below as well as some arguments adapted from the parabolic equation as developed in [25–27]. It is worth mentioning that the well-posedness and some regularity issues for the KFP equation set in the torus have been obtained in [1]. For the whole space setting, we refer to the recent works [12, 13] and the references therein. Finally, the KFP equation in a bounded domain has been considered in [147, 158, 169].

We next consider the first eigenvalue problem and the longtime behavior providing a quantitative answer to the first eigenelements issue.

Theorem 1.1.2 (Long time asymptotic)**.** *Under the assumptions of Theorem 1.1.1, there exist two weight functions* ω_1, m_1 *and an exponent* $r > 2$ *with* $L^r_{\omega_1} \subset (L^2_{m_1})'$ *such* *that there exists a unique eigentriplet* $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times L^r_{\omega_1} \times L^2_{m_1}$ *satisfying the first eigenproblem* (1.1.11) *together with the normalization condition* $\|\phi_1\|_{L^2_{m_1}} = 1$, $\langle \phi_1, f_1 \rangle = \langle \phi_1, f_1 \rangle_{L^2_{m_1}, (L^2_{m_1})'} = 1$. *These eigenfunctions are continuous functions and they also satisfy*

$$0 < f_1 \lesssim \omega^{-1}, \quad 0 < \phi_1 \lesssim \omega \quad on \quad \mathcal{O}, \tag{1.1.23}$$

for any admissible weight function ω. *Furthermore, there exist some constructive constants* $C \geq 1$ *and* $\lambda_2 < \lambda_1$ *such that for any strongly confining admissible weight function* ω, *any exponent* $p \in [1, \infty]$ *and any initial datum* $f_0 \in L^p_\omega$, *the associated solution* f *to the kinetic Fokker-Planck equation* (1.1.1), (1.1.3), (1.1.4) *satisfies*

$$\|f(t) - \langle f_0, \phi_1 \rangle f_1 e^{\lambda_1 t}\|_{L^p_\omega} \leq C e^{\lambda_2 t} \|f_0 - \langle f_0, \phi_1 \rangle f_1\|_{L^p_\omega}, \tag{1.1.24}$$

for any $t \geq 0$.

This result improves the recent work [90, Section 11] (see in particular [90, Theorem 11.6], [90, Theorem 11.8] and [90, Theorem 11.11]) by slightly generalizing the framework to a position dependent wall temperature and by providing a fully constructive approach for the exponential stability of the first eigenfunction. We refer to the previous works [18, 104, 125] (partially based on [114, 124, 153]) where similar results are established for the same kind of equation in a bounded domain with no-flow boundary condition. We also emphasize that in the conservative case, many works have been done related to hypocoercivity and constructive rate of convergence to the steady state in [70, 81, 106, 112, 113, 166] or more recently in [1, 32, 39, 48, 79, 134]. From a technical point of view, this result is a consequence of the abstract version of the Krein-Rutman-Doblin-Harris theorem that will be presented in section 1.7.1 (and which is really in the spirit of the recent work [90]) together with the ultracontractivity property stated below and the Harnack estimates established in [97].

Both the above well-posedness and the longtime behavior results are based on the following ultracontractivity property.

Theorem 1.1.3 (Ultracontractivity)**.** *There exist* $\Theta, C > 0$ *and* $\kappa \geq 0$ *such that any solution* f *to the KFP equation* (1.1.1)–(1.1.3)–(1.1.4) *satisfies*

$$\|f(T, \cdot)\|_{L^\infty_\omega} \leq C \frac{e^{\kappa T}}{T^\Theta} \|f_0\|_{L^1_\omega}, \quad \forall T > 0, \tag{1.1.25}$$

for any strongly confining admissible weight function ω.

This result slightly improves and generalizes [48, Theorem 1.1] which establishes a similar result in the conservative case. The proof is very alike the one of [48, Theorem 1.1] although some steps are slightly simplified. The strategy is based on Nash's gain of integrability argument [145] which is performed however on a time integral inequality as in Moser's work [142], and is then more convenient in order to use the interior gain of integrability deduced from Bouchut's regularity result [29] following the way paved by [97, Theorem 6] for proving a somehow similar local version. Contrary to the last reference, the gain of integrability is not formulated locally in x, v and integrated in time but globally in x, v and pointwisely in time as in the the ultracontractivity theory of Davies and Simon [63, 64]. Exactly as in [48], the key argument consists in exhibiting a suitable twisted weight function which is somehow slightly more elaborated than the one used during the proof of the growth estimates in Theorem 1.1.1. In Theorem 1.1.1 and Theorem 1.1.3, the boundedness assumption on Ω is really only needed in the proof of the uniqueness of the solution in the L^p_ω framework and it is likely that it can be removed. Here, we do not try to generalize these results to the case of an unbounded domain, see however [33] for partial results in that direction.

1.1.4 Organization of this chapter

Section 1.2 is dedicated to the proof of some weighted L^p a priori growth bounds for the primal and the dual problems. These estimates and the well-posedness in a L^2 framework for the same problems (and thus part of Theorem 1.1.1) are then established rigorously in Section 1.3. Section 1.4 is devoted to the proof of the ultracontractivity property as stated in Theorem 1.1.3. In section 1.5 we come back to the well-posedness in a general framework and we end the proof of Theorem 1.1.1. Section 1.6 is dedicated to the proof of the Harnack inequality associated to our equations. In Section 1.7 we state and prove a constructive version of the Krein-Rutman theorem and deduce Theorem 1.1.2.

1.2 Weighted L^p a priori growth estimates

This section is devoted to the proof of some a priori growth estimates in weighted L^p spaces for solutions to the KFP equation (1.1.1)–(1.1.3)–(1.1.4) and its formal adjoint.

1.2.1 A priori estimates for the primal problem

We recall that for two functions $f, \varphi : \mathbb{R}^d \to \mathbb{R}_+$ and $p \in [1, \infty)$, we have

$$\int_{\mathbb{R}^d} (\mathscr{C} f) f^{p-1} \varphi^p dv = -\frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} |\nabla_v (f\varphi)^{p/2}|^2 + \int f^p \varphi^p \varpi^{\mathscr{C}}_{\omega,p}, \tag{1.2.1}$$

with $\varpi^{\mathscr{C}}_{\omega,p}$ defined in (1.1.17), what can be established by mere repeated integrations by part, see for instance [90, Lemma 7.7] and the references therein. From the definition of the admissible weight functions in Section 1.1.2 and for further references, we may observe that the large velocity asymptotic of $\varpi^{\mathscr{C}}_{\omega,p}$ is controlled by

$$\limsup_{|v|\to\infty} (\sup_{\Omega} \varpi^{\mathscr{C}}_{\omega,p} - \varpi^{\sharp}_{\omega,p}) \le 0, \quad \varpi^{\sharp}_{\omega,p} := -b_0^{\sharp} \langle v \rangle^{\varsigma}, \quad \varsigma := \gamma + s - 2, \tag{1.2.2}$$

with $b_0^{\sharp} > 0$ given by

$$\begin{aligned} b_0^{\sharp} &:= (k - k_p) b_0 \ \text{ if } \ s = 0, \\ b_0^{\sharp} &:= b_0 s \zeta \ \text{ if } \ s \in (0, \gamma), \\ b_0^{\sharp} &:= b_0 s \zeta - (s\zeta)^2 \ \text{ if } \ s = \gamma. \end{aligned} \tag{1.2.3}$$

In a more quantitative way, for any $\vartheta \in (0,1)$, there exists $\kappa', R' > 0$ such that

$$\sup_{\Omega} \varpi^{\mathscr{C}}_{\omega,p} \le \kappa' \chi_{R'} + \chi^c_{R'} \vartheta \varpi^{\sharp}_{\omega,p}, \tag{1.2.4}$$

where $\chi_R(v) := \chi(|v|/R)$, $\chi \in C^2(\mathbb{R}_+)$, $\mathbf{1}_{[0,1]} \le \chi \le \mathbf{1}_{[0,2]}$, and $\chi^c_R : 1 - \chi_R$.

Lemma 1.1. *For any admissible weight function ω, there exist $\kappa \ge 0$ and $C \ge 1$ such that for both exponents $p = 1, 2$, any solution f to the KFP equation* (1.1.1)–(1.1.3)–(1.1.4) *satisfies, at least formally,*

$$\|f_t\|_{L^p_\omega} \le C e^{\kappa t} \|f_0\|_{L^p_\omega}, \quad \forall\, t \ge 0. \tag{1.2.5}$$

The proof is based on moment estimates introduced in [49, Proposition 3.3] for the case $p = 2$ and in [48, Lemma 2.3] for the case $p = 1$, which are reminiscent of L^1 hypodissipativity techniques, see e.g. [19, 102, 136], and which are based on the usual multiplier used in order to control the diffusive reflection operator in previous works on the Boltzmann equation, see e.g. [6, 19, 137, 138]. For further references, we define the formal adjoints

$$\mathscr{L}^* := v \cdot \nabla_x + \mathscr{C}^*, \quad \mathscr{C}^* g := \Delta_v g - b \cdot \nabla_v g + (c - \operatorname{div} b) g. \tag{1.2.6}$$

Proof of Lemma 1.1. Consider $0 \le f_0 \in L^p(\omega)$ and $f = f(t,x,v) \ge 0$ a solution to the Cauchy problem (1.1.1)–(1.1.3)–(1.1.4). We introduce the modified weight functions ω_A and $\widetilde{\omega}$ defined by

$$\omega_A^p := \mathscr{M}_x^{1-p} \chi_A + \omega^p (1 - \chi_A), \qquad \widetilde{\omega}^p := \left(1 + \frac{1}{2} \frac{n_x \cdot v}{\langle v \rangle^4} \right) \omega_A^p, \tag{1.2.7}$$

with $A \ge 1$ to be chosen later, $\hat{v} := v / \langle v \rangle$ and $\widetilde{v} := \hat{v} / \langle v \rangle$. It is worth emphasizing that

$$c_A^{-1} \omega \le \tfrac{1}{2} \omega_A \le \widetilde{\omega} \le \tfrac{3}{2} \omega_A \le c_A \omega, \tag{1.2.8}$$

with $c_A \in (0, \infty)$. We then write

$$\frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} f^p \, \widetilde{\omega}^p = \int_{\mathcal{O}} (\mathscr{C} f) f^{p-1} \widetilde{\omega}^p + \frac{1}{p} \int_{\mathcal{O}} f^p \, \mathscr{T}^* \widetilde{\omega}^p - \frac{1}{p} \int_{\Sigma} (\gamma f)^p \, \widetilde{\omega}^p (n_x \cdot v), \tag{1.2.9}$$

and we estimate each term separately below.

Step 1. We first compute separately each contribution of the boundary term in (2.3.9), namely we write

$$- \int_{\Sigma} (\gamma f)^p \, \widetilde{\omega}^p (n_x \cdot v) = B_1 + B_2$$

with

$$B_1 := - \int_{\Sigma} (\gamma f)^p \omega_A^p \, n_x \cdot v, \quad B_2 := - \frac{1}{2} \int_{\Sigma} (\gamma f)^p (n_x \cdot \hat{v})^2 \, \omega_A^p.$$

On the one hand, we have

$$\begin{aligned} B_1 &= - \int_{\Sigma_+} (\gamma_+ f)^p \omega_A^p |n_x \cdot v| + \int_{\Sigma_-} \{ \iota_S \mathscr{S} \gamma_+ f + \iota_D \mathscr{D} \gamma_+ f \}^p \omega_A^p |n_x \cdot v| \\ &\le - \int_{\Sigma_+} (\gamma_+ f)^p \omega_A^p (n_x \cdot v)_+ + \int_{\Sigma_-} \iota_S (\mathscr{S} \gamma_+ f)^p \omega_A^p (n_x \cdot v)_- + \int_{\Sigma_-} \iota_D (\widetilde{\gamma_+ f})^p \mathscr{M}_x^p \omega_A^p (n_x \cdot v)_- \\ &\le - \int_{\Sigma_+} \iota_D (\gamma_+ f)^p \omega_A^p (n_x \cdot v)_+ + \int_{\partial \Omega} \iota_D (\widetilde{\gamma_+ f})^p K_1(\omega_A), \end{aligned}$$

where we have used the convexity of the mapping $s \mapsto s^p$ in the first line, we have made the change of variables $v \mapsto \mathcal{V}_x v$ in the second integral in the second line and we have set

$$K_1(\omega_A) := \int_{\mathbb{R}^d} \mathscr{M}_x^p \, \omega_A^p \, (n_x \cdot v)_- \, dv < \infty. \tag{1.2.10}$$

For $p = 1$, we observe that $\omega_A \ge 1$ and we set $K_2(\omega_A) := 1$. For $p = 2$, using the Cauchy-Schwarz inequality, we have

$$(\widetilde{\gamma_+ f})^2 \le K_2(\omega_A)^{-1} \int_{\mathbb{R}^d} (\gamma_+ f)^2 \omega_A^2 (n_x \cdot v)_+,$$

with

$$K_2(\omega_A)^{-1} := \int_{\mathbb{R}^d} \omega_A^{-2} (n_x \cdot v)_+ \, dv < \infty.$$

In both case we deduce

$$B_1 \le \int_{\partial\Omega} \iota_D (K_1(\omega_A) - K_2(\omega_A)) (\widetilde{\gamma_+ f})^p. \tag{1.2.11}$$

On the other hand, using the boundary condition (1.1.3) and making change of variables $v \mapsto \mathcal{V}_x v$, there holds

$$\begin{aligned} B_2 &= -\frac{1}{2}\int_{\Sigma_+} (\gamma_+ f)^p \omega_A^p (n_x\cdot\hat{v})^2 - \frac{1}{2}\int_{\Sigma_-} (\iota_S \mathscr{S}\gamma_+ f + \iota_D \mathscr{D}\gamma_+ f)^p \omega_A^p (n_x\cdot\hat{v})^2 \\ &= -\frac{1}{2}\int_{\Sigma_+} (\gamma_+ f)^p \omega_A^p (n_x\cdot\hat{v})^2 - \frac{1}{2}\int_{\Sigma_+} (\iota_S \gamma_+ f + \iota_D \mathscr{D}\gamma_+ f)^p \omega_A^p (n_x\cdot\hat{v})^2. \end{aligned}$$

When $p = 1$, denoting

$$K_0(\omega_A) := \frac{1}{2}\int_{\mathbb{R}^d} \mathscr{M}_x (n_x\cdot\hat{v})_+^2\, \omega_A\, dv < \infty, \tag{1.2.12}$$

we therefore have

$$B_2 \quad\le\quad -\frac{1}{2}\int_{\Sigma_+} \iota_D(\mathscr{D}\gamma_+ f)(n_x\cdot\hat{v})^2\omega_A = -\int_{\partial\Omega} \iota_D K_0(\omega_A)(\widetilde{\gamma_+ f}).$$

On the other hand, when $p = 2$, denoting

$$K_0(\omega_A)^{-1} := 2\int_{\Sigma_+^x} \langle v\rangle^2\, \omega_A^{-2}\, dv < \infty, \tag{1.2.13}$$

and thanks to the Cauchy-Schwarz inequality, we have

$$B_2 \le -\frac{1}{2}\int_{\Sigma_+} (\gamma_+ f)^2 (n_x\cdot\hat{v})^2\omega_A^2 \le -\int_{\partial\Omega} K_0(\omega_A)(\widetilde{\gamma_+ f})^2.$$

In both cases $p = 1$ and $p = 2$, we have established

$$B \le \int_{\partial\Omega} \iota_D \left[K_1(\omega_A) - K_2(\omega_A) - K_0(\omega_A)\right] (\widetilde{\gamma_+ f})^p, \tag{1.2.14}$$

and we observe that

$$\lim_{A\to\infty} K_1(\omega_A) = \lim_{A\to\infty} K_2(\omega_A) = 1,$$

thanks to the dominated convergence theorem, the normalization condition on $\mathscr{M}_x$ and the condition (1.1.21). We similarly have

$$\lim_{A\to\infty} K_0(\omega_A) = \frac{1}{2}\int_{\mathbb{R}^d} \mathscr{M}_x (n_x\cdot\hat{v})_+^2\, dv \ge C_1(\Theta_*, \Theta^*) > 0,$$

when $p = 1$, and

$$\lim_{A\to\infty} K_0(\omega_A)^{-1} = 2\int_{\Sigma_-^x} \mathscr{M}_x \langle v\rangle^2\, dv \le C_2(\Theta_*, \Theta^*) < \infty,$$

when $p = 2$. All these convergences being uniform in $x \in \partial\Omega$, we can choose $A > 0$ large enough in such a way that

$$K_1(\omega_A) - K_2(\omega_A) - K_0(\omega_A) \le 0,$$

and thus

$$-\int_\Sigma (\gamma f)^p\, \widetilde{\omega}^p (n_x \cdot v) \le 0.$$

Step 2. We now deal with the first term at the right-hand side of (2.3.9). On the one hand from (1.2.1), we have

$$\int_{\mathbb{R}^d} (\mathscr{C} f) f^{p-1} \widetilde{\omega}^p = -\frac{4(p-1)}{p} \int_{\mathbb{R}^d} |\nabla_v (f^{p/2} \widetilde{\omega}^{p/2})|^2 + \int_{\mathbb{R}^d} f^p \widetilde{\omega}^p \varpi^{\mathscr{C}}_{\widetilde{\omega},p},$$

with

$$\varpi^{\mathscr{C}}_{\widetilde{\omega},p} := 2\left(1-\frac{1}{p}\right)\frac{|\nabla_v \widetilde{\omega}|^2}{\widetilde{\omega}^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v \widetilde{\omega}}{\widetilde{\omega}} - b \cdot \frac{\nabla_v \widetilde{\omega}}{\widetilde{\omega}} + c - \frac{1}{p}\operatorname{div}_v b.$$

Defining, $\wp^p := 1 + \frac{1}{2}\frac{n_x \cdot v}{\langle v\rangle^4}$ and $\wp_A^p := 1 + \chi_A(\mathscr{M}^{1-p}\omega^{-p} - 1)$ so that $\widetilde{\omega} = \wp \omega_A$ and $\omega_A = \wp_A \omega$, we compute

$$\varpi^{\mathscr{C}}_{\widetilde{\omega},p} = \varpi_{A,p} + 2\frac{\nabla_v \omega_A}{\omega_A} \cdot \frac{\nabla_v \wp}{\wp} + 2\left(1-\frac{1}{p}\right)\frac{|\nabla_v \wp|^2}{\wp^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v \wp}{\wp} - b \cdot \frac{\nabla_v \wp}{\wp}, \quad (1.2.15)$$

with

$$\varpi_{A,p} := 2\left(1-\frac{1}{p}\right)\frac{|\nabla_v \omega_A|^2}{\omega_A^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v \omega_A}{\omega_A} - b \cdot \frac{\nabla_v \omega_A}{\omega_A} + c - \frac{1}{p}\operatorname{div}_v b,$$

and next

$$\varpi_{A,p} = \varpi^{\mathscr{C}}_{\omega,p} + 2\frac{\nabla_v \omega}{\omega} \cdot \frac{\nabla_v \wp_A}{\wp_A} + 2\left(1-\frac{1}{p}\right)\frac{|\nabla_v \wp_A|^2}{\wp_A^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v \wp_A}{\wp_A} - b \cdot \frac{\nabla_v \wp_A}{\wp_A}. \quad (1.2.16)$$

Because χ_A has compact support in the velocity variable, the same holds for all the terms except the first one at the right-hand side of (1.2.16), and thus

$$|\varpi_{A,p} - \varpi^{\mathscr{C}}_{\omega,p}| \lesssim \frac{1}{\langle v\rangle^4} \lesssim \frac{|\varpi^\sharp_{\omega,p}|}{\langle v\rangle^{2+\gamma+s}}.$$

Similarly, observing that

$$\frac{\nabla_v \omega_A}{\omega_A} \cdot \frac{\nabla_v \wp}{\wp} = \frac{\nabla_v \omega}{\omega} \cdot \frac{\nabla_v \wp}{\wp} + \frac{\nabla_v \wp_A}{\wp_A} \cdot \frac{\nabla_v \wp}{\wp},$$

where $|\frac{\nabla_v \wp}{\wp}| \lesssim \langle v\rangle^{-4}$ and the second term is compactly supported, we have

$$|\varpi^{\mathscr{C}}_{\widetilde{\omega},p} - \varpi_{A,p}| \lesssim \left(1 + |b| + \frac{|\nabla \omega|}{\omega}\right)\frac{1}{\langle v\rangle^4} \lesssim \frac{|\varpi^\sharp_{\omega,p}|}{\langle v\rangle^2}.$$

Combining the last two estimates together with (1.2.4), we deduce that for any $\vartheta \in (0,1)$, there exists $\tilde\kappa, \tilde R > 0$ such that

$$\sup_\Omega \varpi^{\mathscr{C}}_{\widetilde{\omega},p} \le \tilde\kappa \chi_{\tilde R} + \chi^c_{\tilde R}\vartheta \varpi^\sharp_{\omega,p}. \quad (1.2.17)$$

Step 3. We finally deal with the second term at the right-hand side of (2.3.9). When $p = 1$ we have

$$v \cdot \nabla_x \widetilde{\omega} = \frac{1}{2}(\hat v \cdot D_x n_x \hat v)\frac{\omega_A}{\langle v\rangle^2} \lesssim \frac{1}{\langle v\rangle^2}\widetilde{\omega} \lesssim \frac{|\varpi^\sharp_{\omega,p}|}{\langle v\rangle^{s+\gamma}}\widetilde{\omega},$$

thanks to the regularity assumption on Ω. On the other hand, when $p = 2$, we first compute

$$v \cdot \nabla_x(\widetilde{\omega}^2) = \frac{1}{2}(\hat{v} \cdot D_x n_x \hat{v}) \frac{\omega_A^2}{\langle v \rangle^2} + \left(1 + \frac{1}{2} \frac{n_x \cdot v}{\langle v \rangle^4}\right) v \cdot \nabla_x(\omega_A^2).$$

Since

$$\nabla_x(\omega_A^2) = \chi_A \mathscr{M}_x^{-1} \left[\frac{(d-1)}{2} \frac{\nabla_x \Theta_x}{\Theta_x} - \frac{|v|^2}{2} \frac{\nabla_x \Theta_x}{\Theta_x^2} \right],$$

assumption (1.1.8) together with the fact that χ_A is compactly supported and the regularity assumption on Ω as above imply

$$v \cdot \nabla_x(\widetilde{\omega}^2) \lesssim \frac{1}{\langle v \rangle^2} \widetilde{\omega}^2 \lesssim \frac{|\varpi^\sharp_{\omega,p}|}{\langle v \rangle^{s+\gamma}} \widetilde{\omega}.$$

Step 4. Coming back to (2.3.9) and using Step 1, we deduce that

$$\frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} f^p \widetilde{\omega}^p \leq -\frac{4(p-1)}{p} \int_{\mathcal{O}} |\nabla_v (f\widetilde{\omega})^{p/2}|^2 + \int_{\mathcal{O}} f^p \widetilde{\omega}^p \varpi^{\mathscr{L}}_{\widetilde{\omega},p} \tag{1.2.18}$$

for both $p = 1, 2$ and with

$$\varpi^{\mathscr{L}}_{\widetilde{\omega},p} := \varpi^{\mathscr{C}}_{\widetilde{\omega},p} + \frac{1}{\widetilde{\omega}^p} v \cdot \nabla_x \widetilde{\omega}^p. \tag{1.2.19}$$

Gathering the estimates (1.2.4) and those established in Step 2 and Step 3, we deduce that for any $\vartheta \in (0,1)$, there exists $\kappa, R > 0$ such that

$$\varpi^{\mathscr{L}}_{\widetilde{\omega},p} \leq \kappa \chi_R + \chi_R^c \vartheta \varpi^\sharp_{\omega,p}. \tag{1.2.20}$$

In particular, $\varpi^{\mathscr{L}}_{\widetilde{\omega},p} \leq \kappa$ and we immediately conclude thanks to Grönwall's lemma and the comparison (1.2.8) between ω and $\widetilde{\omega}$. □

Remark 1.2. For later references, we emphasize that the same proof works for establishing (1.2.5) with $p := 1$ and $\omega := \langle v \rangle^k$, $k \geq k_1 := \max(k_1^*, d+2)$.

1.2.2 A priori estimates for the dual problem

We establish now a similar exponential growth a priori estimate in a general weighted L^q framework, $q = 1, 2$, for the dual backward problem associated to (1.1.1)–(1.1.3)–(1.1.4). More precisely we consider the equation

$$\begin{cases} -\partial_t g = v \cdot \nabla_x g + \mathscr{C}^* g & \text{in} \quad (0,T) \times \mathcal{O}, \\ \gamma_+ g = \mathscr{R}^* \gamma_- g & \text{on} \quad (0,T) \times \Sigma_+, \\ g(T) = g_T & \text{in} \quad \mathcal{O}, \end{cases} \tag{1.2.21}$$

for any $T \in (0, \infty)$ and any final datum g_T. The adjoint Fokker-Planck operator $\mathscr{C}^*$ is defined in (1.2.6), and the adjoint reflection operator $\mathscr{R}^*$ is defined by

$$\mathscr{R}^* g(x,v) = \iota_S \mathscr{S} g(x,v) + \iota_D \mathscr{D}^* g(x),$$

with

$$\mathscr{D}^* g(x) = \widetilde{\mathscr{M}_x g}(x) := \int_{\mathbb{R}^d} g(x,w) \mathscr{M}_x(w) (n_x \cdot w)_- \, dw,$$

for any function g with support on Σ_-.

For two solutions f to the forward Cauchy problem (1.1.1)–(1.1.3)–(1.1.4) and g to the dual problem (1.2.21), the usual identity

$$\int_{\mathcal{O}} f(T) g_T = \int_{\mathcal{O}} f_0 g(0) \tag{1.2.22}$$

then holds at least formally, see also Theorem 1.3.5 below. We may indeed formally compute

$$\begin{aligned}
\int_{\mathcal{O}} f(T) g_T &= \int_{\mathcal{O}} f_0 g(0) + \int_0^T \int_{\mathcal{O}} (\partial_t f g + f \partial_t g)\, ds \\
&= \int_{\mathcal{O}} f_0 g(0) - \int_0^T \int_{\mathcal{O}} (v \cdot \nabla_x f g + f v \cdot \nabla_x g)\, ds \\
&= \int_{\mathcal{O}} f_0 g(0) - \int_0^T \int_{\Sigma} (v \cdot n) \gamma f \gamma g\, ds \\
&= \int_{\mathcal{O}} f_0 g(0) - \int_0^T \int_{\Sigma_+} (v \cdot n)(\gamma_+ f)(\mathscr{R}^* \gamma_- g)\, ds \\
&\quad + \int_0^T \int_{\Sigma_-} |v \cdot n| (\mathscr{R} \gamma_+ f)(\gamma_- g)\, ds,
\end{aligned}$$

by using the Green-Ostrogradski formula and the reflection conditions at the boundary. From the very definition of $\mathscr{R}$ and $\mathscr{R}^*$, we then deduce (1.2.22).

Lemma 1.3. *For any admissible weight function ω and any exponent $q = 1$ or $q = 2$, there exist $\kappa \in \mathbb{R}$ and $C \geq 1$ such that for any $T > 0$ and any $g_T \in L^q_m$ with $m := \omega^{-1}$, the associated solution g to the backwards dual problem* (1.2.21) *satisfies*

$$\|g(0)\|_{L^q_m} \leq C e^{\kappa T} \|g_T\|_{L^q_m}. \tag{1.2.23}$$

Proof of Lemma 1.3. Without loss of generality we may suppose that $m \geq \mathscr{M}_x$, otherwise we replace m by cm where $c > 0$ is such that $m \geq c^{-1} \mathscr{M}_x$.

Consider a final time $T \in (0, \infty)$, a final datum $0 \leq g_T \in L^q_m$ and $g = g(t, x, v) \geq 0$ a solution to the backward dual Cauchy problem (1.2.21). For $A \geq 1$, we introduce the weight functions

$$m_A^q := \chi_A \mathscr{M}_x + (1 - \chi_A) m^q, \quad \widetilde{m}^q := \left(1 - \frac{1}{2} \frac{n_x \cdot v}{\langle v \rangle^4}\right) m_A^q, \tag{1.2.24}$$

with the notations introduced in (1.2.4). It is worth emphasizing that

$$\mathscr{M}_x \leq m_A \leq m \quad \text{and} \quad c_A^{-1} m \leq \frac{1}{2} m_A \leq \widetilde{m} \leq \frac{3}{2} m_A \leq \frac{3}{2} m, \tag{1.2.25}$$

with $c_A \in (0, \infty)$. Similarly as in the proof of Lemma 1.1, we compute

$$-\frac{1}{q} \frac{d}{dt} \int_{\mathcal{O}} g^q\, \widetilde{m} = \int_{\mathcal{O}} g^{q-1} \,(\mathscr{C}^* g)\, \widetilde{m}^q + \frac{1}{q} \int_{\Sigma} (\gamma g)^q\, \widetilde{m}^q\, (n_x \cdot v) - \frac{1}{q} \int_{\mathcal{O}} g^q\, (v \cdot \nabla_x \widetilde{m}^q), \tag{1.2.26}$$

and we estimate each term separately.

Step 1. In order to estimate the boundary term in (1.2.26), we split it into

$$\int_{\Sigma} (\gamma g)^q\, \widetilde{m}^q\, (n_x \cdot v) \quad = \quad B_1 - B_2$$

with

$$B_1 = \int_\Sigma (\gamma g)^q \, m_A^q \, (n_x \cdot v) \quad \text{and} \quad B_2 = \frac{1}{2} \int_\Sigma (\gamma g)^q \, m_A^q \, (n_x \cdot \hat{v})^2.$$

For the first term, we have

$$\begin{aligned} B_1 &= \int_{\Sigma_+} (\iota_S \mathscr{S} \gamma_- g + \iota_D \mathscr{D}^* \gamma_- g)^q \; m_A^q \, (n_x \cdot v)_+ - \int_{\Sigma_-} (\gamma_- g)^q \, m_A^q \, (n_x \cdot v)_- \\ &\leq \int_{\Sigma_+} \iota_S \, (\mathscr{S} \gamma_- g)^q \; m_A^q \, (n_x \cdot v)_+ + \int_{\Sigma_+} \iota_D \, (\mathscr{D}^* \gamma_- g)^q \; m_A^q \, (n_x \cdot v)_+ - \int_{\Sigma_-} (\gamma_- g)^q \, m_A^q \, (n_x \cdot v)_- \\ &\leq -\int_{\Sigma_-} \iota_D (\gamma_- g)^q \, m_A^q \, (n_x \cdot v)_- + \int_{\Sigma_+} \iota_D \left(\widetilde{\mathscr{M}_x \gamma_- g} \right)^q \; m_A^q \, (n_x \cdot v)_+, \end{aligned}$$

where we have used the boundary condition in (1.2.21) on the first line, the convexity of the mapping $s \mapsto s^q$ on the second line, and the change of variables $v \to \mathcal{V}_x v$ on the third one. Defining

$$K_1(m_A) := \int_{\mathbb{R}^d} m_A^q (n_x \cdot v)_+ \, dv < \infty,$$

we equivalently have

$$B_1 \leq -\int_{\Sigma_-} \iota_D (\gamma_- g)^q \, m_A^q \, (n_x \cdot v)_- + \int_{\partial\Omega} \iota_D K_1(m_A) (\widetilde{\mathscr{M}_x \gamma_- g})^q.$$

When $q = 1$, we set $K_2(m_A) := 1$ and use the fact $m_A \geq \mathscr{M}_x$ in order to obtain

$$B_1 \leq \int_{\partial\Omega} \iota_D \left\{ K_1(m_A) - 1 \right\} \widetilde{\mathscr{M}_x \gamma_- g}.$$

On the other hand, when $q = 2$, the Cauchy-Schwarz inequality yields

$$(\widetilde{\mathscr{M}_x \gamma_- g})^2 \leq K_2(m_A)^{-1} \int_{\mathbb{R}^d} (\gamma_- g)^2 m_A^2 (n_x \cdot v)_- \, dv,$$

where we have set

$$K_2(m_A)^{-1} := \int_{\mathbb{R}^d} \mathscr{M}_x^2 m_A^{-2} (n_x \cdot v)_- \, dv < \infty,$$

and we thus obtain

$$B_1 \leq \int_{\partial\Omega} \iota_D \left\{ K_1(m_A) - K_2(m_A) \right\} (\widetilde{\mathscr{M}_x \gamma_- g})^2.$$

We now deal with the second term B_2, observing first that

$$B_2 = \frac{1}{2} \int_{\Sigma_-} (\gamma_- g)^q \, m_A^q \, (n_x \cdot \hat{v})^2 + \frac{1}{2} \int_{\Sigma_-} \left\{ \iota_S \gamma_- g + \iota_D \mathscr{D}^* \gamma_- g \right\}^q \; m_A^q \, (n_x \cdot \hat{v})^2,$$

where we have used the boundary condition in (1.2.21) and the change of variables $v \to \mathcal{V}_x v$. If $q = 1$, we through away the two first terms and we have

$$B_2 \geq \frac{1}{2} \int_{\Sigma_-} \iota_D (\widetilde{\mathscr{M}_x \gamma_- g}) \, m_A \, (n_x \cdot \hat{v})^2 \geq \frac{1}{2} \int_{\partial\Omega} \iota_D K_0(m_A) (\widetilde{\mathscr{M}_x \gamma_- g}),$$

where we have set

$$K_0(m_A) := \int_{\mathbb{R}^d} m_A (n_x \cdot \hat{v})_+^2 \, dv < \infty.$$

Otherwise if $q = 2$, we through away the last integral and using the Cauchy-Schwarz inequality and $\iota_D \le 1$, we obtain

$$B_2 \ge \frac{1}{2}\int_{\Sigma_-} (\gamma_- g)^q \, m_A^q \, (n_x \cdot \hat{v})^2 \ge \frac{1}{2}\int_{\partial\Omega} \iota_D K_0(m_A) (\widetilde{\mathscr{M}_x \gamma_- g})^2,$$

where we have set

$$K_0(m_A)^{-1} := \int_{\Sigma_-^x} \mathscr{M}_x^2 m_A^{-2} \langle v\rangle^2 \, dv < \infty.$$

In both cases, gathering previous estimates yields

$$\int_\Sigma (\gamma g)^q \, \tilde{m}^q \, (n_x \cdot v) \le \int_{\partial\Omega} \iota_D \left\{ K_1(m_A) - K_2(m_A) - \frac{1}{2} K_0(m_A) \right\} (\widetilde{\mathscr{M}_x \gamma_- g})^q. \tag{1.2.27}$$

We observe that in both cases $q = 1, 2$, we have

$$\lim_{A\to\infty} K_1(m_A) = \lim_{A\to\infty} K_2(m_A) = 1,$$

thanks to the dominated convergence theorem, the normalization condition on $\mathscr{M}_x$ and the condition (1.1.21). We similarly have

$$\lim_{A\to\infty} K_0(m_A) = \int_{\mathbb{R}^d} \mathscr{M}_x (n_x \cdot \hat{v})_+^2 \, dv \ge C_1(\Theta_*, \Theta^*) > 0,$$

when $q = 1$, and

$$\lim_{A\to\infty} K_0(m_A)^{-1} = \int_{\Sigma_-^x} \mathscr{M}_x \langle v\rangle^2 \, dv \le C_2(\Theta_*, \Theta^*) < \infty,$$

when $q = 2$. All these convergences being uniform in $x \in \partial\Omega$, we can choose $A > 0$ large enough in such a way that

$$K_1(m_A) - K_2(m_A) - \frac{1}{2} K_0(m_A) \le 0,$$

which implies

$$\int_\Sigma (\gamma g)^q \, \tilde{m}^q \, (n_x \cdot v) \le 0. \tag{1.2.28}$$

Step 2. We now estimate the first term at the right-hand side of (1.2.26). First of all, from (1.2.1)-(1.1.17), we have

$$\int_{\mathbb{R}^d} (\mathscr{C}^* g) g^{q-1} \tilde{m}^q = -\frac{4(q-1)}{q} \int_{\mathbb{R}^d} |\nabla_v (g^{q/2} \tilde{m}^{q/2})|^2 + \int g^q \tilde{m}^q \varpi^{\mathscr{C}^*}_{\tilde{m},q},$$

with

$$\varpi^{\mathscr{C}^*}_{\varphi,q} = 2\left(1 - \frac{1}{q}\right) \frac{|\nabla_v \varphi|^2}{\varphi^2} + \left(\frac{2}{q} - 1\right) \frac{\Delta_v \varphi}{\varphi} + b \cdot \frac{\nabla_v \varphi}{\varphi} + c + \left(\frac{1}{q} - 1\right) \mathrm{div}_v \, b.$$

Arguing exactly as in Step 2 of the proof of Lemma 1.1, we can write

$$\varpi^{\mathscr{C}^*}_{\tilde{m},q} = \varpi^{\mathscr{C}^*}_{m,q} + \mathfrak{W},$$

with

$$\varpi^{\mathscr{C}^*}_{m,q} = \varpi^{\mathscr{C}}_{\omega,p}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \mathfrak{W} = o(|\varpi^\sharp_{\omega,p}|).$$

Step 3. We finally deal with the second term at the right-hand side of (1.2.26). We compute

$$v \cdot \nabla_x(\widetilde{m}^q) = \left(1 - \frac{1}{2}\frac{n_x \cdot v}{\langle v \rangle^4}\right) v \cdot \nabla_x(m_A^q) - \frac{1}{2}(\hat{v} \cdot D_x n_x \hat{v}) \frac{m_A^q}{\langle v \rangle^2}$$

and observe that

$$\nabla_x(m_A^q) = \chi_A \mathscr{M}_x \left[\frac{|v|^2}{2} \frac{\nabla_x \Theta_x}{\Theta_x^2} - \frac{(d-1)}{2} \frac{\nabla_x \Theta_x}{\Theta_x} \right].$$

Hence assumption (1.1.8) together with the fact that χ_A is compactly supported and the regularity assumption on Ω as above imply

$$\frac{1}{q} v \cdot \nabla_x(\widetilde{m}^q) \lesssim \frac{1}{\langle v \rangle^2} \widetilde{m}^q \lesssim \frac{|\varpi^\sharp|_{\omega,p}}{\langle v \rangle^{s+\gamma}} \widetilde{m}^q.$$

Step 4. Coming back to (1.2.26) and gathering previous estimates, we deduce that

$$-\frac{1}{q}\frac{d}{dt}\int_{\mathcal{O}} g^q \widetilde{m}^q \le -\frac{4(q-1)}{q} \int_{\mathcal{O}} |\nabla_v (g\widetilde{m})^{q/2}|^2 + \int_{\mathcal{O}} g^q \widetilde{m}^q \varpi^{\mathscr{L}^*}_{\widetilde{m},q} \tag{1.2.29}$$

for both $q = 1, 2$ and with

$$\varpi^{\mathscr{L}^*}_{\widetilde{m},q} := \varpi^{\mathscr{C}^*}_{\widetilde{m},q} + \frac{1}{\widetilde{m}^q} v \cdot \nabla_x \widetilde{m}^q. \tag{1.2.30}$$

Arguing as in the end of the proof of Lemma 1.1, we obtain that for any $\vartheta \in (0,1)$, there exists $\kappa, R > 0$ such that

$$\varpi^{\mathscr{L}^*}_{\widetilde{m},q} \le \kappa \chi_R + \chi_R^c \vartheta \varpi^\sharp_{\omega,p}, \tag{1.2.31}$$

in particular, $\varpi^{\mathscr{L}^*}_{\widetilde{m},q} \le \kappa$, and we immediately conclude thanks to Grönwall's lemma and the fact that $m \lesssim \widetilde{m} \lesssim m$. □

1.3 Well-posedness in a weighted L^2 framework

We briefly discuss the well-posedness in a weighted L^2 framework for both the primal and the dual Cauchy problems, using some material developed in [49, 90].

1.3.1 Trace results in a L^2 framework

In this section, we consider the kinetic Fokker-Planck equation

$$\partial_t g + v \cdot \nabla_x g = \mathbf{L} g + G, \quad \mathbf{L} g := \Delta_v g + b_i \partial_{v_i} g + \eta g \tag{1.3.1}$$

for a vector field $b = b(x,v)$, a function $\eta = \eta(x,v)$ and a source term $G = G(t,x,v)$. We formulate some trace results for solutions to the Vlasov-Fokker-Planck equation developed in [90, Sec. 11] and [49, Sec. 2] (see also [138, Section 4.1]) and and which are mainly a consequence of the two following facts:

(1) If $g \in L^2_{tx} H^1_v$ is a weak solution to (1.3.1), then it is a renormalized solution;

(2) If $g \in L^\infty_{txv}$ and $\nabla_v g \in L^2_{txv}$ is a weak solution to (1.3.1), then it admits a trace $\gamma g \in L^\infty$ in a renormalized sense.

We introduce some notations. We denote

$$d\xi_1 := |n_x \cdot v| \, dv \, d\sigma_x \quad \text{and} \quad d\xi_2 := (n_x \cdot \hat{v})^2 \, dv \, d\sigma_x \tag{1.3.2}$$

the measures on the boundary set Σ. We denote by $\mathscr{B}_1$ the class of renormalized functions $\beta \in W^{2,\infty}_{\rm loc}(\mathbb{R})$ such that β'' has a compact support, by $\mathscr{B}_2$ the class of functions $\beta \in W^{2,\infty}_{\rm loc}(\mathbb{R})$ such that $\beta'' \in L^\infty(\mathbb{R})$ and by $\mathcal{D}_0(\bar{\mathcal{U}})$ the space of test functions $\varphi \in \mathcal{D}(\bar{\mathcal{U}})$ such that $\varphi = 0$ on Γ_0. We finally define the operators

$$\mathbf{M}^*_0 \varphi := -\partial_t \varphi - v \cdot \nabla_x \varphi, \quad \mathbf{M}^*_i \varphi := \partial_{v_i} \varphi - b_i \varphi,$$

and we assume

$$b_i, \eta \in L^\infty_{tx} L^\infty_{{\rm loc},v}. \tag{1.3.3}$$

Theorem 1.3.1. *We consider $g \in L^2((0,T)\times\Omega; H^1_{\rm loc}(\mathbb{R}^d))$, $G \in L^2_{tx}(H^{-1}_{{\rm loc},v})$, b_i, η satisfying* (1.3.3) *and we assume that g is a solution to the kinetic Fokker-Planck equation* (1.3.1) *in the sense of distribution $\mathcal{D}'(\mathcal{U})$.*

(1) There exists $\gamma g \in L^2_{\rm loc}(\Gamma, d\xi_2 dt)$, $g \in C([0,T]; L^2_{\rm loc}(\mathcal{O}))$ and the following Green renormalized formula

$$\begin{aligned} &\int_{\mathcal{U}} (\beta(g)\, \mathbf{M}^*_0 \varphi + \partial_{v_i} \beta(g) \mathbf{M}^*_i \varphi + \beta''(g)\, |\nabla_v g|^2 \varphi)\, dv dx dt \\ &+ \int_\Gamma \beta(\gamma\, g)\, \varphi\; n_x \cdot v\, dv d\sigma_x dt + \Big[\int_{\mathcal{O}} (\beta(g)\varphi)(t,\cdot) dx dv\Big]_0^T = \langle G + \eta g, \beta'(g)\varphi\rangle \end{aligned} \tag{1.3.4}$$

holds for any renormalized function $\beta \in \mathscr{B}_1$ and any test functions $\varphi \in \mathcal{D}(\bar{\mathcal{U}})$, as well as for any renormalized function $\beta \in \mathscr{B}_2$ and any test functions $\varphi \in \mathcal{D}_0(\bar{\mathcal{U}})$. It is worth emphasizing that $\beta'(g)\varphi \in L^2_{tx} H^1_v$ so that the duality product $\langle G, \beta'(g)\varphi\rangle$ is well defined.

(2) If furthermore $g_0 \in L^2_{\rm loc}(\bar{\mathcal{O}})$ and $\gamma_- g \in L^2_{\rm loc}(\Gamma; d\xi_1 dt)$, then $g \in C([0,T]; L^2_{\rm loc}(\bar{\mathcal{O}}))$, $\gamma_+ g \in L^2_{\rm loc}(\Gamma; d\xi_1 dt)$ and (1.3.4) *holds for any renormalized function $\beta \in \mathscr{B}_2$ and any test functions $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$.*

(3) Alternatively to point (2), if furthermore $g \in L^\infty_{\rm loc}(\bar{\mathcal{U}})$, then $\gamma g \in L^\infty_{\rm loc}(\Gamma)$ and (1.3.4) *holds for any renormalized function $\beta \in \mathscr{B}_2$ and any test functions $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$.*

We will also use the following stability result in the spirit of [138, Theorem 5.2].

Theorem 1.3.2. *Let us consider four sequences (g^k), (b^k), (η^k) and (G^k) and four functions g, b, η, G which all satisfy the requirements of Theorem 1.3.1.*

(1) If $g^k \rightharpoonup g$ weakly in $L^2_{\rm loc}(\bar{\mathcal{U}})$, $b^k \to b$ strongly in $L^2_{\rm loc}(\bar{\mathcal{O}})$, $\eta^k \to \eta$ strongly in $L^2_{\rm loc}(\bar{\mathcal{O}})$ and $G^k \rightharpoonup G$ weakly in $L^2_{{\rm loc},x} H^{-1}_{{\rm loc},v}$, then g satisfies (1.3.1) *so that it admits a trace function $\gamma g \in L^2_{\rm loc}(\Gamma; d\xi_2 dt)$ and $\gamma g^k \rightharpoonup \gamma g$ weakly in $L^2_{\rm loc}(\Gamma; d\xi_2 dt)$.*

(2) If $g^k \to g$ strongly in $L^2_{\rm loc}(\bar{\mathcal{U}})$, $b^k \rightharpoonup b$ weakly in $L^2_{\rm loc}(\bar{\mathcal{O}})$, $\eta^k \rightharpoonup \eta$ weakly in $L^2_{\rm loc}(\bar{\mathcal{O}})$ and $G^k \to G$ weakly in $L^2_{{\rm loc},x} H^{-1}_{{\rm loc},v}$, then g satisfies (1.3.1) *so that it admits a trace function $\gamma g \in L^2_{\rm loc}(\Gamma; d\xi_2 dt)$ and $\gamma g^k \rightharpoonup \gamma g$ weakly in $L^2_{\rm loc}(\Gamma; d\xi_2 dt)$.*

1.3.2 Well-posedness for the primal equation

For further reference, for an admissible weight function ω, we define the Hilbert norm $\|\cdot\|_{\mathscr{H}} = \|\cdot\|_{\mathscr{H}_\omega}$ by

$$\|f\|^2_{\mathscr{H}} := \|f\|^2_{L^2_\omega} + \|f\|^2_{H^{1,\dagger}_\omega}, \quad \|f\|^2_{H^{1,\dagger}_\omega} := \int_{\mathcal{U}} \{|\nabla_v g|^2 + \langle \varpi^\sharp_{\omega,2}\rangle g^2\}\omega^2\, dv\, dx\, dt,$$

with $\varpi^\sharp_{\omega,2}$ defined in (1.2.2), and we denote by $\mathscr{H} = \mathscr{H}_\omega$ the associated Hilbert space. We now state the well-posedness result for the primal problem which is nothing but [49, Theorem 2.12].

Proposition 1.3. *We make the regularity assumptions on* Ω, Θ, b *and* c *as presented in Section 1.1, in particular* (1.1.2), (1.1.8), (1.1.12) *and* (1.1.13) *hold. For any admissible weight function* ω *and any* $f_0 \in L^2_\omega(\mathcal{O})$, *there exists a unique global solution* $f \in X_T := L^\infty(0,T;L^2_\omega) \cap C([0,T];L^2_{\rm loc}) \cap \mathscr{H}$, $\forall T > 0$, *to the kinetic Fokker-Planck equation* (1.1.1) *complemented with the Maxwell reflection boundary condition* (1.1.3) *and associated to the initial datum* f_0. *More precisely, the function* f *satisfies equation* (1.3.1) *in the sense of distributions in* $\mathcal{D}'(\mathcal{U})$ *with trace functions, defined thanks to Theorem 1.3.1, satisfying* $\gamma f \in L^2_\omega(\Gamma, d\xi_2)$ *as well as the Maxwell reflection boundary condition* (1.1.3) *pointwisely and* $f(t,\cdot) \in L^2_\omega$, $\forall t \in [0,T]$, *and the initial condition* $f(0,\cdot) = f_0$ *pointwisely.*

Because we will need to adapt it in the next section, we allude the proof and we refer to [49] for details.

Proof of Proposition 1.3. We split the proof into four steps.
Step 1. Given $\mathfrak{f} \in L^2_\omega(\Gamma_-; d\xi_1)$, we solve the inflow problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathscr{C} f & \text{in} \quad (0,\infty) \times \mathcal{O} \\ \gamma_- f = \mathfrak{f} & \text{on} \quad (0,\infty) \times \Sigma_- \\ f_{|t=0} = f_0 & \text{in} \quad \mathcal{O}, \end{cases} \tag{1.3.5}$$

thanks to Lions' variant of the Lax-Milgram theorem [126, Chap III, §1]. More precisely, we define $\widetilde{\omega}$ as during the proof of Lemma 1.1 and the bilinear form $\mathscr{E} : \mathscr{H} \times C^1_c([0,T) \times \mathcal{O} \cup \Gamma_-) \to \mathbb{R}$ by

$$\mathscr{E}(f,\varphi) \;:=\; \int_{\mathcal{U}} (\lambda f - \mathscr{C} f)\varphi \widetilde{\omega}^2 - \int_{\mathcal{U}} f(\partial_t + v \cdot \nabla_x)(\varphi \widetilde{\omega}^2).$$

Using the Green-Ostrogradski formula, we observe that

$$\begin{aligned} \mathscr{E}(\varphi,\varphi) \;=\; & \int_{\mathcal{U}} (\lambda \varphi - \mathscr{C}\varphi)\varphi \widetilde{\omega}^2 + \frac{1}{2}\int_{\mathcal{O}} \varphi(0,\cdot)^2 \widetilde{\omega}^2 \\ & -\frac{1}{2}\int_{\mathcal{U}} \varphi^2 v \cdot \nabla_x \widetilde{\omega}^2 + \frac{1}{2}\int_{\Gamma_-} (\gamma_- \varphi)^2 \widetilde{\omega}^2 \, d\xi_1, \end{aligned}$$

for any $\varphi \in C^1_c([0,T) \times \mathcal{O} \cup \Gamma_-)$. The same computations as presented during the proof of Lemma 1.1 imply that

$$\mathscr{E}(\varphi,\varphi) \geq (\lambda - \lambda_0)\|\varphi\|^2_{L^2_{\widetilde{\omega}}} + \|\varphi\|^2_{H^{1,\dagger}_{\widetilde{\omega}}} + \tfrac{1}{2}\|\varphi(0)\|^2_{L^2_{\widetilde{\omega}}} + \tfrac{1}{2}\|\gamma_- \varphi\|^2_{L^2_{\widetilde{\omega}}(\Gamma_-; d\xi_1)}, \tag{1.3.6}$$

for some $\lambda_0 \in \mathbb{R}$. For $\lambda > \lambda_0$, the bilinear form $\mathscr{E}$ is thus coercive and the above mentioned Lions' theorem implies the existence of a function $f_\lambda \in \mathscr{H}$ which satisfies the variational equation

$$\mathscr{E}(f_\lambda,\varphi) = \int_{\Gamma_-} \mathfrak{f} e^{-\lambda t} \varphi \widetilde{\omega}^2 \, d\xi_1 + \int_{\mathcal{O}} f_0 \varphi(0,\cdot) \widetilde{\omega}^2 \, dv \, dx, \quad \forall \, \varphi \in C^1_c([0,T) \times \mathcal{O} \cup \Gamma_-). \tag{1.3.7}$$

Defining $f := f_\lambda e^{\lambda t}$ and using Theorem 1.3.1, we deduce that $f \in \mathscr{H} \cap C([0,T]; L^2_\omega(\mathcal{O}))$ is a renormalized solution to the inflow problem (1.3.5) and that $\gamma f \in L^2_{\widetilde{\omega}}(\Gamma; d\xi_1)$. From the renormalization formulation, we have the uniqueness of such a solution. Because of the trace Theorem 1.3.1-(2), we can take $\beta(s) = s^2$ in (1.3.4) and we get

$$\begin{aligned} & \int_{\mathcal{U}} (f^2(-v \cdot \nabla_x \varphi^2 - 2\varphi^2 \varpi^{\mathscr{C}}_{\varphi,2}) + 2|\nabla_v (f\varphi)|^2) \, dv dx dt \\ & \quad + \int_{\Gamma} (\gamma \, f)^2 \, \varphi^2 \; n_x \cdot v \, dv d\sigma_x dt + \Big[\int_{\mathcal{O}} (f^2 \varphi^2)(t,\cdot) dx dv \Big]_0^T = 0, \end{aligned}$$

for any $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$. Taking $\varphi := \tilde{\omega}$ in that last identity thanks to an approximation procedure and next using (1.3.7), the same computations as presented during the proof of Lemma 1.1 and the Gronwall lemma, we also deduce the energy estimate

$$\begin{aligned} \|f_t\|^2_{L^2_{\tilde{\omega}}} + \int_0^t \left(\|\gamma_+ f_s\|^2_{L^2_{\tilde{\omega}}(\Gamma_+;d\xi_1)} + 2\|f\|^2_{H^{1,\dagger}_{\tilde{\omega}}} \right) e^{\lambda_0(t-s)}\, ds \\ \le \|f_0\|^2_{L^2_{\tilde{\omega}}} e^{\lambda_0 t} + \int_0^t \|\mathfrak{f}_s\|^2_{L^2_{\tilde{\omega}}(\Gamma_-;d\xi_1)} e^{\lambda_0(t-s)}\, ds. \end{aligned}$$

Step 2. For any $\alpha \in (0,1)$ and $h \in \mathscr{H} \cap C([0,T];L^2_\omega(\mathcal{O}))$ solution to the problem (1.3.5) for some $\mathfrak{h} \in L^2_{\tilde{\omega}}(\Gamma_-;d\xi_1)$, and thus $\gamma h \in L^2_\omega(\Gamma;d\xi_1)$, we then consider the modified Maxwell reflection boundary condition problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathscr{C} f & \text{in} \quad (0,T) \times \mathcal{O} \\ \gamma_- f = \alpha \mathscr{R} \gamma_+ h & \text{on} \quad (0,T) \times \Sigma_- \\ f(t=0,\cdot) = f_0 & \text{in} \quad \mathcal{O}, \end{cases} \tag{1.3.8}$$

for which a solution $f \in \mathscr{H} \cap C([0,T];L^2_\omega(\mathcal{O}))$ such that $\gamma f \in L^2_\omega(\Gamma;d\xi_1)$ is given by the first step. Repeating the arguments of Step 1 in the proof of Lemma 1.1, we have

$$\begin{aligned} \|\mathscr{R}\gamma_+ h\|^2_{L^2_{\tilde{\omega}}(\Sigma_-;d\xi_1)} &\le \int_{\Sigma_+} \iota_S (\gamma_+ h)^2 \omega_A^2 d\xi_1 + \int_{\partial\Omega} \iota_D (K_1(\omega_A) - K_0(\omega_A))(\widetilde{\gamma_+ h})^2 \\ &\le \int_{\Sigma_+} (1-\iota_D)(\gamma_+ h)^2 \omega_A^2 d\xi_1 + \int_{\partial\Omega} \iota_D K_2(\omega_A)(\widetilde{\gamma_+ h})^2 \\ &\le \|\gamma_+ h\|^2_{L^2_{\tilde{\omega}}(\Gamma_+;d\xi_1)}. \end{aligned}$$

Thanks to the energy estimate stated in the first step, we immediately deduce that the mapping $h \mapsto f$ is α-Lipschitz for the norm defined by

$$\sup_{t\in[0,T]} \left\{ \|f_t\|^2_{L^2_{\tilde{\omega}}} e^{-\lambda_0 t} + \int_0^t \|\gamma f_s\|^2_{L^2_{\tilde{\omega}}(\Gamma_+;d\xi_1)} e^{-\lambda_0 s}\, \mathrm{d}s \right\}.$$

From the Banach fixed point theorem, we deduce the existence of a unique fixed point, and that provides a solution to (1.3.8).

Step 3. For a sequence $\alpha_k \in (0,1)$, $\alpha_k \nearrow 1$, we next consider the sequence (f_k) obtained in Step 2 as the solution to the modified Maxwell reflection boundary condition problem

$$\begin{cases} \partial_t f_k + v \cdot \nabla_x f_k = \mathscr{C} f_k & \text{in} \quad (0,T) \times \mathcal{O} \\ \gamma_- f_k = \alpha_k \mathscr{R} \gamma_+ f_k & \text{on} \quad (0,T) \times \Sigma_- \\ f_k(t=0,\cdot) = f_0 & \text{in} \quad \mathcal{O}. \end{cases} \tag{1.3.9}$$

From the fact that $\mathscr{R} : L^2_{\tilde{\omega}}(\Sigma_+;d\xi_1) \to L^2_{\tilde{\omega}}(\Sigma_+;d\xi_1)$ with norm less than 1 as established in Step 2 and the energy estimate stated at the end of Step 1, f_k satisfies

$$\|f_{kt}\|^2_{L^2_{\tilde{\omega}}} + \int_0^t \left\{ (1-\alpha_k^2)\|\gamma f_{ks}\|^2_{L^2_{\tilde{\omega}}(\Gamma_+;\mathrm{d}\xi_1)} + 2\|f_{ks}\|^2_{H^{1,\dagger}_{\tilde{\omega}}} \right\} e^{\lambda_0(t-s)}\, \mathrm{d}s \le \|f_0\|^2_{L^2_{\tilde{\omega}}} e^{\lambda_0 t} \tag{1.3.10}$$

for any $t \in (0,T)$ and any $k \ge 1$. Choosing $\beta(s) := s^2$ and $\varphi := (n_x \cdot v)\langle v\rangle^{-2}\omega^2(v)$ in the Green formula (1.3.4), we additionally have

$$\int_\Gamma (\gamma f_k)^2 \omega^2 \, \mathrm{d}\xi_2 \, \mathrm{d}t \lesssim \|f_0\|^2_{L^2_\omega} e^{\lambda_0 T}. \tag{1.3.11}$$

From the above estimates, we deduce that, up to the extraction of a subsequence, there exist $f \in \mathscr{H} \cap L^\infty(0,T; L^2_\omega(\mathcal{O}))$ and $\mathfrak{f}_\pm \in L^2_\omega(\Gamma_\pm; d\xi_2 dt)$ such that

$$f_k \rightharpoonup f \text{ weakly in } \mathscr{H} \cap L^\infty(0,T; L^2_\omega(\mathcal{O})), \quad \gamma_\pm f_k \rightharpoonup \mathfrak{f}_\pm \text{ weakly in } L^2_\omega(\Gamma; d\xi_2 dt).$$

Because $\langle v\rangle \omega^{-1} \in L^2(\mathbb{R}^d)$, we have $L^2_\omega(\Gamma; d\xi_2) \subset L^1(\Gamma; d\xi_1)$. On the other hand, we recall that from the very definition (1.1.3), we have

$$\mathscr{R} : L^1(\Sigma_+; d\xi_1) \to L^1(\Sigma_-; d\xi_1), \quad \|\mathscr{R}\|_{L^1(\Sigma; d\xi_1)} \le 1. \tag{1.3.12}$$

These three pieces of information together imply that $\mathscr{R}(\gamma f_{k+}) \rightharpoonup \mathscr{R}(\mathfrak{f}_+)$ weakly in $L^1(\Gamma_-; d\xi_1)$. On the other hand, from Theorem 1.3.2, we have $\gamma f_k \rightharpoonup \gamma f$ weakly in $L^2_{\text{loc}}(\Gamma; d\xi_2)$. Using both convergences in the boundary condition $\gamma_- f_k = \mathscr{R}(\gamma_+ f_k)$, we obtain $\gamma_- f = \mathscr{R}(\gamma_+ f)$. We may thus pass to the limit in equation (1.3.9) and we obtain that $f \in X_T$ is a renormalized solution to the KFP equation (1.1.1) complemented with the Maxwell reflection boundary condition (1.1.3) and associated to the initial datum f_0. Passing to the limit in (1.3.10), we also have

$$\|f_t\|^2_{L^2_{\widetilde\omega}} + 2\int_0^t \|f_{ks}\|^2_{H^{1,\dagger}_{\widetilde\omega}} e^{\lambda_0(t-s)}\, ds \le \|f_0\|^2_{L^2_{\widetilde\omega}} e^{\lambda_0 t}, \tag{1.3.13}$$

for any $t \in (0,T)$.

Step 4. We consider now two solutions f_1 and $f_2 \in X_T$ to the KFP equation (1.1.1)-(1.1.3) associated to the same initial datum f_0, so that the function $f := f_2 - f_1 \in X_T$ is a solution to the KFP equation (1.1.1)-(1.1.3) associated to the initial datum $f(0) = 0$. We write (1.3.4) with the choice $\varphi := \widetilde\omega_1 \chi_R$, with $\widetilde\omega_1$ given by (1.2.7) associated to $p = 1$, $A > 0$ large enough, $\omega_1 := \langle v\rangle^{k_1}$ defined in Remark 1.2, with $\chi_R(v) := \chi(v/R)$, $\mathbf{1}_{B_1} \le \chi \in \mathcal{D}(\mathbb{R}^d)$, and with the choice $\beta \in C^2(\mathbb{R})$, $\beta(0) = 0$, β'' with compact support. We get

$$\int_{\mathcal{O}} \beta(f_T)\varphi \,dv\,dx + \int_\Gamma \beta(\gamma f)\,\varphi\,(n_x\cdot v)\,dv\,d\sigma_x\,dt + \int_{\mathcal{U}} \beta''(f)\,|\nabla_v f|^2 \varphi\,dv\,dx\,dt$$
$$= \int_{\mathcal{U}} \{\beta(f)\,\mathscr{T}^*\varphi + \beta(f)\,(\Delta_v\varphi - \partial_{v_i}(b_i\varphi)) + cf\beta'(f)\varphi\}\,dv\,dx\,dt.$$

We assume $0 \le \beta(s) \le |s|$, $|\beta'(s)| \le 1$, and $\beta'' \ge 0$ so that we may get rid of the last term at the left-hand side of the above identity. Recalling that Ω is bounded, we observe that

$$\begin{aligned} |\beta(f)\,(\Delta_v\varphi - \partial_{v_i}(b_i\varphi)) + cf\beta'(f)\varphi| &\le |f|(|\Delta_v\varphi| + |\partial_{v_i}(b_i\varphi)| + |c\varphi|) \\ &\lesssim |f|\omega_1(1+|v|^{\gamma-2}) \in L^1(\mathcal{U}), \end{aligned}$$

because of the condition $k_* > k_1 + (d+\gamma-2)/2$ and the bound $(1+\langle v\rangle^{s/2})\,\omega f \in L^2(\mathcal{U})$. The same uniform estimate holds for the term $\beta(f)\,\mathscr{T}^*\varphi$. We also observe that

$$|\beta(\gamma f)\,\varphi\,(n_x\cdot v)| \lesssim |\gamma f|\omega_1|n_x\cdot v| \in L^1(\Gamma),$$

because of the condition $k_* > k_1 + d/2 + 1$ and the bound $\gamma f\omega \in L^2(\Gamma; d\xi_2 dt)$ provided by (1.3.11). We may thus pass to the limit $R \to \infty$ and $\beta(s) \nearrow |s|$ such that $0 \le s\beta'(s) \nearrow |s|$, and we deduce

$$\int_{\mathcal{O}} |f_T|\,\widetilde\omega_1 dv\,dx + \int_\Gamma |\gamma f|\widetilde\omega_1\,(n_x\cdot v)\,dv\,d\sigma_x\,dt \le \int_{\mathcal{U}} |f|\varpi^{\mathscr{L}}_{\widetilde\omega,1}\widetilde\omega_1 dv\,dx\,dt.$$

Using finally the estimate of Step 1 in the proof of Lemma 1.1 in order to get rid of the boundary term as well as the estimates obtained in Step 2 and Step 3 in the proof of Lemma 1.1 in order to deal with the RHS term, we get

$$\int_{\mathcal{O}} |f_T|\widetilde\omega_1\,dv\,dx \le \kappa \int_0^T\int_{\mathcal{O}} |f|\widetilde\omega_1\,dv\,dx\,dt,$$

and we conclude to $f = 0$ thanks to Grönwall's lemma. □

1.3.3 Well-posedness for the dual equation and conclusions

We first establish the well-posedness of the dual KFP equation in a L^2 framework.

Proposition 1.4. *Under the assumptions of Proposition 1.3, for any admissible weight function ω, any final time $T > 0$ and any final datum $g_T \in L^2_m(\mathcal{O})$, $m := \omega^{-1}$, there exists a unique $g \in Y_T := L^\infty(0,T; L^2_m) \cap C([0,T]; L^2_{\rm loc}) \cap \mathscr{H}_m$ solution to the backward dual kinetic Fokker-Planck equation* (1.2.21) *in a similar sense as stated in Proposition 1.3.*

Sketch of the proof of Proposition 1.4. We follow the same strategy as during the proof of Pro-position 1.3.

Step 1. Given $\mathfrak{g} \in L^2_m(\Gamma_-; d\xi_1)$, we consider the backward inflow problem

$$\begin{cases} -\partial_t g - v \cdot \nabla_x g = \mathscr{C}^* g & \text{in} \quad (0,T) \times \mathcal{O} \\ \gamma_+ g = \mathfrak{g} & \text{on} \quad (0,T) \times \Sigma_+ \\ g_{|t=T} = g_T & \text{in} \quad \mathcal{O}, \end{cases} \tag{1.3.14}$$

We define $\widetilde{m}$ as during the proof of Lemma 1.3 and the bilinear form $\mathscr{E} : \mathscr{H}_m \times C^1_c((0,T] \times \mathcal{O} \cup \Gamma_+) \to \mathbb{R}$, by

$$\mathscr{E}(g,\varphi) \quad := \quad \int_{\mathcal{U}} (\lambda g - \mathscr{C}^* g)\varphi \widetilde{m}^2 + \int_{\mathcal{U}} g(\partial_t + v \cdot \nabla_x)(\varphi \widetilde{m}^2),$$

which is coercive for λ large enough thanks to Lemma 1.3 (and more precisely (1.2.29)-(1.2.30)). Using Lions' variant of the Lax-Milgram theorem [126, Chap III, §1], we obtain a variational solution $g \in \mathscr{H}_m$ to (1.3.14), and more precisely

$$\mathscr{E}(g,\varphi) = \int_{\Gamma_+} \mathfrak{g}\varphi \widetilde{m}^2 \, d\xi_1 + \int_{\mathcal{O}} g_T \varphi(T,\cdot)\widetilde{m}^2 \, dv \, dx, \quad \forall \, \varphi \in C^1_c((0,T] \times \mathcal{O} \cup \Gamma_+).$$

Thanks to the trace Theorem 1.3.1 and the dissipativity property (1.2.29) of $\mathscr{L}^*$, we deduce that $g \in C([0,T]; L^2_m) \cap \mathscr{H}_m$.

Step 2. For a sequence $\alpha_k \in (0,1)$, $\alpha_k \nearrow 1$, we build a sequence (g_k) of solutions to the modified Maxwell reflection boundary condition problem

$$\begin{cases} -\partial_t g_k - v \cdot \nabla_x g_k = \mathscr{C}^* g_k & \text{in} \quad (0,T) \times \mathcal{O} \\ \gamma_+ g_k = \alpha_k \mathscr{R}^* \gamma_- f_k & \text{on} \quad (0,T) \times \Sigma_+ \\ g_k(t=T,\cdot) = g_T & \text{in} \quad \mathcal{O}, \end{cases} \tag{1.3.15}$$

by using Step 1, the fact that $\mathscr{R}^* : L^2_{\widetilde{m}}(\Sigma_-; d\xi_1) \to L^2_{\widetilde{m}}(\Sigma_-; d\xi_1)$ from Step 1 in Lemma 1.3 and the Banach fixed point Theorem. This sequence satisfies

$$\sup_{[0,T]} \|g_{kt}\|^2_{L^2_{\widetilde{m}}} + \int_0^T \left\{ \|\gamma g_{ks}\|^2_{L^2_m(\Gamma_+; d\xi_2)} + \|g_{ks}\|^2_{H^{1,\dagger}_{\widetilde{m}}} \right\} \mathrm{d}s \le C_T \|g_T\|^2_{L^2_{\widetilde{m}}}$$

for some constant C_T and any $k \geq 1$. We may extract converging subsequences $(g_{k'})$ and $(\gamma g_{k'})$ with associated limits g and $\bar{\gamma}$, and passing to the limit in (1.3.15) with the help of Theorem 1.3.2, we deduce that $\bar{\gamma} = \gamma g$ and that g is a renormalized solution to (1.2.21).

Step 3. We may assume $\omega \lesssim e^{\zeta \langle v \rangle^{s_*}}$ with $s_* = \min(2,\gamma)$ and $\zeta \in (0,\zeta^*)$, defining $\zeta^* := b_0/2$ if $\gamma < 2$, $\zeta^* := \min(b_0, 1/\Theta^*)/2$ if $\gamma = 2$, $\zeta^* := 1/(2\Theta^*)$ if $\gamma > 2$. We set $m_1 := e^{-\zeta_1 \langle v \rangle^{s_*}}$, with $\zeta_1 \in (\zeta, \zeta_*)$. We consider two solutions g_1 and $g_2 \in C([0,T]; L^2_m) \cap \mathscr{H}_m$ to the backward dual KFP equation (1.2.21) associated to the same final datum g_T, so that the function $g := g_2 - g_1 \in C([0,T]; L^2_m) \cap \mathscr{H}_m$ is a solution to the KFP equation (1.1.1)-(1.1.3) associated to the initial datum $g(T) = 0$. Choosing $\varphi := \widetilde{m}_1 \chi_R$ in (1.3.4), with the notations of Step 4 of the proof of Proposition 1.3, and proceeding similarly, we get

$$\int_{\mathcal{O}} \beta(g(0))\varphi \,\mathrm{d}v\,\mathrm{d}x + \int_{\Gamma} \beta(\gamma g)\,\varphi\,(n_x \cdot v)\,\mathrm{d}v\,\mathrm{d}\sigma_x\,\mathrm{d}t + \int_{\mathcal{U}} \beta''(f)\,|\nabla_v f|^2 \varphi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\mathcal{U}} \beta(g)\,\mathscr{T}\varphi + \int_{\mathcal{U}} \{\beta(g)\,(\Delta_v \varphi + \mathrm{div}_v(b\varphi)) + (c - \mathrm{div}_v\, b) f \beta'(f)\varphi\}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.$$

Taking advantage of the fact that all the terms in the interior and at the boundary are now well defined, we may argue as for the proof of the $L^1_{\widetilde{m}_1}$ estimate performed in Lemma 1.3, and we conclude that $g \equiv 0$ as in Step 4 of the proof of Proposition 1.3. $\square$

We conclude the section by reformulating and slightly improving the two previous well-posedness results.

Theorem 1.3.5. *Consider an admissible weight function ω and set $m := \omega^{-1}$.*

(1) There exists a semigroup $S_{\mathscr{L}}$ on $L^2_\omega(\mathcal{O})$ such that for any $f_0 \in L^2_\omega(\mathcal{O})$, the function $f_t := S_{\mathscr{L}}(t) f_0$ is the unique solution in $C([0,T]; L^2_\omega) \cap \mathscr{H}$, $\forall T > 0$, to the KFP equation (1.1.1)–(1.1.3)–(1.1.4)*. Furthermore* (1.2.5) *holds if additionally $f_0 \in L^p_\omega$ for some $p \in [1,\infty]$.*

(2) Similarly, there exists a semigroup $S_{\mathscr{L}^*}$ on $L^2_m(\mathcal{O})$ such that for any $g_T \in L^2_m(\mathcal{O})$, the function $g_t := S_{\mathscr{L}^*}(T-t) g_T$ is the unique solution in $C([0,T]; L^2_m) \cap \mathscr{H}_m$, $\forall T > 0$, to the dual KFP problem (1.2.21)*. Furthermore* (1.2.23) *holds if additionally $g_T \in L^q_m$ for some $q \in [1,\infty]$.*

(3) The semigroups $S_{\mathscr{L}}$ and $S_{\mathscr{L}^*}$ are dual one toward the other. In other words, the equation (1.2.22) *holds for any $f_0 \in L^2_\omega(\mathcal{O})$ and $g_T \in L^2_m(\mathcal{O})$.*

Proof of Theorem 1.3.5. The proof is split into four steps.

Step 1. We may define the semigroup $S_{\mathscr{L}}$ by setting $S_{\mathscr{L}}(t) f_0 := f_t$ for any $f_0 \in L^2_\omega$ and $t \geq 0$, where f_t is the unique solution in X_T, $\forall T > 0$, to the KFP equation (1.1.1)–(1.1.3)–(1.1.4) provided by Proposition 1.3. In particular (1.2.5) holds for $p = 2$. Proceeding as in Step 4 during the proof of Proposition 1.3, we may justify the computations performed during the proof of Lemma 1.1 and we get that (1.2.5) holds for $p = 1$ when $f_0 \in L^2_\omega \cap L^1_\omega$. By interpolation, we obtain that (1.2.5) holds for any $p \in [1,2]$ when $f_0 \in L^2_\omega \cap L^p_\omega$.

Step 2. From the well-posedness result of Proposition 1.4, we may define the semigroup $S_{\mathscr{L}^*}$ by setting $S_{\mathscr{L}^*}(t) g_T := g(T-t)$ for any $g_T \in L^2_m$ and any $0 \leq t \leq T$, where g is unique solution in Y_T to the KFP problem (1.2.21). We obtain as in Step 1 that furthermore (1.2.23) holds for any $q \in [1,2]$ if $g_0 \in L^2_m \cap L^q_m$.

Step 3. We change ι by $\iota_n := \iota_{S,n} + \iota_{D,n} \leq 1 - 1/n$, with $\iota_{S,n} := \iota_S(1-1/n)$, $\iota_{D,n} := \iota_D(1-1/n)$, and we denote by $\mathscr{R}_n$ and $\mathscr{R}^*_n$ the corresponding reflection operators. Denoting by f_n the solution associated to the KFP equation, the reflection operator $\mathscr{R}_n$ and the initial datum f_0 given by Step 1 (or Proposition 1.3), we have the additional property $\gamma f_n \in L^2_\omega(\Gamma; d\xi_1)$. The solution g_n associated to the dual problem (1.2.21) for the reflection operator $\mathscr{R}^*_n$ and the final datum g_T given by Step 2 (or Proposition 1.4) also satisfies the additional property $\gamma g_n \in L^2_m(\Gamma; d\xi_1)$. Because of these additional estimates on the boundary, we may justify the computations leading to the identity (1.2.22) by starting from (1.3.4) applied with $g := f_n$, $\varphi := g_n$ and $\beta(s) := s$ or by applying (a variant of) Theorem 1.3.1 to the function $\beta(f_n g_n)$, noticing that

$$\partial_t(f_n g_n) + v \cdot \nabla_x(f_n g_n) = \Delta_v f_n g_n - f_n \Delta_v g_n + \mathrm{div}_v(b f_n g_n) \ \text{ in } \ \mathcal{D}'(\mathcal{U}).$$

In that way, we obtain

$$\int_{\mathcal{O}} f_n(T) g_T = \int_{\mathcal{O}} f_0 g_n(0), \quad \forall\, n \geq 1. \tag{1.3.16}$$

Because (f_n) is bounded in $L^\infty(0,T;L^2_\omega)$ and in $W^{1,\infty}(0,T;\mathcal{D}'(\mathcal{O}))$, we deduce that $f_n(T) \rightharpoonup f(T) := S_{\mathcal{L}}(T) f_0$ weakly in L^2_ω. Similarly, we have $g_n(0) \rightharpoonup g(0) = S_{\mathscr{L}^*}(T) g_T$ weakly in L^2_m. We may thus pass to the limit $n \to \infty$ in (1.3.16) and we deduce that (1.2.22) holds, which exactly means that $(S_{\mathscr{L}^*})^* = S_{\mathscr{L}}$.

Step 4. For any $p \in (2,\infty]$, we know from Step 2 that for any $g_T \in L^2_m \cap L^{p'}_m$, $T > 0$, there holds

$$\|g(t,\cdot)\|_{L^{p'}_m} \leq C_1 e^{C_2(T-t)} \|g_T\|_{L^{p'}_m}, \quad \forall\, t \in [0,T]. \tag{1.3.17}$$

Now, for $f_0 \in L^p_\omega$, we have

$$\begin{aligned} \|f(T)\|_{L^p_\omega} &= \sup_{g_T \in L^2_m, \|g_T\|_{L^{p'}_m} \leq 1} \int f(T) g_T \\ &= \sup_{g_T \in L^2_m, \|g_T\|_{L^{p'}_m} \leq 1} \int f_0 g(0) \\ &\leq \|f_0\|_{L^p_\omega} \sup_{g_T \in L^2_m, \|g_T\|_{L^{p'}_m} \leq 1} \|g(0)\|_{L^{p'}_m} \\ &\leq \|f_0\|_{L^p_\omega} \sup_{g_T \in L^2_m, \|g_T\|_{L^{p'}_m} \leq 1} C_1 e^{C_2 T} \|g_T\|_{L^{p'}_m} = C_1 e^{C_2 T} \|f_0\|_{L^p_\omega}, \end{aligned}$$

where we have successively used the Riesz representation theorem, the duality identity (1.2.22), the Holder inequality, the estimate (1.3.17) and the Riesz representation theorem again. We have thus established that (1.2.5) holds for any $f_0 \in L^p_\omega \cap L^2_\omega$, $p \in [1,\infty]$. We establish in the same way that (1.2.23) holds for any $g_T \in L^q_m \cap L^2_m$, $q \in [1,\infty]$.

Step 5. For $f_0 \in L^2_\omega$, let us introduce a sequence $f_{0,n} \in L^2_{\omega^\sharp} \cap L^\infty_\omega$, such that $\omega^\sharp$ is an admissible weight function satisfying $\omega^\sharp/\omega \to \infty$ and such that $f_{0,n} \to f_0$ in L^2_ω. From the previous analysis, the solution f_n to the KFP equation (1.1.1)–(1.1.3) associated to the initial datum $f_{0,n}$ satisfies $f_n \in L^\infty(0,T;L^2_{\omega^\sharp} \cap L^\infty_\omega) \cap C([0,T];L^2_{\text{loc}}) \subset C([0,T];L^2_\omega)$. From (1.2.5), (f_n) is a Cauchy sequence in $C([0,T];L^2_\omega)$ and thus converges to the solution f to the KFP equation (1.1.1)–(1.1.3) associated to the initial datum f_0. We have established that $f \in C([0,T];L^2_\omega)$ and the same argument holds for the dual problem. □

1.4 Ultracontractivity

In this section, we explain how to adapt to the KFP equation in a domain the De Giorgi-Nash-Moser theory developed for parabolic equations, in particular in [65, 141, 142, 145], and generalized recently to the KFP equation in the whole space, in particular in [97, 149]. The gain of integrability $L^1 \to L^2$ essentially follows and slightly simplifies the proofs presented in [48, 49] which are very in the spirit of Nash approach [145].

1.4.1 An improved weighted L^2 estimate at the boundary

Let us observe that for a solution f to the KFP equation (1.1.1), we may write

$$\mathcal{T} \frac{f^2}{2} = f\, \mathcal{T} f = f\, \mathscr{C} f,$$

where we define

$$\mathcal{T} := \partial_t + v \cdot \nabla_x, \tag{1.4.1}$$

and we recall that $\mathscr{C}$ has been defined in (1.1.9). Multiplying that equation by $\Phi^2 := \varphi^2 \widetilde{\omega}^2$ with a time truncation function $\varphi \in \mathcal{D}(0,T)$ and a weight function $\widetilde{\omega} : \mathcal{O} \to (0,\infty)$, and next integrating in all the variables with the help of (1.2.1), we obtain

$$\frac{1}{2}\int_\Gamma (\gamma f)^2 \Phi^2 n_x \cdot v - \frac{1}{2}\int_{\mathcal{U}} f^2 \mathcal{T}\Phi^2 = -\int_{\mathcal{U}} |\nabla_v (f\Phi)|^2 + \int_{\mathcal{U}} f^2 \Phi^2 \widetilde{\varpi}, \tag{1.4.2}$$

with $\mathcal{U} := (0,T) \times \mathcal{O}$, $\Gamma := (0,T) \times \Sigma$, $T \in (0,\infty)$ and $\widetilde{\varpi} := \varpi^{\mathscr{C}}_{\widetilde{\omega},2}$ is defined in (1.1.17), or equivalently by

$$\widetilde{\varpi} = \frac{1}{4}\frac{|\nabla_v \widetilde{\omega}^2|^2}{\widetilde{\omega}^4} - \frac{1}{2} b \cdot \frac{\nabla_v \widetilde{\omega}^2}{\widetilde{\omega}^2} + c - \frac{1}{2}\operatorname{div}_v b. \tag{1.4.3}$$

We first establish a key estimate on the KFP equation (1.1.1)–(1.1.3)–(1.1.4) which makes possible to control a solution near the boundary. The proof is based on the introduction of an appropriate weight function which combines the twisting term used in the previous section and the twisting term used in [90, Section 11], that last one being in the spirit of moment arguments used in [130, 139].

Proposition 1.1. *For any admissible weight function ω there exists $C = C(\omega,\Omega) > 0$ such that for any solution f to the KFP equation* (1.1.1)–(1.1.3)*, any $T > 0$ and any smooth function $0 \le \varphi \in \mathcal{D}((0,T))$, there holds*

$$\int_{\mathcal{U}} f^2 \omega^2 \left\{ \frac{(n_x \cdot \hat{v})^2}{\delta^{1/2}} + \langle v \rangle^{\varsigma} \right\} \varphi^2 + \int_{\mathcal{U}} |\nabla_v (f\omega)|^2 \varphi^2 \le C \int_{\mathcal{U}} f^2 \omega^2 \left[|\partial_t \varphi^2| + \varphi^2 \right],$$

with $\varsigma := \gamma + s - 2$.

It is worth emphasizing that an admissible weight function ω is strongly confining if and only if the associated parameter $\varsigma \in \mathbb{R}$ satisfies $\varsigma > 0$.

Proof of Proposition 1.1. We introduce the function

$$\Phi^2 := \varphi^2 \, \widetilde{\omega}^2, \quad \widetilde{\omega}^2 := \left(1 + \frac{1}{4}\frac{n_x \cdot v}{\langle v \rangle^4} + \frac{1}{4D^{1/2}} \, \delta(x)^{1/2} \, \frac{n_x \cdot v}{\langle v \rangle^2} \right) \omega_A^2,$$

where the weight function ω_A is defined in (1.2.7) for $A \ge 1$ large enough (to be fixed below) and where $D = \sup_{x \in \Omega} \delta(x)$ is half the diameter of Ω, so that in particular an estimate similar to (1.2.25) holds. From (1.4.2) we have

$$\begin{aligned} 2\int_{\mathcal{U}} |\nabla_v (f\Phi)|^2 + \int_\Gamma (\gamma f)^2 \Phi^2 (n_x \cdot v) - \int_{\mathcal{U}} f^2 v \cdot \nabla_x \Psi_2 \\ = \int_{\mathcal{U}} f^2 v \cdot \nabla_x \Psi_1 + 2\int_{\mathcal{U}} f^2 \Phi^2 \widetilde{\varpi} + \int_{\mathcal{U}} f^2 \partial_t (\Phi^2), \end{aligned} \tag{1.4.4}$$

for $\widetilde{\varpi}$ as defined on (1.4.3) and we denote

$$\Psi_1 := \varphi^2 \omega_A^2 \left(1 + \frac{1}{4}\frac{n_x \cdot v}{\langle v \rangle^4} \right), \quad \Psi_2 := \frac{\varphi^2 \omega_A^2}{4D^{1/2}} \, \delta(x)^{1/2} \, \frac{n_x \cdot v}{\langle v \rangle^2}.$$

We now compute each term separately.

Step 1. Observing that $\widetilde{\omega}^2 = \left(1 + \frac{1}{4}\, n_x \cdot \frac{v}{\langle v\rangle^4}\right)\omega_A^2$ on the boundary $\Gamma = (0,T)\times\Sigma\times\mathbb{R}^d$, we can argue as in Step 1 in the proof of Lemma 1.1 to deduce that, choosing $A \geq 1$ large enough, the contribution of the boundary term in (1.4.4) is nonnegative, that is

$$\int_\Gamma (\gamma f)^2 \Phi^2 (n_x \cdot v) \geq 0.$$

Step 2. In order to deal with the third term at the left-hand side of (1.4.4), we define $\psi := \delta(x)^{1/2}(n_x \cdot v)\langle v\rangle^{-2}$. Observing that $\langle v\rangle\psi \in L^\infty(\mathcal{O})$, $\nabla_v\psi \in L^\infty(\mathcal{O})$ and

$$-v\cdot\nabla_x\psi = \frac{1}{2}\frac{1}{\delta(x)^{1/2}}(n_x\cdot v)^2\langle v\rangle^{-2} - \delta(x)^{1/2}(v\cdot D_x n_x v)\langle v\rangle^{-2},$$

we have

$$-\int_\mathcal{U} f^2 v\cdot\nabla_x\Psi_2 = \frac{1}{4D^{1/2}}\int_\mathcal{U} f^2\varphi^2\omega_A^2\left\{\frac{1}{2}\frac{1}{\delta(x)^{1/2}}(n_x\cdot\hat{v})^2 - \delta(x)^{1/2}(\hat{v}\cdot D_x n_x\hat{v})\right\}.$$

For the first term at the right-hand side of (1.4.4), a direct computation gives

$$\int_\mathcal{U} f^2 v\cdot\nabla_x\Psi_1 = \frac{1}{4}\int_\mathcal{U} f^2\varphi^2\omega_A^2\langle v\rangle^{-2}(\hat{v}\cdot D_x n_x\hat{v}).$$

Step 3. Writing $\widetilde{\varpi} \leq 2\langle\widetilde{\varpi}_+\rangle - \langle\widetilde{\varpi}_-\rangle$ and gathering previous estimates yields

$$\begin{aligned}
&2\int_\mathcal{U}|\nabla_v(f\widetilde{\omega})|^2\varphi^2 + \frac{1}{8D^{1/2}}\int_\mathcal{U} f^2\varphi^2\omega_A^2\frac{(n_x\cdot\hat{v})^2}{\delta(x)^{1/2}} + 2\int_\mathcal{U} f^2\varphi^2\widetilde{\omega}^2\langle\widetilde{\varpi}_-\rangle\\
&\leq \int_\mathcal{U} f^2\varphi^2\omega_A^2(\hat{v}\cdot D_x n_x\hat{v})\left\{\frac{1}{4\langle v\rangle^2} + \frac{\delta^{1/2}}{4D^{1/2}}\right\} + 4\int_\mathcal{U} f^2\varphi^2\omega_A^2\langle\widetilde{\varpi}_+\rangle + \int_\mathcal{U} f^2\widetilde{\omega}^2\partial_t(\varphi^2)\\
&\leq C_{\Omega,A}\int_\mathcal{U} f^2\omega^2\langle\widetilde{\varpi}_+\rangle\varphi^2 + C_A\int_\mathcal{U} f^2\omega^2|\partial_t\varphi^2|,
\end{aligned}$$

where we recall that $\delta \in W^{2,\infty}(\Omega)$. Using that $\langle\widetilde{\varpi}_-\rangle \geq \kappa_0\langle v\rangle^\varsigma$ with $\kappa_0 > 0$, because of (1.2.2)-(1.2.19), and also that $\widetilde{\varpi}_+$ is bounded, we deduce

$$\begin{aligned}
&2\int_\mathcal{U}|\nabla_v(f\widetilde{\omega})|^2\varphi^2 + \frac{1}{8D^{1/2}}\int_\mathcal{U} f^2\varphi^2\omega_A^2\frac{(n_x\cdot\hat{v})^2}{\delta(x)^{1/2}} + 2\kappa_0\int_\mathcal{U} f^2\varphi^2\widetilde{\omega}^2\langle v\rangle^\varsigma\\
&\leq C_{\Omega,A}\int_\mathcal{U} f^2\omega^2\varphi^2 + C_A\int_\mathcal{U} f^2\omega^2|\partial_t\varphi^2|.
\end{aligned}$$

We then conclude by observing that

$$\int_\mathcal{O}|\nabla_v(f\omega)|^2 + f^2\omega^2 \lesssim \int_\mathcal{O}|\nabla_v(f\widetilde{\omega})|^2 + f^2\widetilde{\omega}^2$$

and using that $\omega \lesssim \omega_A \lesssim \widetilde{\omega} \lesssim \omega_A \lesssim \omega$. □

We may write the above weighted L^2 estimate in a more convenient way, where the penalization of a neighborhood of the boundary is made clearer. For that purpose we state the following interpolation result which formalizes and improves some estimates used during the proof of [90, Lemma 11.9] (see also [48, Lemma 3.2]).

Lemma 1.2. *For $d \geq 1$, for any $\varsigma > 0$ and any function $g : \mathcal{O} \to \mathbb{R}$, there holds*

$$\int_{\mathcal{O}} \frac{g^2}{\delta^\beta} \lesssim \int_{\mathcal{O}} |\nabla_v g|^2 + \int_{\mathcal{O}} g^2 \left(\frac{(n_x \cdot \hat{v})^2}{\delta^{1/2}} + \langle v \rangle^\varsigma \right), \tag{1.4.5}$$

with $\beta = [2(d' + 1 + 2d'/\varsigma)]^{-1}$, $d' := \max(d, 3)$.

Proof of Lemma 1.2. For $\alpha, \beta, \eta > 0$, we start writing

$$\begin{aligned} \int_{\mathcal{O}} \frac{g^2}{\delta^\beta} &= \int_{\mathcal{O}} \frac{g^2}{\delta^\beta} \mathbf{1}_{\langle v \rangle \delta^\alpha \geq 1} + \int_{\mathcal{O}} \frac{g^2}{\delta^\beta} \mathbf{1}_{(n_x \cdot v)^2 > \delta^{2\eta}} \mathbf{1}_{\langle v \rangle \delta^\alpha < 1} \\ &\quad + \int_{\mathcal{O}} \frac{g^2}{\delta^\beta} \mathbf{1}_{|n_x \cdot v| \leq \delta^\eta} \mathbf{1}_{\langle v \rangle \delta^\alpha < 1} =: T_1 + T_2 + T_3. \end{aligned}$$

For the first term, we have

$$T_1 \leq \int_{\mathcal{O}} \frac{g^2}{\delta^\beta} \langle v \rangle^\varsigma \delta^{\varsigma\alpha} = \int_{\mathcal{O}} g^2 \langle v \rangle^\varsigma,$$

by choosing $\alpha\varsigma = \beta$. For the second term, we have

$$T_2 \leq \int_{\mathcal{O}} \frac{g^2}{\delta^\beta} \frac{1}{\langle v \rangle^2 \delta^{2\beta/\varsigma}} \frac{(n_x \cdot v)^2}{\delta^{2\eta}} = \int_{\mathcal{O}} g^2 \frac{(n_x \cdot \hat{v})^2}{\delta^{1/2}},$$

by choosing $2\eta + \beta(1 + 2/\varsigma) = 1/2$. For the third term, we define $2^* := 2d'/(d' - 2)$ the Sobolev exponent in dimension $d' \geq 3$, and we compute

$$\begin{aligned} T_3 &\leq \int_\Omega \delta^{-\beta} \Big(\int_{\mathbb{R}^d} |g|^{2^*} \Big)^{2/2^*} \Big(\int_{\mathbb{R}^d} \mathbf{1}_{|n_x \cdot v| \leq \delta^\eta} \mathbf{1}_{\langle v \rangle < \delta^{-\beta/\varsigma}} \Big)^{2/d} \\ &\lesssim \int_\Omega \delta^{-\beta} (\delta^\eta \delta^{-(d-1)\beta/\varsigma})^{2/d} \int_{\mathbb{R}^d} (|\nabla_v g|^2 + g^2) \end{aligned}$$

where we have used the Hölder inequality in the first line and the Sobolev inequality in the second line together with the observation that $\langle v' \rangle \leq \langle v \rangle$ in the orthonormal representation $v = v_1 n_x + v'$. Choosing η such that

$$-\beta - 2(1 - 1/d)\beta/\varsigma + \eta 2/d = 0,$$

we deduce that $\eta = (d/2 + (d-1)/\varsigma)\beta$, then $\beta = [2(d + 1 + 2d/\varsigma)]^{-1}$ and we conclude to (1.4.5). □

As an immediate consequence of Proposition 1.1 and the interpolation inequality (1.4.5), we get the following estimate which holds for strongly confining weight functions.

Corollary 1.3. *Consider a strongly confining weight function ω and recall that $\varsigma := \gamma + s - 2 > 0$ and $\beta := [2(d' + 1 + 2d'/\varsigma)]^{-1}$. Under the conditions of Proposition 1.1, there exists C_Ω such that*

$$\int_{\mathcal{U}} \frac{f^2}{\delta^\beta} \omega^2 \varphi^2 + \int_{\mathcal{U}} f^2 \langle v \rangle^\varsigma \omega^2 \varphi^2 + \int_{\mathcal{U}} |\nabla_v (f\omega)|^2 \varphi^2 \lesssim \int_{\mathcal{U}} f^2 \omega^2 [|\partial_t \varphi^2| + C_\Omega \varphi^2].$$

For a weakly confining admissible weight function, we obtain the following weaker estimate which is similar to [48, Proposition 3.3] and [49, Proposition 5.3].

Corollary 1.4. *Let us consider a weakly confining admissible weight function* ω *corresponding to the case when* $\varsigma := \gamma + s - 2 \leq 0$. *We set* $\beta := [2(d'+1)]^{-1}$, *recalling that* $d' := \max(d, 3)$. *Under the conditions of Proposition 1.1, there holds*

$$\int_{\mathcal{U}} \frac{f^2}{\delta^\beta} \frac{\omega^2}{\langle v\rangle^2} \varphi^2 + \int_{\mathcal{U}} |\nabla_v (f\omega)|^2 \varphi^2 \lesssim \int_{\mathcal{U}} f^2 \omega^2 [|\partial_t \varphi^2| + \varphi^2].$$

Proof of Corollary 1.4. We just use the inequality

$$\int_{\mathcal{O}} \frac{g^2}{\langle v\rangle^2 \delta^\beta} \lesssim \int_{\mathcal{O}} (|\nabla_v g|^2 + g^2) + \int_{\mathcal{O}} g^2 \frac{(n_x \cdot \hat{v})^2}{\delta^{1/2}},$$

that we may establish proceeding exactly as in [48, Lemma 3.2], with $g := f\omega$ and the conclusion of Proposition 1.1. □

1.4.2 A downgraded weighted $L^2 - L^p$ estimate

Taking advantage of a known $L^2 - L^p$ estimate available for the KFP equation set in the whole space, and thus in the interior of the domain, we deduce a downgrade weighted $L^2 - L^p$ estimate.

Proposition 1.5. *We set* $\nu := \max(2, \gamma - 1)$. *There exists* $p > 2$, $\alpha > p$ *and* $C \in (0, \infty)$ *such that any solution* f *to the KFP equation* (1.1.1)–(1.1.3) *satisfies*

$$\left\| f\varphi \frac{\omega}{\langle v\rangle^\nu} \delta^{\alpha/p} \right\|_{L^p(\mathcal{U})} \leq C \|(\varphi + |\varphi'|) f\omega\|_{L^2(\mathcal{U})}, \tag{1.4.6}$$

for any $0 \leq \varphi \in \mathcal{D}((0,T))$, *any* $T > 0$ *and any admissible weight function* ω.

We follow the proof of [97, Lemma 10] and of Step 2 in the proof of [48, Proposition 3.5].

Proof of Proposition 1.5. We split the proof into two steps.

Step 1. Consider a subset Ω' of Ω such that $\overline{\Omega'} \subset \Omega$. We introduce a truncation function $\chi \in \mathcal{D}(\Omega)$ such that $\mathbf{1}_{\Omega'} \leq \chi \leq 1$, and the function $\omega_0 := \langle v\rangle^{-\nu}\omega$. We define the function $\bar{f} := \varphi\chi\omega_0 f$ which satisfies

$$\partial_t \bar{f} + v \cdot \nabla_x \bar{f} = F \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d,$$

where $F := F_0 + \operatorname{div}_v F_1$ with

$$F_0 := f\omega_0(\varphi'\chi + \varphi v \cdot \nabla_x \chi) + \varphi\chi(\omega_0 b - \nabla_v \omega_0) \cdot \nabla_v f + c\bar{f}$$

and

$$F_1 := \varphi\chi\omega_0 \nabla_v f.$$

From [29, Theorem 1.3] with $p = 2$, $r = 0$, $\beta = 1$, $m = 1$, $\kappa = 1$ and $\Omega = 1$, we have

$$\|D_t^{1/3} \bar{f}\|^2_{L^2(\mathbb{R}^{1+2d})} + \|D_x^{1/3} \bar{f}\|^2_{L^2(\mathbb{R}^{1+2d})} \lesssim \|\bar{f}\|^2_{L^2(\mathbb{R}^{1+2d})} + \|\nabla_v \bar{f}\|^2_{L^2(\mathbb{R}^{1+2d})} + \|\langle v\rangle F_0\|^2_{L^2(\mathbb{R}^{1+2d})} + \|\langle v\rangle^2 F_1\|^2_{L^2(\mathbb{R}^{1+2d})}.$$

On the one hand, a straightforward computation gives

$$\|\bar{f}\|_{L^2(\mathbb{R}^{1+2d})} \lesssim \|\varphi\omega\langle v\rangle^{-\nu} f\|_{L^2(\mathcal{U})}$$

and

$$\|\nabla_v \bar f\|_{L^2(\mathbb{R}^{1+2d})} \lesssim \|\varphi\omega\langle v\rangle^{-1-\nu} f\|_{L^2(\mathcal{U})} + \|\varphi\langle v\rangle^{-\nu}\nabla_v(f\omega)\|_{L^2(\mathcal{U})}.$$

On the other hand, we have

$$\begin{aligned}\|\langle v\rangle F_0\|_{L^2(\mathbb{R}^{1+2d})} &\lesssim \|\varphi'\omega\langle v\rangle^{1-\nu} f\|_{L^2(\mathcal{U})} + \|\nabla\chi\|_{L^\infty(\Omega)}\|\varphi\omega\langle v\rangle^{2-\nu} f\|_{L^2(\mathcal{U})} \\ &\quad + \|\varphi\omega\langle v\rangle^{\gamma-2-\nu} f\|_{L^2(\mathcal{U})} + \|\varphi\omega\langle v\rangle^{\max(\gamma-1,s)-\nu}\nabla_v f\|_{L^2(\mathcal{U})}\end{aligned}$$

using the growth conditions (1.1.12)-(1.1.13), and

$$\|\langle v\rangle^2 F_1\|_{L^2(\mathbb{R}^{1+2d})} \lesssim \|\varphi\omega\langle v\rangle^{2-\nu}\nabla_v f\|_{L^2(\mathcal{U})}.$$

Observing that $|\omega\nabla_v f| \lesssim |\nabla_v(f\omega)| + \langle v\rangle^{s-1}\omega|f|$, it follows

$$\begin{aligned}&\|\langle v\rangle F_0\|_{L^2(\mathbb{R}^{1+2d})} + \|\langle v\rangle^2 F_1\|_{L^2(\mathbb{R}^{1+2d})} \\ &\quad\lesssim \|\varphi'\omega\langle v\rangle^{1-\nu} f\|_{L^2(\mathcal{U})} + \|\nabla_x\chi\|_{L^\infty(\Omega)}\|\varphi\omega\langle v\rangle^{2-\nu} f\|_{L^2(\mathcal{U})} \\ &\qquad + \|\varphi\langle v\rangle^{\max(2,\gamma-1,s)-\nu} f\|_{L^2(\mathcal{U})} + \|\varphi\langle v\rangle^{\max(2,\gamma-1,s)-\nu}\nabla_v(f\omega)\|_{L^2(\mathcal{U})}.\end{aligned}$$

As $s \le 2$, we have $\max(2,\gamma-1,s) = \max(2,\gamma-1) = \nu$. Therefore, Corollary 1.3 and the Sobolev embedding $H^{1/3}(\mathbb{R}^{1+2d}) \subset L^p(\mathbb{R}^{1+2d})$, with $p := 6d/(3d-2) > 2$, yield

$$\begin{aligned}\|\bar f\|_{L^p(\mathbb{R}^{1+2d})} &\lesssim \|D_t^{1/3}\bar f\|_{L^2(\mathbb{R}^{1+2d})} + \|D_x^{1/3}\bar f\|_{L^2(\mathbb{R}^{1+2d})} + \|\nabla_v\bar f\|_{L^2(\mathbb{R}^{1+2d})} + \|\bar f\|_{L^2(\mathbb{R}^{1+2d})} \\ &\lesssim \|\varphi'\omega f\|_{L^2(\mathcal{U})} + \|\nabla_x\chi\|_{L^\infty(\Omega)}\|\varphi\omega f\|_{L^2(\mathcal{U})} + \|\varphi\omega f\|_{L^2(\mathcal{U})}.\end{aligned}$$

Step 2. Choosing $\chi_k \in \mathcal{D}(\Omega)$ such that $\mathbf{1}_{\Omega_{k+1}} \le \chi_k \le \mathbf{1}_{\Omega_k}$, with $\Omega_k := \{x \in \Omega \mid \delta > 2^{-k}\}$, and $2^{-k}\|\nabla_x\chi_k\|_{L^\infty} \lesssim 1$ uniformly in $k \ge 1$, we deduce from the last estimate that

$$\|f\varphi\omega_0\|_{L^p(\mathcal{U}_{k+1})} \lesssim \|\varphi' f\omega\|_{L^2(\mathcal{U})} + 2^k\|\varphi f\omega\|_{L^2(\mathcal{U})},$$

for any $k \ge 1$, where we denote $\mathcal{U}_k := (0,T)\times\Omega_k\times\mathbb{R}^d$. Summing up, we obtain

$$\begin{aligned}\int_{\mathcal{U}} \delta^\alpha(\varphi f\omega_0)^p &= \sum_k \int_{\mathcal{U}_{k+1}\backslash\mathcal{U}_k} \delta^\alpha(\varphi f\omega_0)^p \\ &\lesssim \sum_k 2^{-k\alpha}\int_{\mathcal{U}_{k+1}}(\varphi f\omega_0)^p \\ &\lesssim \sum_k 2^{k(p-\alpha)}(\|\varphi' f\omega\|_{L^2(\mathcal{U})} + \|\varphi f\omega\|_{L^2(\mathcal{U})})^p \\ &\lesssim (\|\varphi' f\omega\|_{L^2(\mathcal{U})} + \|\varphi f\omega\|_{L^2(\mathcal{U})})^p\end{aligned}$$

because $\alpha > p$, what is nothing but (1.4.6). □

1.4.3 The $L^1 - L^r$ estimate up to the boundary

We are now in position for stating our weighted $L^1 - L^r$ estimate up to the boundary which is the well-known cornerstone step in the proof of De Giorgi-Nash-Moser gain of integrability estimate.

Proposition 1.6. *There exists an exponent $r > 2$ such that any solution f to the KFP equation* (1.1.1)–(1.1.3) *satisfies*

$$\|f\varphi\omega\|_{L^2(\mathcal{U})} \lesssim \|(\varphi+|\varphi'|)^{2\frac{r-1}{r-2}}\varphi^{-\frac{r}{r-2}} f\omega\|_{L^1(\mathcal{U})}, \tag{1.4.7}$$

for any $\varphi \in C_c^1((0,T))$, any $T > 0$ and any strongly confining admissible weight function ω.

The proof is a variant of the proof of [48, Proposition 3.7].

Proof of Proposition 1.6. We set $\Delta := \delta^{\alpha/(p\nu)}\langle v\rangle^{-1}$ and we observe that Proposition 1.5 writes

$$\|f\varphi\omega\Delta^\nu\|_{L^p(\mathcal{U})} \lesssim \|(\varphi+|\varphi'|)f\omega\|_{L^2(\mathcal{U})}. \tag{1.4.8}$$

From Corollary 1.3 and Hölder's inequality, we have next

$$\begin{aligned}\left\|f\varphi\omega\langle v\rangle^{\theta\varsigma/2}\delta^{-\beta(1-\theta)/2}\right\|_{L^2(\mathcal{U})} &\le \left\|f\varphi\omega\langle v\rangle^{\varsigma/2}\right\|^{\theta}_{L^2(\mathcal{U})}\left\|f\varphi\omega\delta^{-\beta/2}\right\|^{1-\theta}_{L^2(\mathcal{U})}\\ &\lesssim \left\|(\varphi+\sqrt{\varphi|\varphi'|})f\omega\right\|_{L^2(\mathcal{U})},\end{aligned}$$

for any $\theta\in(0,1)$. Choosing $\theta=\theta_0$ such that

$$\frac{\beta}{\varsigma}\left(\frac{1}{\theta_0}-1\right)=\frac{\alpha}{p\nu},$$

and setting $\mu := \theta_0\varsigma/2$, we thus deduce

$$\|f\varphi\omega\Delta^{-\mu}\|_{L^2(\mathcal{U})} \lesssim \|(\varphi+\sqrt{\varphi|\varphi'|})f\omega\|_{L^2(\mathcal{U})}. \tag{1.4.9}$$

The Hölder inequality

$$\|h\|_{L^r(\mathcal{U})} \le \|h\|^{1-\theta}_{L^p_{\sigma_p}(\mathcal{U})}\|h\|^{\theta}_{L^2_{\sigma_2}(\mathcal{U})}, \tag{1.4.10}$$

with $1/r := (1-\theta)/p+\theta/2$ and $1=\sigma_p^{1-\theta}\sigma_2^{\theta}$ for any $\theta\in(0,1)$, implies

$$\begin{aligned}\|f\varphi\omega\|_{L^r(\mathcal{U})} &\lesssim \|f\varphi\omega\Delta^{\mu(\theta/(1-\theta))}\|^{1-\theta}_{L^p(\mathcal{U})}\|f\varphi\omega\Delta^{-\mu}\|^{\theta}_{L^2(\mathcal{U})}\\ &\lesssim \|(\varphi+|\varphi'|)f\omega\|^{1-\theta_1}_{L^2(\mathcal{U})}\|(\varphi+\sqrt{\varphi|\varphi'|})f\omega\|^{\theta_1}_{L^2(\mathcal{U})},\\ &\lesssim \|(\varphi+|\varphi'|)f\omega\|_{L^2(\mathcal{U})},\end{aligned}$$

where we have chosen $\theta=\theta_1$ such that $\mu(\theta_1/(1-\theta_1))=\nu$, we have used the two inequalities (1.4.8) and (1.4.9) in the second line, and finally the most classical form of the Young inequality in the last line. We now use the Hölder inequality

$$\|(\varphi+|\varphi'|)f\omega\|_{L^2(\mathcal{U})} \le \|\varphi f\omega\|^{\theta_2}_{L^r(\mathcal{U})}\|(\varphi+|\varphi'|)^{2\frac{r-1}{r-2}}\varphi^{-\frac{r}{r-2}}f\omega\|^{1-\theta_2}_{L^1(\mathcal{U})}$$

with $\theta_2 := r/(2r-2)$ at the RHS of the last estimate. After simplification of the terms involving the L^r norm and taking the power $(1-\theta_2)^{-1}$ of the resulting inequality, we conclude to (1.4.7). □

For a weakly confining weight function ω, we have the following slightly weaker estimate.

Proposition 1.7. *Let $T>0$ and ω a weakly confining admissible weight function, then there is $r>2$ such that any solution f to the KFP equation* (1.1.1)–(1.1.3) *satisfies*

$$\|(\varphi+|\varphi'|)f\omega\|_{L^2(\mathcal{U})} \lesssim \|(\varphi+|\varphi'|)^{2\frac{r-1}{r-2}}\varphi^{-\frac{r}{r-2}}f\omega\langle v\rangle^K\|_{L^1(\mathcal{U})}, \tag{1.4.11}$$

for any $\varphi\in C^1_c((0,T))$ and $K:=2(3d+1)(2d+3)$.

Proof of Proposition 1.6. We observe that for a weakly confining weight function ω, there always holds $\gamma \le 2$, $s \le 2-\gamma$, and thus $\nu = \max(2, \gamma-1) = 2$. The estimates established in Corollary 1.4 and Proposition 1.5 then write for instance

$$\left\| f\varphi\omega\langle v\rangle^{-1}\delta^{-\beta/2}\right\|_{L^2(\mathcal{U})} \le C\|(\varphi+|\varphi'|)f\omega\|_{L^2(\mathcal{U})},$$

with $\beta := [2(d+1)]^{-1}$, and

$$\left\| f\varphi\omega\langle v\rangle^{-2}\delta^{1+1/p}\right\|_{L^p(\mathcal{U})} \le C\|(\varphi+|\varphi'|)f\omega\|_{L^2(\mathcal{U})},$$

with $1/p = 1/2 - 1/(2(2d+1))$. Using an interpolation argument, we get

$$\|f\varphi\omega\langle v\rangle^{-\mu}\|_{L^r(\mathcal{U})} \le C\|(\varphi+|\varphi'|)f\omega\|_{L^2(\mathcal{U})}, \tag{1.4.12}$$

by choosing $r \in (2,p)$ and $\theta_1 \in (0,1)$ in such a way that

$$\frac{1}{r} = \frac{\theta_1}{p} + \frac{1-\theta_1}{2}, \quad (1-\theta_1)\beta/2 = \theta_1(1+1/p)$$

and thus $\mu := 2\theta_1 + (1-\theta_1) = \theta_1 + 1$. From the Holder inequality, we also have

$$\|(\varphi+|\varphi'|)f\omega\|_{L^2(\mathcal{U})} \le \|\varphi f\omega\langle v\rangle^{-\mu}\|^{\theta_2}_{L^r(\mathcal{U})}\|(\varphi+|\varphi'|)^{2\frac{r-1}{r-2}}\varphi^{-\frac{r}{r-2}}f\omega\langle v\rangle^K\|^{1-\theta_2}_{L^1(\mathcal{U})} \tag{1.4.13}$$

with $\theta_2 := r/(2r-2)$ and $K = \frac{\theta_2}{1-\theta_2}\mu$. We now compute

$$p = \frac{2d+1}{d}, \quad \theta_1 = \frac{2d+1}{12d^2+20d+5}, \quad r = \frac{12d^2+20d+5}{6d^2+10d+2},$$

so that

$$\begin{aligned} K &= \frac{\theta_2}{1-\theta_2}\mu = \frac{r}{r-2}(1+\theta_1) = (12d^2+20d+5)\left(1+\frac{2d+1}{12d^2+20d+5}\right) \\ &= 12d^2+22d+6 = 2(3d+1)(2d+3). \end{aligned}$$

The both last estimates together imply (1.4.11). □

1.4.4 The $L^1 - L^p$ estimate on the dual problem

We consider the dual backward problem (1.2.21) for which we establish the same kind of estimate as for the forward KFP problem (1.1.1)–(1.1.3). In order to make the discussion simpler, we separate the analysis for strongly and weakly confining admissible weight function in this section.

Proposition 1.8. *There exist some exponent $r > 2$ such that any solution g to the dual backward problem* (1.2.21) *satisfies*

$$\|g\varphi m\|_{L^r(\mathcal{U})} \lesssim \|(\varphi+|\varphi'|)^{2\frac{r-1}{r-2}}\varphi^{-\frac{r}{r-2}}gm\|_{L^1(\mathcal{U})}, \tag{1.4.14}$$

for any test function $0 \le \varphi \in \mathcal{D}((0,T))$, any $T > 0$ and any function $m = \omega^{-1}$ which is the inverse of a strongly confining admissible weight function ω.

Proof of Proposition 1.8. The proof follows the same steps as for the proof of Proposition 1.6 and we thus repeat it without too much details.

Step 1. An improved weighted L^2 estimate at the boundary. Let ω be an admissible weight function and define $m = \omega^{-1}$.

We define the modified weight function $\widetilde{m}$ by

$$\widetilde{m}^2 := m_A^2 \left(1 + \frac{1}{4} \frac{n_x \cdot v}{\langle v \rangle^4} - \frac{1}{4D^{1/2}} \delta(x)^{1/2} \frac{n_x \cdot v}{\langle v \rangle^2} \right),$$

where m_A has been defined in (1.2.24) and $D = \sup_{x \in \Omega} \delta(x)$. Considering a solution g to the dual backward problem (1.2.21), multiplying the equation by $\Phi^2 := \varphi^2 \widetilde{m}^2$ with $\varphi \in \mathcal{D}(0,T)$, and integrating in all the variables, we obtain

$$-\frac{1}{2} \int_\Gamma (\gamma g)^2 \Phi^2 (n_x \cdot v) + \frac{1}{2} \int_\mathcal{U} g^2 \mathcal{T} \Phi^2 = \int_\mathcal{U} g (\mathscr{C}^* g) \Phi^2,$$

with $\mathcal{T}$ defined in (1.4.1).

Since $\widetilde{m}^2 := m_A^2 (1 + \frac{1}{4} \frac{n_x \cdot v}{\langle v \rangle^4})$ on the boundary Γ, Step 1 of the proof of Lemma 1.3 implies that

$$-\frac{1}{2} \int_\Gamma (\gamma g)^2 \Phi^2 (n_x \cdot v) \geq 0.$$

Arguing exactly as in the proof of Proposition 1.1 and using the estimates from Step 2 of the proof of Lemma 1.3, we obtain

$$\int g^2 m^2 \left[\frac{(n_x \cdot v)^2}{\langle v \rangle \delta^{1/2}} + \langle v \rangle^\varsigma \right] \varphi^2 + \int |\nabla_v (gm)|^2 \varphi^2 \lesssim \int g^2 m^2 [|\partial_t \varphi^2| + \varphi^2].$$

Proceeding next as for Corollary 1.3 with the help of the interpolation Lemma 1.2, we deduce

$$\int g^2 \Big(\frac{1}{\delta^\beta} + \langle v \rangle^\varsigma \Big) m^2 \varphi^2 + \int |\nabla_v (gm)|^2 \varphi^2 \lesssim \int g^2 m^2 [|\partial_t \varphi^2| + \varphi^2], \tag{1.4.15}$$

for some $\beta > 0$.

Step 2. Downgraded weighted $L^2 - L^p$ estimate, $p > 2$. For $0 \leq \varphi \in \mathcal{D}(0,T)$, $0 \leq \chi \in \mathcal{D}(\Omega)$ and the weight function $m_0 = m \langle v \rangle^{-\nu}$ with $\nu > 0$ to be chosen later, the function $\bar{g} := g \varphi \chi m_0$ satisfies

$$-\partial_t \bar{g} - v \cdot \nabla_x \bar{g} = G$$

where $G = G_0 + \operatorname{div}_v G_1$ with

$$G_0 := -g m_0 (\varphi' \chi + \varphi \nabla_x \chi \cdot v) - \varphi \chi (m_0 b - \nabla_v m_0) \cdot \nabla_v g + (c - \operatorname{div}_v b) \bar{g}$$

and

$$G_1 := \varphi \chi m_0 \nabla_v g.$$

Proceeding as in the proof of Proposition 1.5, we get first

$$\|\bar{g}\|^2_{L^p(\mathbb{R}^{2d+1})} \lesssim \|\varphi' m g\|^2_{L^2(\mathcal{U})} + \|\nabla \chi\|^2_{L^\infty} \|\varphi m g\|^2_{L^2(\mathcal{U})} + \|\varphi m g\|^2_{L^2(\mathcal{U})}.$$

for some $p \in (2, \infty)$, and next, by interpolation, we conclude with

$$\left\| g \varphi \frac{m}{\langle v \rangle^\nu} \delta^{\alpha/p} \right\|_{L^p(\mathcal{U})} \lesssim \|(\varphi + |\varphi'|) g m\|_{L^2(\mathcal{U})} \tag{1.4.16}$$

for some $\alpha > p > 2$ and $\nu \geq 0$.

Step 3. Up to the boundary $L^2 - L^r$ estimate, $r > 2$. Proceeding as during the proof of Proposition 1.6, we may use the estimates (1.4.15) and (1.4.16) together with the Hölder inequality (1.4.10) in order to obtain that there exists an exponent $r > 2$ such that any solution g to the dual problem (1.2.21) satisfies

$$\|g\varphi m\|_{L^r(\mathcal{U})} \lesssim \|(\varphi + |\varphi'|)gm\|_{L^2(\mathcal{U})}.$$

for any $\varphi \in C_c^1((0,T))$, $T > 0$. We conclude to (1.4.14) by using the Hölder inequality once more, exactly as during the proof of Proposition 1.6. □

We now prove an estimate similar to Proposition 1.8 for weakly confining admissible weights in the spirit of Proposition 1.7

Proposition 1.9. *Let ω be a weakly confining admissible weight function and define $m = \omega^{-1}$. Any solution g to the dual backward problem* (1.2.21) *satisfies*

$$\|(\varphi + |\varphi'|)g\varphi m\|_{L^2(\mathcal{U})} \lesssim \|(\varphi + |\varphi'|)^{2\frac{r-1}{r-2}}\varphi^{-\frac{r}{r-2}} gm\langle v\rangle^K\|_{L^1(\mathcal{U})}, \tag{1.4.17}$$

for any test function $0 \leq \varphi \in \mathcal{D}((0,T))$, any $T > 0$ and $K = 2(3d+1)(2d+3)$.

Proof of Proposition 1.9. The proof follows the same steps as for the proof of Proposition 1.8.

Step 1. A weighted L^2 estimate at the boundary. Let ω be an admissible weight function and define $m = \omega^{-1}$.

We define the modified weight function $\widetilde{m}$ by

$$\widetilde{m}^2 := m_A^2 \left(1 + \frac{1}{4}\frac{n_x \cdot v}{\langle v\rangle^4} - \frac{1}{4D^{1/2}}\delta(x)^{1/2}\frac{n_x \cdot v}{\langle v\rangle^2}\right),$$

as done in the Step 1 of Proposition 1.8. Considering a solution g to the dual backward problem (1.2.21), indeed multiplying the equation by $\Phi^2 := \varphi^2\widetilde{m}^2$ with $\varphi \in \mathcal{D}(0,T)$, and integrating in all the variables, we obtain

$$-\frac{1}{2}\int_\Gamma (\gamma g)^2\Phi^2(n_x \cdot v) + \frac{1}{2}\int_{\mathcal{U}} g^2\mathcal{T}\Phi^2 = \int_{\mathcal{U}} g(\mathscr{C}^*g)\Phi^2,$$

with $\mathcal{T}$ defined in (1.4.1).

Since $\widetilde{m}^2 := m_A^2(1 + \frac{1}{4}\frac{n_x\cdot v}{\langle v\rangle^4})$ on the boundary Γ, Step 1 of the proof of Proposition 1.8 implies that

$$-\frac{1}{2}\int_\Gamma (\gamma g)^2\Phi^2(n_x \cdot v) \geq 0.$$

Arguing similarly as in the proof of Proposition 1.1 and using the estimates from Step 2 of the proof of Lemma 1.3 with the difference that, since ω is a weakly confining weight function, we will have $|\varpi^{\mathscr{C}^*}_{\widetilde{m},2}| \lesssim |\varpi^{\mathscr{C}^*}_{m,2}| < \infty$. Then we obtain

$$\int g^2m^2\frac{(n_x \cdot v)^2}{\langle v\rangle\delta^{1/2}}\varphi^2 + \int |\nabla_v(gm)|^2\varphi^2 \lesssim \int g^2m^2[|\partial_t\varphi^2| + \varphi^2].$$

Proceeding next as during Corollary 1.4 we deduce

$$\int \frac{g^2}{\delta^\beta}m^2\varphi^2 + \int |\nabla_v(gm)|^2\varphi^2 \lesssim \int g^2m^2[|\partial_t\varphi^2| + \varphi^2], \tag{1.4.18}$$

for some $\beta = [2(d+1)]^{-1} > 0$.

Step 2. Downgraded weighted $L^2 - L^p$ estimate, $p > 2$. We remark that the computations from Proposition 1.8 hold for any admissible weight so proceeding similarly we get that there is some $p \in (2, \infty)$ such that

$$\left\| g\varphi \frac{m}{\langle v\rangle^\nu} \delta^{\alpha/p} \right\|_{L^p(\mathcal{U})} \lesssim \|(\varphi + |\varphi'|) g m\|_{L^2(\mathcal{U})} \tag{1.4.19}$$

for any $0 \le \varphi \in \mathcal{D}(0,T)$ and some $\alpha > p > 2$ and $\nu \ge 0$.

Step 3. Up to the boundary $L^2 - L^r$ estimate, $r > 2$. Proceeding as during the proof of Proposition 1.7, we may use the estimates (1.4.18) and (1.4.19) together with the Hölder inequality (1.4.12) in order to obtain that there exists an exponent $r > 2$ such that any solution g to the dual problem (1.2.21) satisfies

$$\|\varphi g m \langle v\rangle^{-\mu}\|_{L^r(\mathcal{U})} \lesssim \|(\varphi + |\varphi'|) g m\|_{L^2(\mathcal{U})}.$$

for any $\varphi \in C^1_c((0,T))$, $T > 0$ and some $\mu > 0$. We conclude to (1.4.17) by using the Hölder inequality once more, exactly as in (1.4.13) during the proof of Proposition 1.7. □

1.4.5 Conclusion of the proof of the ultracontractivity property

We now conclude the proof of Theorem 1.1.3 in several elementary and classical (after Nash's work) steps.

Proof of Theorem 1.1.3. Consider a strongly confining admissible weight function ω and denote by $m = \omega^{-1}$ its inverse. We split the proof into four steps.

Step 1. We first establish that there exist a constant $\eta > 0$ such that any solution f to the KFP equation (1.1.1)–(1.1.3)–(1.1.4) satisfies

$$\|f(T,\cdot)\|_{L^2_\omega(\mathcal{O})} \le C_{12} T^{-\eta} \|f_0\|_{L^1_\omega(\mathcal{O})}, \quad \forall T \in (0,1). \tag{1.4.20}$$

First indeed, from Proposition 1.6, there exist an exponent $r > 2$ such that

$$\|f\varphi\omega\|_{L^r(\mathcal{U})} \lesssim \left\| (\varphi + |\varphi'|)^{2\frac{r-1}{r-2}} \varphi^{-\frac{r}{r-2}} f\omega \right\|_{L^1(\mathcal{U})}.$$

Thanks to the estimate (1.2.5) from Lemma 1.1 for $p = 1$ and $p = r$ provided by Theorem 1.3.5, we have

$$\|\varphi\|_{L^r(0,T)} \|f_T\omega\|_{L^r(\mathcal{O})} \lesssim e^{\kappa T} \|f\varphi\omega\|_{L^r(\mathcal{U})}$$

and

$$\left\| (\varphi + |\varphi'|)^{2\frac{r-1}{r-2}} \varphi^{-\frac{r}{r-2}} f\omega \right\|_{L^1(\mathcal{U})} \lesssim e^{\kappa T} \left\| (\varphi + |\varphi'|)^{2\frac{r-1}{r-2}} \varphi^{-\frac{r}{r-2}} \right\|_{L^1(0,T)} \|f_0\omega\|_{L^1(\mathcal{O})}.$$

We choose $\varphi(t) := \psi(t/T)$ with $\psi \in C^1_c((0,1))$ such that $0 \le \psi \le \mathbf{1}_{[1/4,3/4]}$, $\psi \not\equiv 0$ and $|\psi'|^{2\frac{r-1}{r-2}} \psi^{-\frac{r}{r-2}} \in L^1(0,1)$, which is possible by taking $\psi(s) := (s - 1/4)^n_+ (3/4 - s)^n_+$ for $n > 0$ large enough, and we easily compute

$$\|\varphi\|_{L^r(0,T)} = T^{1/r} \|\psi\|_{L^r(0,1)}, \quad \|\varphi\|_{L^1(0,T)} = T \|\psi\|_{L^1(0,1)},$$
$$\left\| |\varphi'|^{2\frac{r-1}{r-2}} \varphi^{-\frac{r}{r-2}} \right\|_{L^1(0,T)} = T^{1-2\frac{r-1}{r-2}} \left\| |\psi'|^{2\frac{r-1}{r-2}} \psi^{-\frac{r}{r-2}} \right\|_{L^1(0,1)}.$$

Gathering the three last estimates and the three last identities, we finally obtain

$$T^{1/r}\|f_T\omega\|_{L^r(\mathcal{O})} \lesssim \left(T + T^{1-2\frac{r-1}{r-2}}\right)\|f_0\omega\|_{L^1(\mathcal{O})},$$

which implies

$$\begin{aligned}\|f_T\omega\|_{L^r(\mathcal{O})} &\lesssim T^{-\frac{1}{r}-1-\frac{2}{r-2}}\left(T^{1+\frac{2}{r-2}}+1\right)\|f_0\omega\|_{L^1(\mathcal{O})}\\ &\lesssim T^{-\frac{1}{r}-1-\frac{2}{r-2}}\|f_0\omega\|_{L^1(\mathcal{O})}.\end{aligned}$$

Then by using again Lemma 1.1 for $p=1$ and an interpolation argument as before, choosing $\theta := \frac{r}{2(r-1)} > 0$ such that $1/2 = 1-\theta+\theta/r$, we deduce

$$\begin{aligned}\|f_T\omega\|_{L^2(\mathcal{O})} &\le \|f_T\omega\|^{\theta}_{L^r(\mathcal{O})}\|f_T\omega\|^{1-\theta}_{L^1(\mathcal{O})}\\ &\lesssim T^{\theta(-\frac{1}{r}-1-\frac{2}{r-2})}\|f_0\omega\|_{L^1(\mathcal{O})}\end{aligned}$$

from what we immediately conclude to (1.4.20) with $\eta := \frac{r}{2(r-1)}\left(\frac{1}{r}+1+\frac{2}{r-2}\right)$.

Step 2. Arguing in a similar fashion as above but using the estimates from Lemma 1.3 and Proposition 1.8 (instead of Lemma 1.1 and Proposition 1.6), we deduce a similar result for the dual problem. More precisely, there exists $\eta' > 0$ such that any solution g to the dual backward problem (1.2.21) satisfies

$$\|g(0,\cdot)\|_{L^2_m} \le C^*_{12}T^{-\eta'}\|g_T\|_{L^1_m}, \quad \forall T \in (0,1). \tag{1.4.21}$$

Step 3. As the dual counterpart of (1.4.21), we have that any solution f to the KFP equation (1.1.1)–(1.1.3)–(1.1.4) satisfies

$$\|f(T,\cdot)\|_{L^\infty_\omega} \le C^*_{12}T^{-\eta'}\|f_0\|_{L^2_\omega}, \quad \forall T \in (0,1). \tag{1.4.22}$$

Indeed we may argue by duality in the following way: For $f_0 \in L^2_\omega$ and $g_T \in L^1_m$ we denote respectively by f and g the solution to the primal (1.1.1)–(1.1.3)–(1.1.4) and dual (1.2.21) KFP problem, then we have

$$\begin{aligned}\|f(T,\cdot)\|_{L^\infty_\omega} &= \sup_{g_T\in L^1_m, \|g_T\|_{L^1_m}\le 1}\int f(T,\cdot)g_T\\ &= \sup_{g_T\in L^1_m, \|g_T\|_{L^1_m}\le 1}\int f_0 g(0,\cdot)\\ &\le \|f_0\|_{L^2_\omega}\sup_{g_T\in L^1_m, \|g_T\|_{L^1_m}\le 1}\|g(0,\cdot)\|_{L^2_m}\\ &\le \|f_0\|_{L^2_\omega}C^*_{12}T^{-\eta'},\end{aligned}$$

where we have used an usual representation formula in the first line, the duality formula (1.2.22) in the second line, the Cauchy-Schwarz inequality in the third line and estimate (1.4.21) in the last line.

Step 4. For $T \in (0,1]$, the estimates (1.4.20) and (1.4.22) also write

$$\|f(T,\cdot)\|_{L^\infty_\omega} \le C_{12}(T/2)^{-\eta'}\|f(T/2,\cdot)\|_{L^2_\omega}, \quad \|f(T/2,\cdot)\|_{L^2_\omega} \le C^*_{12}(T/2)^{-\eta}\|f_0\|_{L^1_\omega},$$

so that

$$\|f(T,\cdot)\|_{L^\infty_\omega} \le C_{12}C^*_{12}2^{\eta+\eta'}T^{-\eta-\eta'}\|f_0\|_{L^1_\omega}. \tag{1.4.23}$$

which is nothing but (1.1.25) for $T \in (0,1]$ with $\Theta := \eta + \eta'$, $\kappa = (\alpha + \alpha')/2$ and $C \geq C_{12} C_{12}^* 2^{\eta+\eta'}$. For $T > 1$ we write $f(T) = S_{\mathscr{L}}(1) f(T-1)$, then we compute

$$\begin{aligned} \|f(T,\cdot)\|_{L^\infty_\omega} &= \|S_{\mathscr{L}}(1) f(T-1,\cdot)\|_{L^\infty_\omega} \\ &\lesssim \|f(T-1,\cdot)\|_{L^1_\omega} \\ &\lesssim e^{\kappa(T-1)} \|f_0\|_{L^1_\omega}, \end{aligned}$$

where we have used (1.4.23) at the second line and Lemma 1.1 at the third one, which concludes the proof of (1.1.25) for $T > 1$. □

We end this section by formulating a variant of Theorem 1.1.3 for weakly confining weight functions.

Proposition 1.10. *Let ω be a weakly confining admissible weight function such that $s > 0$ or $s = 0$ and $k > K + k^*$ for $K := 4(3d+1)(2d+3)$. Define $\omega_\infty := \omega^{1/2}$ if $s > 0$, $\omega_\infty := \omega \langle v \rangle^{-K}$ if $s = 0$. For any $T > 0$, there exists $\kappa, \eta > 0$ such that any solution f to the KFP equation* (1.1.1)–(1.1.3) *satisfies*

$$\|f(T,\cdot)\|_{L^\infty_{\omega_\infty}(\mathcal{O})} \lesssim e^{\kappa T} T^{-\eta} \|f_0\|_{L^1_\omega(\mathcal{O})} \tag{1.4.24}$$

Proof of Proposition 1.10. We adapt the first step of the proof of Theorem 1.1.3 by using the estimate established in Proposition 1.7 instead of Proposition 1.6 as well as the estimate (1.2.5) from Lemma 1.1 for $p = 1$ and $p = 2$ provided by Theorem 1.3.5.

Step 1. We set $\omega_2 = \omega^{3/4}$ if $s > 0$, $\omega_2 = \omega \langle v \rangle^{-K/2}$ if $s = 0$ and we prove first that

$$\|f(T,\cdot)\|_{L^2_{\omega_2}(\mathcal{O})} \lesssim T^{-\eta} \|f_0\|_{L^1_\omega(\mathcal{O})} \quad \forall T \in (0,T]. \tag{1.4.25}$$

Indeed from Proposition 1.7 and the definitions of ω_∞ and ω_2 we have that there is $r > 2$, from Proposition 1.7, such that

$$\begin{aligned} \|(\varphi + |\varphi'|) f \omega_2\|_{L^2(\mathcal{U})} &\lesssim \left\| (\varphi + |\varphi'|)^{2\frac{r-1}{r-2}} \varphi^{-\frac{r}{r-2}} f \omega_2 \langle v \rangle^{K/2} \right\|_{L^1(\mathcal{U})}, \\ &\lesssim \left\| (\varphi + |\varphi'|)^{2\frac{r-1}{r-2}} \varphi^{-\frac{r}{r-2}} f \omega \right\|_{L^1(\mathcal{U})} \end{aligned} \tag{1.4.26}$$

for any $0 < \varphi \in C^1_c((0,T))$ and any $T > 0$.

From Lemma 1.1 for $p = 1$ and $p = 2$ we have

$$\|\varphi + |\varphi'|\|_{L^2(0,T)} \|f_T \omega_2\|_{L^2(\mathcal{O})} \leq e^{\kappa T} \|(\varphi + |\varphi'|) f \omega_2\|_{L^2(\mathcal{U})} \tag{1.4.27}$$

and

$$\left\| (\varphi + |\varphi'|)^{2\frac{r-1}{r-2}} \varphi^{-\frac{r}{r-2}} f \omega \right\|_{L^1(\mathcal{U})} \lesssim e^{\kappa T} \|(\varphi + |\varphi'|)^{2\frac{r-1}{r-2}} \varphi^{-\frac{r}{r-2}}\|_{L^1(0,T)} \|f_0 \omega\|_{L^1(\mathcal{O})} \tag{1.4.28}$$

As done during the proof of Theorem 1.1.3 we choose $\varphi(t) := \psi(t/T)$ with $\psi \in C^1_c((0,1))$ such that $0 \leq \psi \leq \mathbf{1}_{[1/4,3/4]}$, $\psi \not\equiv 0$ and $|\psi'|^{2\frac{r-1}{r-2}} \psi^{-\frac{r}{r-2}} \in L^1(0,1)$, which is possible by taking $\psi(s) := (s - 1/4)^n_+ (3/4 - s)^n_+$ for $n > 0$ large enough, and we easily compute

$$\|\varphi\|_{L^2(0,T)} \geq T^{1/2} \|\psi + |\psi'|\|_{L^2(0,1)}, \quad \|\varphi\|_{L^1(0,T)} = T \|\psi\|_{L^1(0,1)},$$
$$\left\| |\varphi'|^{2\frac{r-1}{r-2}} \varphi^{-\frac{r}{r-2}} \right\|_{L^1(0,T)} = T^{1-2\frac{r-1}{r-2}} \left\| |\psi'|^{2\frac{r-1}{r-2}} \psi^{-\frac{r}{r-2}} \right\|_{L^1(0,1)}.$$

Then we deduce

$$\|f_T\omega_2\|_{L^2(\mathcal{O})} \lesssim T^{-\frac{1}{2}-\frac{r}{r-2}}\left(1+T^{1+\frac{r}{r-2}}\right)\|f_0\overline{\omega}\|_{L^1(\mathcal{O})} \tag{1.4.29}$$

which is nothing but (1.4.25) for $T\in(0,1]$ with $\eta=\frac{1}{2}+\frac{r}{r-2}$.

Step 2. Now we prove a similar result for the dual problem, for this we define $m_\infty=\omega_\infty^{-1}$ and $m_2=\omega_2^{-1}$. Indeed by using Lemma 1.3 and Proposition 1.9 we proof, by arguing similarly as in the Step 1, that there exists $\eta'>0$ such that any solution g to the dual backward problem (1.2.21) satisfies

$$\|g(0,\cdot)\|_{L^2_{m_2}} \le C_{12}^* T^{-\eta'}\|g_T\|_{L^1_{m_\infty}}, \quad \forall T\in(0,1). \tag{1.4.30}$$

Step 3. As the dual counterpart of (1.4.30), we have that any solution f to the KFP equation (1.1.1)–(1.1.3)–(1.1.4) satisfies

$$\|f(T,\cdot)\|_{L^\infty_{\omega_\infty}} \le C_{12}^* T^{-\eta'}\|f_0\|_{L^2_{\omega_2}}, \quad \forall T\in(0,1). \tag{1.4.31}$$

Indeed we may argue by duality as in the Step 3 of the proof of Theorem 1.1.3: For $f_0\in L^2_{\omega_2}$ and $g_T\in L^1_{m_\infty}$ we denote respectively by f and g the solution to the primal (1.1.1)–(1.1.3)–(1.1.4) and dual (1.2.21) KFP problem, then we have

$$\begin{aligned}
\|f(T,\cdot)\|_{L^\infty_{\omega_\infty}} &= \sup_{g_T\in L^1_{m_\infty},\|g_T\|_{L^1_{m_\infty}}\le 1}\int f(T,\cdot)g_T\\
&= \sup_{g_T\in L^1_{m_\infty},\|g_T\|_{L^1_{m_\infty}}\le 1}\int f_0 g(0,\cdot)\\
&\le \|f_0\|_{L^2_{\omega_2}} \sup_{g_T\in L^1_m,\|g_T\|_{L^1_m}\le 1}\|g(0,\cdot)\|_{L^2_{m_2}}\\
&\le \|f_0\|_{L^2_{\omega_2}} C_{12}^* T^{-\eta'},
\end{aligned}$$

where we have used an usual representation formula in the first line, the duality formula (1.2.22) in the second line, the Cauchy-Schwarz inequality in the third line and estimate (1.4.30) in the last line.

Step 4. For $T\in(0,1]$, the estimates (1.4.20) and (1.4.22) also write

$$\|f(T,\cdot)\|_{L^\infty_{\omega_\infty}} \le C_{12}(T/2)^{-\eta'}\|f(T/2,\cdot)\|_{L^2_{\omega_2}}, \quad \|f(T/2,\cdot)\|_{L^2_{\omega_2}} \le C_{12}^*(T/2)^{-\eta}\|f_0\|_{L^1_\omega},$$

so that

$$\|f(T,\cdot)\|_{L^\infty_{\omega_\infty}} \le C_{12}C_{12}^* 2^{\eta+\eta'}T^{-\eta-\eta'}\|f_0\|_{L^1_\omega}. \tag{1.4.32}$$

which is nothing but (1.1.25) for $T\in(0,1]$ with $\Theta:=\eta+\eta'$ and $C\ge C_{12}C_{12}^*2^{\eta+\eta'}$. For $T>1$ we write $f(T)=S_{\mathscr{L}}(1)f(T-1)$, then we compute

$$\begin{aligned}
\|f(T,\cdot)\|_{L^\infty_\omega} &= \|S_{\mathscr{L}}(1)f(T-1,\cdot)\|_{L^\infty_\omega}\\
&\lesssim \|f(T-1,\cdot)\|_{L^1_\omega}\\
&\lesssim e^{\kappa(T-1)}\|f_0\|_{L^1_\omega},
\end{aligned}$$

where we have used (1.4.32) at the second line and Lemma 1.1 at the third one, which concludes the proof of (1.4.24) for $T>1$. □

1.5 Well-posedness in a general framework

In this section, we establish the well-posedness of the KFP equation in a general weighted Lebesgue space framework and in a weighted Radon measures space framework. We deduce the existence and uniqueness of a family of fundamental solutions.

1.5.1 Additional a priori estimates in the L^1 framework.

We recall that any solution f to the KFP equation (1.1.1) satisfies

$$\|f_t\|_{L^p_\omega(\mathcal{O})} \lesssim \|f_0\|_{L^p_\omega(\mathcal{O})}, \quad \forall\, t \in [0,T], \tag{1.5.1}$$

$$\|f_t\|_{L^1_{\omega\langle v\rangle^\varsigma}(\mathcal{U})} \lesssim \|f_0\|_{L^1_\omega(\mathcal{O})}, \tag{1.5.2}$$

$$\|\nabla_v f\|_{L^2_\omega(\mathcal{U})} \lesssim \|f_0\|_{L^2_\omega(\mathcal{O})}, \tag{1.5.3}$$

$$\|f_t\|_{L^r_{\omega_r}(\mathcal{O})} \le C_t \|f_0\|_{L^p_\omega(\mathcal{O})}, \quad \forall\, t \in (0,T], \tag{1.5.4}$$

for any admissible weight function ω, any exponent $1 \le p \le r \le \infty$ and some admissible weight function ω_r from (1.2.5), from (1.2.18) and (1.2.20), from (1.3.13) and from Theorem 1.1.3 with $\omega_r = \omega$ in the case of a strongly confining weight function ω or from Proposition 1.10 and a standard interpolation argument in the case of a weakly confining weight function ω. The two last estimates together immediately give

$$\|\nabla_v f\|_{L^2((t_0,T)\times\mathcal{O})} \le C_{t_0,T} \|f_0\|_{L^p_\omega(\mathcal{O})}, \quad \forall\, T > t_0 > 0. \tag{1.5.5}$$

We will use an additional a priori estimate that we establish now. For further references, for $k > 0$, we define the functions T_k by

$$T_k(s) := \max(\min(s,k), -k).$$

Lemma 1.1. *Consider an admissible weight function ω and a solution f to the KFP equation* (1.1.1)–(1.1.3)–(1.1.4) *associated to an initial datum $0 \le f_0 \in L^1_\omega$. There (at least formally) hold*

$$\|\nabla_v T_K(f/\mathscr{M})\mathscr{M}^{1/2}\|_{L^2(\mathcal{U})} \le C_{T,K} \|f_0\|_{L^1_\omega} \tag{1.5.6}$$

and

$$\sup_{t\in[0,T]} \|f \mathbf{1}_{f \ge (k+1)\mathscr{M}}\|_{L^1(\mathcal{O})} \le \|f_0 \mathbf{1}_{f_0 \ge k\mathscr{M}}\|_{L^1(\mathcal{O})} + C_T \|f_0\|_{L^1_\omega}, \tag{1.5.7}$$

for any $T, K, k > 0$ and some constants $C_{T,K}$ and C_T.

Proof of Lemma 1.1. For a renormalizing function β, a positive weight function m and a nonnegative test function φ, we (at least formally) compute

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{O}} \beta(\frac{f}{m})\varphi = \int \beta'(\frac{f}{m})\frac{\varphi}{m}\{-v\cdot\nabla_x f + \Delta_v f + b\cdot\nabla_v f + cf\} \\
&= \int_\Sigma (-v\cdot n_x)\beta(\frac{f}{m})\varphi - \int \beta''(\frac{f}{m})|\nabla_v(\frac{f}{m})|^2\varphi \\
&\quad + \int \{(v\cdot\nabla_x\varphi)\beta(\frac{f}{m}) - (v\cdot\nabla_x m)\frac{f}{m}\beta'(\frac{f}{m})\frac{\varphi}{m}\} + \int \frac{f}{m}\beta'(\frac{f}{m})c\varphi \\
&\quad + \int \{\alpha(\frac{f}{m})\operatorname{div}_v(\frac{\varphi}{m}\nabla_v m) + \beta(\frac{f}{m})\operatorname{div}_v(m\nabla_v\frac{\varphi}{m}) - \beta'(\frac{f}{m})\frac{f}{m}\nabla_v\frac{\varphi}{m}\cdot\nabla_v m\} \\
&+ \int \{\beta(\frac{f}{m})\varphi[-\operatorname{div}_v b] - (b\cdot\nabla_v\varphi)\beta(\frac{f}{m}) + (b\cdot\nabla_v m)\frac{f}{m}\beta'(\frac{f}{m})\frac{\varphi}{m}\},
\end{aligned}$$

with $\alpha'(s) = s\beta''(s)$. Assuming $\beta : \mathbb{R} \to \mathbb{R}_+$ even and convex such that $\beta(0) = \beta'(0) = 0$, the boundary term can be handled in the following way

$$
\begin{aligned}
\int_\Sigma (-v\cdot n_x)\beta(\frac{\gamma f}{m})\varphi &= \int_{\Sigma_-} (v\cdot n_x)_- \beta(\frac{\mathscr{R}\gamma_+ f}{m})\varphi - \int_{\Sigma_+} (v\cdot n_x)_+ \beta(\frac{\gamma_+ f}{m})\varphi \\
&\le \int_{\Sigma_-} (v\cdot n_x)_- \Big\{\iota_D \beta(\frac{\mathscr{D}\gamma_+ f}{m}) + \iota_S \beta(\frac{\mathscr{S}\gamma_+ f}{m})\Big\}\varphi - \int_{\Sigma_+} (v\cdot n_x)_+ \beta(\frac{\gamma_+ f}{m})\varphi \\
&\le \int_{\Sigma_+} (v\cdot n_x)_+ \iota_D \Big\{\beta(\mathscr{M}\frac{\widetilde{\gamma_+ f}}{m\circ\mathcal{V}_x})\varphi\circ\mathcal{V}_x - \beta(\frac{\gamma_+ f}{m})\varphi\Big\} \\
&\quad + \int_{\Sigma_+} (v\cdot n_x)_+ \iota_S \Big\{\beta(\frac{\gamma_+ f}{m\circ\mathcal{V}_x})\varphi\circ\mathcal{V}_x - \beta(\frac{\gamma_+ f}{m})\varphi\Big\},
\end{aligned}
$$

where we have used the convexity of β in the second line and the change of variable $v \mapsto \mathcal{V}_x v$ on the last equality. Taking $m = \varphi := \mathscr{M}$, we get

$$
\int_\Sigma (-v\cdot n_x)\beta(\frac{\gamma f}{\mathscr{M}})\mathscr{M} \le \int_{\Sigma_+} (v\cdot n_x)_+ \iota_D \Big\{\beta(\widetilde{\gamma_+ f}) - \beta(\frac{\gamma_+ f}{\mathscr{M}})\Big\}\mathscr{M} \le 0,
$$

where we have classically used the very definition of $\widetilde{\gamma_+ f}$ and the Jensen inequality in order to get the last inequality. With these choices of functions β, m and φ, the first identity simplifies

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{O}} \beta(\frac{f}{\mathscr{M}})\mathscr{M} \\
&= \int_\Sigma (-v\cdot n_x)\beta(\frac{f}{\mathscr{M}})\mathscr{M} + \int [v\cdot\nabla_x\mathscr{M}](\beta(\frac{f}{\mathscr{M}}) - \frac{f}{\mathscr{M}}\beta'(\frac{f}{\mathscr{M}})) \\
&- \int \beta''(\frac{f}{\mathscr{M}})|\nabla_v(\frac{f}{\mathscr{M}})|^2\mathscr{M} + \int \alpha(\frac{f}{\mathscr{M}})\Delta_v\mathscr{M} \\
&+ \int \{\beta(\frac{f}{\mathscr{M}})\mathscr{M}[-\operatorname{div}_v b] - (b\cdot\nabla_v\mathscr{M})\beta(\frac{f}{\mathscr{M}}) + (b\cdot\nabla_v\mathscr{M} + c\mathscr{M})\frac{f}{\mathscr{M}}\beta'(\frac{f}{\mathscr{M}})\}.
\end{aligned}
$$

We finally particularize $\beta'' = \mathbf{1}_{[k,k+1]}$, $k \ge 0$, so that

$$
0 \le \beta(s),\ s\beta'(s),\ \alpha(s) \le s,\ |s\beta'(s) - \beta(s)| \le s \quad \forall\, s \in \mathbb{R},\ k \ge 0.
$$

Observing that

$$
\frac{|v\cdot\nabla_x\mathscr{M}|}{\mathscr{M}} + \frac{|b\cdot\nabla_v\mathscr{M}|}{\mathscr{M}} + \frac{|\Delta_v\mathscr{M}|}{\mathscr{M}} + |c| + |\operatorname{div}_v b| \le C\langle v\rangle^3,
$$

we deduce with this choice of β that

$$
\frac{d}{dt}\int_{\mathcal{O}} \beta(\frac{f}{\mathscr{M}})\mathscr{M} + \int \beta''(\frac{f}{\mathscr{M}})|\nabla_v(\frac{f}{\mathscr{M}})|^2\mathscr{M} \le C\int |f|\langle v\rangle^3.
$$

Because $\omega \gtrsim \langle v\rangle^3$, we may use (1.1) with $p = 1$ in order to bound the above RHS term and then both (1.5.6) and (1.5.7) follow. □

1.5.2 Well-posedness in a L^p_ω framework

For further references, we note

$$
\mathfrak{L}f := \partial_t f + \mathscr{L}f, \quad \mathfrak{L}^*\varphi := -\partial_t\varphi + \mathscr{L}^*\varphi,
$$

where $\mathscr{L}$ is defined in (1.1.10) and $\mathscr{L}^*$ is defined in (1.2.6). We also denotes $\mathscr{B}_3$ the set of functions $\beta \in C^2(\mathbb{R})$ such that β' has compact support.

Theorem 1.5.2. *We consider an admissible weight function* ω *and an exponent* $p \in [1,\infty]$. *For any* $f_0 \in L^p_\omega(\mathcal{O})$, *there exists a unique function* $f \in C(\mathbb{R}_+; L^p_\omega(\mathcal{O}))$ *satisfying the estimates* (1.5.1), (1.5.4), (1.5.5), (1.5.6), (1.5.7) *and which is a renormalized solution to* (1.1.1), *that is*

$$\int_{\mathcal{U}} \{\beta(f)\mathfrak{L}^*\varphi + \beta''(f)|\nabla_v f|^2\varphi\} + \int_\Gamma \beta(\gamma f)\,\varphi\, n_x\cdot v\, dvd\sigma_x dt = \int_{\mathcal{O}} \beta(f_0)\varphi(0,\cdot)dvdx, \quad (1.5.8)$$

for any $\varphi \in \mathcal{D}(\bar{\mathcal{U}})$ *and* $\beta \in \mathscr{B}_3$. *Furthermore, the one parameter family of mappings* $S_{\mathcal{L}}(t) : L^p_\omega \to L^p_\omega$, *defined by* $S_{\mathcal{L}}(t)f_0 := f(t,\cdot)$ *for* $t \geq 0$ *and* $f_0 \in L^p_\omega$, *is a positive semigroup of linear and bounded operators.*

It is worth emphasizing that because $\beta \in \mathscr{B}_3$, we have $\operatorname{supp}\beta'' \subset [-K,K]$ for some $K > 0$ and thus

$$\beta''(g)\,|\nabla_v g|^2 = \beta''(g)\,\mathbf{1}_{|g|\leq K}|\nabla_v g|^2 = \beta''(g)\,|\nabla_v T_K(g)|^2. \quad (1.5.9)$$

Together with (1.5.6), that implies that the second term in (1.5.8) makes sense. Also observe that $\gamma f \in L^2_\omega((t_0,T)\times\Sigma, d\xi_2)$ for any $0 < t_0 < T$, thanks to the trace Theorem 1.3.1, so that $\beta(\gamma f) \in L^\infty(\Gamma)$ and the boundary term makes sense. A similar result holds for the dual KFP equation (1.2.21).

Proof of Theorem 1.5.2. We split the proof into two steps.

Step 1. Existence part. Because of the linearity of the equation and the weak maximum principle, we may only consider nonnegative initial data (and solutions). We first assume $1 \leq p < 2$. We take $0 \leq f_{0,n} \in L^1_\omega \cap L^2_\omega$ such that $f_{n,0} \to f_0$ in L^p_ω. Thanks to Theorem 1.3.5, we may associate a sequence (f_n) of solutions in $C_t(^2_\omega) \cap L^2_t H^1_\omega$ such that (f_n) satisfies uniformly in $n \geq 1$ the estimates (1.5.1), (1.5.3), (1.5.4) and (1.5.5). Because the equation is linear the function $f_n - f_m$ satisfies (1.5.1), namely

$$\sup_{t\in[0,T]} \|(f_n - f_m)(t)\|_{L^p_\omega} \leq \|f_{0,n} - f_{0,m}\|_{L^p_\omega},$$

and therefore (f_n) is a Cauchy sequence in $C([0,T]; L^p_\omega)$. We define $f := \lim f_n$. Similarly from (1.5.5), the sequence (f_n) is a Cauchy sequence in $L^2(t_0,T;H^1_\omega)$ for any $t_0 \in (0,T)$. We thus have

$$\int_{\mathcal{U}_{t_0}} \beta(f)\mathfrak{L}\varphi + \int_{\mathcal{U}_{t_0}} \beta''(f)|\nabla_v f|^2\varphi + \int_{\Gamma_{t_0}} \beta(\gamma f)\varphi n_x\cdot v = \int_{\mathcal{O}} \beta(f(t_0))\varphi(t_0,\cdot),$$

for any $t_0 > 0$ and $\varphi \in \mathcal{D}([0,T]\times\bar{\mathcal{O}})$, where $\mathcal{U}_{t_0} := (t_0,T)\times\mathcal{O}$ and $\Gamma_{t_0} := (t_0,T)\times\Sigma$. Because of (1.5.1) and (1.5.6), we may pass to the limit $t_0 \to 0$ and we get

$$\int_{\mathcal{U}} \beta(f)\mathfrak{L}\varphi + \int_{\mathcal{U}} \beta''(f)|\nabla_v f|^2\varphi + \int_{\Gamma} \beta(\gamma f)\varphi n_x\cdot v = \int_{\mathcal{O}} \beta(f_0)\varphi(0,\cdot)$$

for any $\varphi \in \mathcal{D}([0,T]\times\bar{\mathcal{O}})$. The same holds for the dual KFP problem for $g_0 \in L^p_m$, $1 \leq p \leq 2$. By duality, we obtain the existence of a solution for any $1 \leq p \leq \infty$ for both problems.

Step 2. Uniqueness part. We consider two solutions f_i to the KFP equation in the sense of Theorem 1.5.2 and we set $f := f_2 - f_1$.

• Take $g_T \in C_c(\mathcal{O})$ and let us consider $g \in L^\infty([0,T]\times\mathcal{O}) \cap C([0,T)\times\mathcal{O})$ the solution associated to the backward dual problem (1.2.21) which existence is given by Theorem 1.3.5

and regularity is given by Theorem 1.3.5 and [97, Theorem 3]. Because $f_t \in L^2(\mathcal{O})$ for $t > 0$, we may use the Theorem 1.3.5 and thus write

$$\int_{\mathcal{O}} f(T) g_T dx dv = \int_{\mathcal{O}} f(t) g(t) dx dv, \quad \forall\, t \in (0,T).$$

• By construction, we have $f \in C([0,T]; L^1)$. We indeed have $f(t_k) \to f(t)$ a.e. on $\mathcal{O}$ as a consequence of the continuity result in Theorem 1.3.1 applied to the function $g := \beta(f)$ with $0 \le \beta(s) \le |s|^{1/2}$. On the other hand, $\{f(t);\, t \in [0,T]\}$ is weakly L^1 relatively compact as a consequence of the L^1_ω bound (1.5.1) (with $p = 1$) and the equi-integrability estimate (1.5.7) (recall that Ω is bounded). The claimed strong continuity follows. We may thus easily pass to the limit $t \to 0$ in the above formula in order to get

$$\int_{\mathcal{O}} f(T) g_T dx dv = \lim_{t \to 0} \int_{\mathcal{O}} f(t) g(t) dx dv = 0,$$

by using $f(0) = 0$ and $g \in L^\infty(0,T; L^\infty(\mathcal{O}))$. Because $g_T \in C_c(\mathcal{O})$ is arbitrary, we deduce that $f(T) = 0$ for any $T > 0$ and the uniqueness is proved. □

1.5.3 Additional a priori estimates in a Radon measures framework

We present an additional a priori estimate which holds for nonnegative solutions in a $M^1_{\omega,0}$ framework. More precisely, we claim that any nonnegative solution f_t associated to an initial datum $0 \le f_0 \in M^1_{\omega,0}$ satisfies, at least formally,

$$\limsup_{t \to 0} \int_{\mathcal{O}} f_t \phi^c \le \int_{\mathcal{O}} f_0 \phi^c, \tag{1.5.10}$$

for any $\phi \in \mathcal{D}(\Omega)$ such that $0 \le \phi \le 1$, and we denote $\phi^c := 1 - \phi$. We indeed first observe that

$$\frac{d}{dt} \int_{\mathcal{O}} f \phi^c = h_1 - h_2,$$

with

$$h_1 := \int_{\mathcal{O}} \{(v \cdot \nabla_x \phi^c) + (c - \mathrm{div}_v\, b)\phi^c) f$$

and

$$h_2 := \int_{\Sigma} \gamma f n_x \cdot v = \int_{\Sigma_+} (1 - \iota_S - \iota_D) \gamma_+ f n_x \cdot v,$$

so that $h_1 \in L^2(0,T)$ and $h_2 \ge 0$. For the first bound, on the one hand, we observe that $\omega \gtrsim \langle v \rangle^{r_0}$, $r_0 := 3d/2 + 1 + \gamma/2$ and $\omega \langle v \rangle^{\varsigma} \gtrsim \langle v \rangle^{r_2}$, $r_2 := 3\gamma/2$, so that from (1.5.1) and (1.5.2), we have

$$f \in L^\infty(0,T; L^1_\omega) \cap L^1(0,T; L^1_{\omega\langle v \rangle^\varsigma}) \subset L^\infty(0,T; L^1_{\omega_0}) \cap L^1(0,T; L^1_{\omega_2}) \subset L^2(0,T; L^1_{\omega_1}),$$

with $\omega_1 := \langle v \rangle^\gamma$. On the other hand, we have $\langle v \rangle + |c - \mathrm{div}_v\, b| \lesssim \omega_1$. Integrating in time the above differential equation and using the Cauchy-Schwarz inequality, we obtain

$$\int_{\mathcal{O}} f_t \phi^c = \int_{\mathcal{O}} f_0 \phi^c + H_1 - H_2,$$

with $H_1 \in C^{0,1/2}([0,T])$, $H_1(0) = 0$, and $H_2 \ge 0$. More precisely

$$\int_{\mathcal{O}} f_t \phi^c \le \int_{\mathcal{O}} f_0 \phi^c + t^{1/2} C \|\phi\|_{W^{1,\infty}} \|f\|_{L^2_{\omega_1}(\mathcal{U})},$$

for any $t \in (0,T)$, and thus (1.5.10) follows. Now, for any weight function $m : \mathbb{R}^d \to (0,\infty)$ such that m is decreasing and $m\omega$ is increasing and any function $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\mathbf{1}_{B_R} \le \psi \le 1$, we set $\chi := \phi\psi$, $\chi^c := 1-\chi$ and we observe that

$$\frac{\chi^c}{m} = \frac{\phi^c}{m}\mathbf{1}_{B_R} + \Big(\frac{\phi^c}{m} + \frac{\phi}{m}\psi^c\Big)\mathbf{1}_{B_R^c},$$

so that

$$\int_{\mathcal{O}} f_t \frac{\chi^c}{m} \le \frac{1}{m(R)}\int_{\mathcal{O}} f_t \phi^c + \frac{2}{(m\omega)(R)}\int_{\mathcal{O}} f_t\omega, \quad \forall\, t \ge 0 \tag{1.5.11}$$

As a consequence of (1.5.10) and (1.5.1), we thus deduce

$$\limsup_{t\to 0} \int_{\mathcal{O}} f_t \frac{\phi^c}{m} \le \frac{1}{m(R)}\int_{\mathcal{O}} f_0\phi^c + \frac{2}{(m\omega)(R)}\int_{\mathcal{O}} f_0\omega. \tag{1.5.12}$$

1.5.4 Radon measures solutions and fundamental solutions

Theorem 1.5.3. *For any admissible weight function ω and any $f_0 \in M^1_{\omega,0}$, there exists a unique solution $f \in C(\mathbb{R}_+; M^1_\omega) \cap C((0,\infty); L^\infty_\omega)$ associated to the KFP equation* (1.1.1)-(1.1.3) *in a sense that we discuss below.*

As a consequence, for any $z_0 := (x_0, v_0) \in \mathcal{O}$, there exists a unique fundamental solution $F \in C(\mathbb{R}_+; M^1_\omega) \cap C((0,\infty); L^\infty_\omega)$ associated to the KFP equation (1.1.1)-(1.1.3) *and the initial datum δ_{z_0}.*

Proof of Theorem 1.5.3. Step 1. Existence. Because of the a priori estimates (1.5.1), (1.5.4), (1.5.5), (1.5.6), (1.5.7), we may proceed exactly as during the proof of Theorem 1.5.2, and we obtain without difficulty the existence of

$$f \in C([0,T]; \mathcal{D}'(\mathcal{O})) \cap L^\infty(0,T; L^1_\omega(\mathcal{O})) \cap L^\infty(t_0,T; L^\infty_\omega(\mathcal{O})), \quad \nabla_v f \in L^2(t_0,T; L^2_\omega(\mathcal{O})),$$

for any $0 < t_0 < T$ and any admissible weight function ω, which is a renormalisation solution on $(0,T)\times\bar{\mathcal{O}}$ and a weak solution $[0,T)\times\mathcal{O}$ corresponding to the initial condition f_0. More precisely, we have both

$$\int_{\mathcal{U}} \{\beta(f)\mathfrak{L}^*\varphi + \beta''(f)|\nabla_v f|^2\varphi\} + \int_\Gamma \beta(\gamma\, f)\,\varphi\; n_x\cdot v\, dv d\sigma_x dt = 0, \tag{1.5.13}$$

for any $\varphi \in \mathcal{D}((0,T)\times\bar{\mathcal{O}})$ and any $\beta \in \mathscr{B}_3$, and

$$\int_{\mathcal{U}} f\mathfrak{L}^*\varphi = \int_{\mathcal{O}} \varphi(0,\cdot) f_0(dxdv), \tag{1.5.14}$$

for any $\varphi \in \mathcal{D}([0,T)\times\mathcal{O})$. By construction, we may also assume that f satisfies (1.5.10). Because of Theorem 1.3.5, for any weight function $m : \mathbb{R}^d \to (0,\infty)$ such that m is decreasing and $m\omega$ is increasing and converges to ∞, and any

$$\varphi \in C([0,T]; L^1_m(\mathcal{O})) \cap L^\infty(0,T; L^\infty_m(\mathcal{O})), \quad \nabla_v\varphi \in L^2(0,T; L^2_m(\mathcal{O})), \tag{1.5.15}$$

solution to the backward dual KFP equation associated to a final datum $\varphi_T \in C_c(\mathcal{O})$, we have

$$\int f(T)\varphi_T = \int f(t)\varphi(t), \quad \forall\, t \in (0,T). \tag{1.5.16}$$

Step 2. Improved identity. We claim that (1.5.16) also holds for $t = 0$. From the weak formulation, we have $t \mapsto f(t) \in C([0,T]; \mathcal{D}'(\mathcal{O}))$. Because of the $L^\infty(0,T; M^1_\omega)$ bound, we deduce that

$$f(t) \rightharpoonup f_0 \quad \text{in} \quad (C_c(\mathcal{O}))' \quad \text{as} \quad t \to 0. \tag{1.5.17}$$

Because the solution to the backward dual problem also satisfies $0 \le \varphi \in C([0,T)\times\mathcal{O})$ as a consequence of [97, Theorem 3] when $0 \le \varphi_T \in C_c(\mathcal{O})$ and with the notations of Section 1.5.3, we may next write

$$\Big|\int_{\mathcal{O}} f(t)\varphi(t) - \int_{\mathcal{O}} f_0\varphi(0)\Big| \;\le\; \Big|\int_{\mathcal{O}} f(t)\varphi(t)\chi - \int_{\mathcal{O}} f_0\varphi(0)\chi\Big| + \int_{\mathcal{O}} f(t)\varphi(t)\chi^c + \int_{\mathcal{O}} f_0\varphi(0)\chi^c,$$

for any $\chi \in \mathcal{D}(\mathcal{O})$ such that $0 \le \chi \le 1$. For the first term, we have

$$\Big|\int_{\mathcal{O}} f(t)\varphi(t)\chi - \int_{\mathcal{O}} f_0\varphi(0)\chi\Big| \le \Big|\int_{\mathcal{O}} f(t)(\varphi(t)\chi - \varphi(0)\chi)\Big| + \Big|\int_{\mathcal{O}} (f(t)-f_0)\varphi(0)\chi\Big| \to 0,$$

as $t\to\infty$, thanks to $\varphi\chi \in C_c([0,T)\times\mathcal{O})$ and (1.5.17). Particularizing $\chi := \phi\psi$, with ϕ and ψ defined in Section 1.5.3, and using (1.5.15) and (1.5.12), we have

$$\begin{aligned}\limsup_{t\to 0}\int_{\mathcal{O}} f(t)\varphi(t)\chi^c &\lesssim \limsup_{t\to 0}\int_{\mathcal{O}} f(t)m^{-1}\chi^c \\ &\lesssim \frac{1}{m(R)}\int_{\mathcal{O}} f_0\phi^c + \frac{1}{(m\omega)(R)}\|f_0\|_{L^1_\omega}.\end{aligned}$$

Using the above estimates as well as (1.5.11) at time $t = 0$, we conclude with

$$\limsup_{t\to 0}\Big|\int_{\mathcal{O}} f(t)\varphi(t) - \int_{\mathcal{O}} f_0\varphi(0)\Big| \;\le\; \frac{1}{m(R)}\int_{\mathcal{O}} f_0\phi^c + \frac{1}{(m\omega)(R)}\|f_0\|_{L^1_\omega}.$$

Assuming now that $\phi \ge \mathbf{1}_{\Omega_\varepsilon}$ and using the very definition of $f_0 \in M^1_{\omega,0}$, the RHS term vanishes in the limit $\varepsilon \to 0$ and $R\to\infty$. We may thus pass to the limit in (1.5.16), and we obtain

$$\int f(T)\varphi_T = \int f_0\varphi(0). \tag{1.5.18}$$

Step 3. Uniqueness. We consider two solutions f_1, f_2 associated to the same initial datum $0 \le f_0 \in M^1_{\omega,0}$. From Step 2, we have

$$\int_{\mathcal{O}} f_1(T)\varphi_T = \int_{\mathcal{O}} f_2(T)\varphi_T,$$

for any $T > 0$ and $\varphi_T \in C_c(\mathcal{O})$, and thus $f_1 = f_2$. □

1.6 About the Harnack inequality

In this section we establish a strong maximum principle for the solutions of the kinetic Fokker-Planck equation in the form of a Harnack inequality, which is very similar to those in [124, Theorem 2.15] and [4].

Theorem 1.6.1. *Consider a weak solution* $0 \le f \in L^2((0,T)\times\mathcal{O}) \cap L^2((0,T)\times\Omega; H^1(\mathbb{R}^d))$ *to the Kinetic Fokker-Planck equation* (1.1.1). *For any* $0 < T_0 < T_1 < T$ *and* $\varepsilon > 0$, *there holds*

$$\sup_{\mathcal{O}_\varepsilon} f_{T_0} \le C \inf_{\mathcal{O}_\varepsilon} f_{T_1}, \tag{1.6.1}$$

for some constant $C = C(T_0, T_1, \varepsilon) > 0$, *where we recall that* $\mathcal{O}_\varepsilon$ *is defined in* (1.1.22).

The proof will be obtained in two steps. In a first step we shall obtain a local version of the Harnack inequality, and then in a second step we shall use a chain argument in order to get (1.6.1). The local Harnack inequality is a direct consequence of [103, Theorem 5 & Proposition 12], see also [97] for previous results in that direction, which apply to super- and sub-solutions to the Kinetic Fokker-Planck equation with vanishing damping term

$$\partial_t g = \mathfrak{M} g := -v \cdot \nabla_x g + \Delta_v g + b \cdot \nabla_v g. \tag{1.6.2}$$

For the reader's convenience, we state these results now. For that purpose, for $r > 0$ and $z_0 := (t_0, x_0, v_0) \in \mathbb{R}^{1+2d}$ we define the set

$$Q_r(z_0) := \{(t,x,v) \in \mathbb{R}^{1+2d} \mid -r^2 < t - t_0 \le 0,\ |x - x_0 - (t-t_0)v_0| < r^3,\ |v - v_0| < r\}$$

as well as the map

$$\mathcal{T}_{r,z_0} : (t,x,v) \mapsto (t_0 + r^2 t, x_0 + r^3 x + r^2 t v_0, v_0 + r v),$$

and we observe that $\mathcal{T}_{R,z_0}(Q_r(0,0,0)) = Q_{rR}(z_0)$.

Theorem 1.6.2. *Let $T > 0$.*

(1) There exist $\zeta \in (0,1)$ and $C_1 \in (0,\infty)$ such that for any $z_0 \in \mathcal{U}$, any $0 < R \le 1$ with $\overline{Q_R(z_0)} \subset \mathcal{U}$, and any nonnegative weak super-solution g to (1.6.2)*, there holds*

$$\|g\|_{L^\zeta(\tilde{Q}^-_{\eta R}(z_0))} \le C_1 \inf_{Q_{\eta R}(z_0)} g, \tag{1.6.3}$$

where $\eta = 1/40$ and $\tilde{Q}^-_{\eta R}(z_0) := \mathcal{T}_{R,z_0}(Q_\eta(-\tau,0,0)) = Q_{\eta R}(z_0 - (R^2\tau, R^2\tau v_0, 0))$ with $\tau := 19\eta^2/2$.

(2) For any $z_0 \in \mathcal{U}$, any $0 < r < r' \le 1$ such that $\overline{Q_{r'}(z_0)} \subset \mathcal{U}$, and any $\zeta > 0$, there is $C_3 > 0$ such that any nonnegative weak sub-solution h to (1.6.2) *satisfies*

$$\|h\|_{L^\infty(Q_r(z_0))} \le C_3 \|h\|_{L^\zeta(Q_{r'}(z_0))}. \tag{1.6.4}$$

Proof of Theorem 1.6.1. We split the proof into two steps.

Step 1: Local Harnack inequality. We claim that for any $z_0 \in \mathcal{U}$ and $0 < R \le 1$ such that $\overline{Q_R(z_0)} \subset \mathcal{U}$, there exists a constant $C > 0$ such that

$$\sup_{\tilde{Q}^-_{\eta R/2}(z_0)} f \le C \inf_{Q_{\eta R/2}(z_0)} f, \tag{1.6.5}$$

where $\tilde{Q}^-_{\eta R/2}(z_0) := Q_{\eta R/2}(z_0 - (R^2\tau, R^2\tau v_0, 0))$.

On the one hand, we take $\lambda > \|c\|_{L^\infty(Q_R(z_0))}$ and we set $g := e^{\lambda t} f$. The function g satisfies

$$\partial_t g = \mathfrak{M} g + (\lambda + c) g \ge \mathfrak{M} g \quad \text{in} \quad Q_R(z_0),$$

so that g is a nonnegative weak super-solution to (1.6.2). We deduce from Theorem 1.6.2–(1) that

$$e^{\lambda t_0} e^{-\lambda R^2(\eta^2+\tau)} \|f\|_{L^\zeta(\tilde{Q}^-_{\eta R}(z_0))} \le \|g\|_{L^\zeta(\tilde{Q}^-_{\eta R}(z_0))} \le C_1 \inf_{Q_{\eta R}(z_0)} g \le C_1 e^{\lambda t_0} \inf_{Q_{\eta R}(z_0)} f.$$

On the other hand, the function $h := e^{-\lambda t} f$ with $\lambda > \|c\|_{L^\infty(Q_{\eta R}(z_0 - (R^2\tau, R^2\tau v_0, 0)))}$ satisfies

$$\partial_t h = \mathfrak{M} h + (c - \lambda) h \le \mathfrak{M} h \quad \text{in} \quad \tilde{Q}^-_{\eta R}(z_0)) = Q_{\eta R}(z_0 - (R^2\tau, R^2\tau v_0, 0)).$$

Therefore h is a nonnegative weak sub-solution to (1.6.2), and thus we deduce from Theorem 1.6.2–(3) that

$$\begin{aligned} e^{-\lambda(t_0-R^2\tau)}\|f\|_{L^\infty(Q_{\eta R/2}(z_0-(R^2\tau,R^2\tau v_0,0))} &\le \|h\|_{L^\infty(Q_{\eta R/2}(z_0-(R^2\tau,R^2\tau v_0,0)))} \\ &\le C_3\|h\|_{L^\varsigma(Q_{\eta R}(z_0-(R^2\tau,R^2\tau v_0,0)))} \\ &\le e^{-\lambda(t_0-R^2\tau)}e^{\lambda\eta^2R^2}C_3\|f\|_{L^\varsigma(Q_{\eta R}(z_0-(R^2\tau,R^2\tau v_0,0)))}. \end{aligned}$$

We conclude the local Harnack inequality (1.6.5) by gathering the two previous estimates.

Step 2: Proof of (1.6.1). Once the local Harnack inequality (1.6.5) holds, one can deduce (1.6.1) by following the second step in the proof of [124, Theorem 2.15], which uses the Harnack chain from [4]. □

1.7 Constructive asymptotic estimate

1.7.1 An abstract constructive Krein-Rutman-Doblin-Harris theorem

We formulate a general abstract constructive Krein-Rutman-Doblin-Harris theorem in the spirit of the ones presented in the recent work [90, Section 6].

We consider a positive semigroup $S = (S_t) = (S(t))$ on a Banach lattice X, which means that X is a Banach space endowed with a closed positive cone X_+ (we write $f \ge 0$ if $f \in X_+$) and that S_t is a bounded linear mapping such that $S_t : X_+ \to X_+$ for any $t \ge 0$. We also assume that S is in duality with a dual semigroup S^* defined on a dual Banach lattice Y, with closed positive cone Y_+. More precisely, we assume that $X \subset Y'$ or $Y \subset X'$, so that the bracket $\langle \phi, f\rangle$ is well defined for any $f \in X$, $\phi \in Y$, that $f \in X_+$ (resp. $\phi \ge 0$) iff $\langle \psi, f\rangle \ge 0$ for any $\psi \in Y_+$ (resp. iff $\langle \phi, g\rangle \ge 0$ for any $g \in X_+$) and that $\langle \phi, S(t)f\rangle = \langle S^*(t)\phi, f\rangle$, for any $f \in X$, $\phi \in Y$ and $t \ge 0$. We denote by $\mathcal{L}$ the generator of S with domain $D(\mathcal{L})$ and by $\mathcal{L}^*$ the generator of S^* with domain $D(\mathcal{L}^*)$. We are interested in the existence of positive eigenvectors for both $\mathcal{L}$ and $\mathcal{L}^*$, and in their quantified exponential stability.

When $\|\cdot\|_k$ is a norm on X (resp. Y), we denote $X_k := (X, \|\cdot\|_k)$ (resp. $Y_k := (Y, \|\cdot\|_k)$). For $\psi \in Y_+$ and $g \in X_+$, we define the seminorms

$$[f]_\psi := \langle \psi, |f|\rangle,\ \forall\, f \in X, \quad [\phi]_g := \langle |\phi|, g\rangle,\ \forall\, \phi \in Y.$$

In order to obtain a very accurate and constructive description of the longtime asymptotic behavior of the semigroup S, we introduce additional assumptions.

• We first make the strong dissipativity assumption

$$\|S(t)f\|_k \le C_0 e^{\lambda t}\|f\|_k + C_1\int_0^t e^{\lambda(t-s)}\|S(s)f\|_0 ds, \tag{1.7.1}$$

$$\|S^*(t)\phi\|_k \le C_0 e^{\lambda t}\|\phi\|_k + C_1\int_0^t e^{\lambda(t-s)}\|S^*(s)\phi\|_0 ds, \tag{1.7.2}$$

for any $f \in X$, $\phi \in Y$, $t > 0$ and $k = 0, 1$, where $\lambda \in \mathbb{R}$, $C_i \in (0,\infty)$ and $\|\cdot\|_k$, $k = 0, 1$ denote two families of dual norms on X and Y such that $X_1 \subset X_0$ and $Y_1 \subset X_0$. More precisely, we assume

$$\|f\|_0 \le \|f\|_1, \quad \|\phi\|_0 \le \|\phi\|_1, \quad |\langle\phi, f\rangle| \le \|\phi\|_0\|f\|_1, \quad |\langle\phi, f\rangle| \le \|\phi\|_1\|f\|_0, \tag{1.7.3}$$

for any $f \in X$ and $\phi \in Y$.

- We also assume for instance one of the two following conditions

$$\exists\, \lambda_0 \in \mathbb{R},\ \lambda_0 > \lambda,\ \exists\, t_0 > 0,\ \exists\, f_0 \in X_+ \backslash \{0\}, \quad S(t_0) f_0 \ge e^{\lambda_0 t_0} f_0 \tag{1.7.4}$$

or

$$\exists\, \lambda_0 \in \mathbb{R},\ \lambda_0 > \lambda,\ \exists\, t_0 > 0,\ \exists\, \phi_0 \in Y_+ \backslash \{0\}, \quad S^*(t_0) \phi_0 \ge e^{\lambda_0 t_0} \phi_0, \tag{1.7.5}$$

and we refer to [90, Lemma 2.4] for variants of these conditions regarding the existence of positive supereigenvectors.

- Next, we make a slightly relaxed Doblin-Harris positivity assumption

$$S_T f \ge \eta_{\varepsilon,T} g_\varepsilon [S_{T_0} f]_{\psi_\varepsilon}, \quad \forall\, f \in X_+, \tag{1.7.6}$$

$$S^*_T \phi \ge \eta_{\varepsilon,T} \psi_\varepsilon [S^*_{T_0} \phi]_{g_\varepsilon}, \quad \forall\, \phi \in Y_+, \tag{1.7.7}$$

for any $T \ge T_1 > T_0 \ge 0$ and $\varepsilon > 0$, where $\eta_{\varepsilon,T} > 0$ and where (g_ε) and (ψ_ε) are two bounded and decreasing families of X_+ and Y_+. We say that the above condition is relaxed because we possibly have $T_0 > 0$ while in the usual Doblin-Harris condition (1.7.6) or (1.7.7) holds with $T_0 = 0$.

- We finally assume the following compatibility interpolation like conditions

$$\|f\|_0 \le \xi_\varepsilon \|f\|_1 + \Xi_\varepsilon [f]_{\psi_\varepsilon},\ \forall\, f \in X,\ \varepsilon \in (0,1], \tag{1.7.8}$$

$$\|\phi\|_0 \le \xi_\varepsilon \|\phi\|_1 + \Xi_\varepsilon [\phi]_{g_\varepsilon},\ \forall\, \phi \in Y,\ \varepsilon \in (0,1], \tag{1.7.9}$$

for two positive families (ξ_ε) and (Ξ_ε) such that $\xi_\varepsilon \searrow 0$ and $\Xi_\varepsilon \nearrow \infty$ as $\varepsilon \searrow 0$.

It is worth pointing out that the above assumptions are written in a very symmetric way between the primal and dual spaces and semigroups. They are yet too rough for addressing the issue of the existence of positive eigenvectors. This existence problem is not our main purpose since it has been widely treated for instance in [90] (see also the references therein). Nevertheless, for keeping the presentation as self-contained as possible, we consider some strengthened (and quite natural) assumptions that allow us to derive the existence part, keeping in mind that many variants are possible and referring the interested reader to Sections 2 and 3 in [90].

- On the one hand, we assume that X_1 is a Banach space and

$$\|S(t) f\|_1 \le C_0' e^{\lambda' t} \|f\|_1, \tag{1.7.10}$$

for some $\lambda' \in \mathbb{R}$, some $C_0' \ge 1$, any $f \in X$ and any $t \ge 0$. Of course this is a consequence of (1.7.1) and the Gronwall inequality with $\lambda' := \lambda + C_1$ and $C_0' := C_0$ under the mild assumption that $t \mapsto \|S(t)f\|_1$ is a (everywhere defined) measurable and locally bounded function on $\mathbb{R}_+$.

- On the other hand, instead of (1.7.2), we rather assume that

$$S^* = V + W * S^* \tag{1.7.11}$$

with

$$\|V(t)\phi\|_0 \le C_0 e^{\lambda t} \|\phi\|_0, \quad \|W(t)\phi\|_1 \le C_1 e^{\lambda t} \|\phi\|_0, \quad W \ge 0, \tag{1.7.12}$$

which is a variant of (1.7.2) for $k = 1$ and which obviously implies (1.7.2) for $k = 0$. We also assume that bounded sequences of Y_1 are weakly compact sequences in the $\sigma(Y_0, X_1)$ sense.

Theorem 1.7.1. *Consider a semigroup S on a Banach lattice X which satisfies the above conditions. Then, there exists a unique eigentriplet $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times Y$ such that*

$$\mathcal{L} f_1 = \lambda_1 f_1, \quad f_1 \geq 0, \quad \mathcal{L}^* \phi_1 = \lambda_1 \phi_1, \quad \phi_1 \geq 0,$$

together with the normalization conditions $\|\phi_1\|_0 = 1$, $\langle \phi_1, f_1 \rangle = 1$. Furthermore, there exist some constructive constants $C \geq 1$ and $\lambda_2 < \lambda_1$ such that

$$\|S(t)f - \langle f, \phi_1 \rangle f_1 e^{\lambda_1 t}\|_1 \leq C e^{\lambda_2 t} \|f - \langle f, \phi_1 \rangle f_1\|_1 \tag{1.7.13}$$

for any $f \in X$ and $t \geq 0$.

Let us make some few comments.

• The above result is a variant, and in some sense a consequence, of [90, Theorem 6.3], see also [136, Theorem 5.3] and [14, Theorem 2.1]. However, the set of assumptions here only involves the semigroups S and S^* and not the eigenelements (λ_1, f_1, ϕ_1) as it was the case in [90, Theorem 6.3]. That makes clearer the properties on the semigroup S really necessary to get the conclusions. The framework is very general and in particular it is not restricted to the measures space in duality with the bounded measurable functions space as it is the case in [14]. Our result is truly constructive what was not the case in the approach developed in [136].

• In the conservative case, namely $\lambda_1 = 0$, $\phi_1 \equiv 1 \in Y_1 \subset L^\infty$, and solely assuming (1.7.1) with $\lambda < 0$, (1.7.6), (1.7.8) and X_1 is a Banach space, the same conclusion (1.7.13) holds true by just following the same proof. Such a result is a general Banach lattice variant of the classical Doblin-Harris theorem available in the measures space in duality with the bounded measurable functions space framework, see [14, 38, 91, 111] for more details and references.

• It is also worth emphasizing that (1.7.7) with $\phi := \psi_\varepsilon$ implies

$$S^*_T \psi_\varepsilon \geq \eta_{\varepsilon,T} [S^*_{T_0} \psi_\varepsilon]_{g_\varepsilon} \psi_\varepsilon =: e^{\lambda'_0 T} \psi_\varepsilon,$$

what is a condition similar to (1.7.5). We may thus alternatively first assume (1.7.6), (1.7.7) and next assume that (1.7.1), (1.7.2) hold for some $\lambda < \lambda'_0$. In other words, our constructive Krein-Rutman-Doblin-Harris theorem is really a consequence of a suitable strong dissipativity condition and of a suitable positivity condition on both primal and dual semigroups together with a compatibility conditions over the several involved norms and seminorms. This strong dissipativity condition is automatically satisfied when the semigroup has appropriate smoothing effects (measured in terms of gain of regularity, exponent of integrability or weight function) as it is the case here for the kinetic Fokker-Planck equation (see Section 1.7.2 below) but can be not true for less regularizing semigroup as for the linear Boltzmann model for instance.

• An alternative natural way to formulate the Doblin-Harris positivity conditions (1.7.6), (1.7.7) is to rather assume a family of weak Harnack conditions

$$S_T f \geq g_\varepsilon \int_{T_0}^T [S_t f]_{\psi_\varepsilon} dt \ \text{ if } \ f \geq 0, \tag{1.7.14}$$

$$S^*_T \phi \geq \psi_\varepsilon \int_{T_0}^T [S^*_t \phi]_{g_\varepsilon} dt \ \text{ if } \ \phi \geq 0, \tag{1.7.15}$$

for some constants $T > T_0 \geq 0$, together with a family of supereigenvectors (or barrier) conditions

$$S^*_t \psi_\varepsilon \geq e^{\nu_\varepsilon t} \psi_\varepsilon, \quad S_t g_\varepsilon \geq e^{\nu_\varepsilon t} g_\varepsilon, \quad \forall\, t \geq T_0, \tag{1.7.16}$$

for any $\varepsilon > 0$ and some $\nu_\varepsilon \in \mathbb{R}$. Using the first inequality in (1.7.16), we find for $f \in X_+$

$$\int_{T_0}^T [S_t f]_{\psi_\varepsilon} dt = \int_{T_0}^T \langle f, S_t^* \psi_\varepsilon \rangle dt \geq \int_{T_0}^T \langle f, e^{\nu_\varepsilon t} \psi_\varepsilon \rangle dt = \int_{T_0}^T e^{\nu_\varepsilon t} dt [f]_{\psi_\varepsilon},$$

and we thus immediately deduce (1.7.6) (with $T_0 = 0$) from (1.7.14) and (1.7.16). We may similarly deduce (1.7.7) (with $T_0 = 0$) from (1.7.15) and (1.7.16).

• We briefly discuss the link between our set of hypotheses and the strong maximum principle which is also classically used in the Krein-Rutman theory. For that purpose, we introduce the notion of strict positivity by writing $f \in X_{++}$ or $f > 0$ (resp. $\phi \in Y_{++}$ or $\phi > 0$) if $\langle \psi, f \rangle > 0$ for any $\psi \in Y_+ \backslash \{0\}$ (resp. $\langle \phi, g \rangle > 0$ for any $g \in X_+ \backslash \{0\}$). Under assumptions (1.7.8) and (1.7.9), we claim that (1.7.6) with $T_0 = 0$, or (1.7.6) and (1.7.7) together with (1.7.4), imply the classical strong maximum principle, and we recall that this last one classically writes

$$f \in D(\mathcal{L}) \cap X_+ \backslash \{0\}, \ \mu \in \mathbb{R}, \ (\mu - \mathcal{L}) f =: g \geq 0 \quad \text{implies} \quad f > 0. \tag{1.7.17}$$

Before proving this claim we establish the following elementary facts

(i) $f \in X_+ \backslash \{0\}$ iff $f \in X_+$ and $\langle \psi_\varepsilon, f \rangle > 0$ for any $\varepsilon \in (0, \varepsilon_f)$, $\varepsilon_f > 0$ small enough,

(ii) $f \geq \alpha_\varepsilon g_\varepsilon$, $\alpha_\varepsilon > 0$, for any $\varepsilon > 0$, imply $f > 0$,

as consequences of (1.7.8) and (1.7.9).

One the one hand, for any fixed $f \in X \backslash \{0\}$ the family of interpolation estimates (1.7.8) implies

$$0 < \frac{1}{2} \|f\|_0 \leq \|f\|_0 - \xi_\varepsilon \|f\|_1 \leq \Xi_\varepsilon [f]_{\psi_\varepsilon},$$

$\varepsilon \in (0, \varepsilon_f)$, $\varepsilon_f > 0$ small enough, which gives (i). In particular, $\psi_\varepsilon \neq 0$ for $\varepsilon > 0$ small enough (what can also be added as an assumption in the definition of ψ_ε!). We similarly have $[\phi]_{g_\varepsilon} > 0$ for any $\phi \in Y_+ \backslash \{0\}$ and any $\varepsilon \in (0, \varepsilon_\phi)$, $\varepsilon_\phi > 0$ small enough, and thus $g_\varepsilon \neq 0$ for $\varepsilon > 0$ small enough. In particular, we have

$$\langle \psi_\varepsilon, g_\varepsilon \rangle \geq \langle \psi_\varepsilon, g_{\varepsilon_0} \rangle \geq \Xi_\varepsilon^{-1} \frac{1}{2} \|g_{\varepsilon_0}\|_0 > 0,$$

for any $\varepsilon \in (0, \varepsilon_{g_{\varepsilon_0}})$ and $\varepsilon_0 > 0$ such that $g_{\varepsilon_0} \neq 0$. Assume now $f \geq \alpha_\varepsilon g_\varepsilon$, $\alpha_\varepsilon > 0$, for any $\varepsilon > 0$. For any $\phi \in Y_+ \backslash \{0\}$, we then have

$$\langle \phi, f \rangle \geq \alpha_{\varepsilon_\phi} \langle \phi, g_{\varepsilon_\phi} \rangle > 0.$$

We have established that $f > 0$, and thus (ii) is proved.

We come now to the proof of the strong maximum principle and thus consider (μ, f, g) satisfying the requirements of (1.7.17). We fix ν strictly larger than μ and strictly larger than the growth bound of S so that we may write

$$f = (\nu - \mathcal{L})^{-1} ((\nu - \mu) f + g) = \int_0^\infty e^{-\nu t} S_t ((\nu - \mu) f + g) dt$$

to get by positivity of g

$$f \geq (\nu - \mu) \int_0^\infty e^{-\nu t} S_t f dt. \tag{1.7.18}$$

From (1.7.6) with $T_0 = 0$, we deduce

$$f \geq g_\varepsilon (\nu - \mu) \int_{T_1}^\infty e^{-\nu t} \eta_{\varepsilon,t} dt \, \langle \psi_\varepsilon, f \rangle, \quad \forall \, \varepsilon > 0. \tag{1.7.19}$$

From (i) above and because $f \in X_+\backslash\{0\}$, there exists $\varepsilon_f > 0$ such that $\langle \psi_\varepsilon, f\rangle > 0$ for any $\varepsilon \in (0, \varepsilon_f)$. Together with (1.7.19), we deduce that $f \geq \alpha_\varepsilon g_\varepsilon$, with $\alpha_\varepsilon > 0$, for any $\varepsilon \in (0, \varepsilon_f)$, and that in turn implies $f > 0$ from (ii) above.

When (1.7.6) and (1.7.7) are satisfied with $T_0 > 0$, we may derive (1.7.17) by using the additional condition (1.7.4). We apply (1.7.6) with $T = t - T_1 + T_0$ to the vector $S_{T_1 - T_0} f$ to get that

$$S_t f \geq \eta_{\varepsilon, t-T_1+T_0} g_\varepsilon \langle \psi_\varepsilon, S_{T_1} f\rangle$$

for any $t \geq 2T_1 - T_0$. Injecting this inequality in (1.7.18), we obtain

$$\begin{aligned} f &\geq g_\varepsilon(\nu - \mu) \int_{2T_1 - T_0}^\infty e^{-\nu t} \eta_{\varepsilon, t-T_1+T_0} dt \, \langle \psi_\varepsilon, S_{T_1} f\rangle \\ &\geq g_\varepsilon(\nu - \mu) \int_{2T_1 - T_0}^\infty e^{-\nu t} \eta_{\varepsilon, t-T_1+T_0} dt \, \langle S^*_{T_1} \psi_\varepsilon, f\rangle, \end{aligned}$$

for any $\varepsilon > 0$. Together with (1.7.7), that implies

$$f \geq g_\varepsilon(\nu - \mu) \int_{2T_1 - T_0}^\infty e^{-\nu t} \eta_{\varepsilon, t-T_1+T_0} dt \, \eta_{\varepsilon', T_1} \langle S^*_{T_0} \psi_\varepsilon, g_{\varepsilon'}\rangle \langle \psi_{\varepsilon'}, f\rangle. \qquad (1.7.20)$$

On the other hand, taking $n \in \mathbb{N}$ large enough so that $nt_0 \geq T_0$ and iterating (1.7.4), we get that $S_{nt_0} f_0 \geq e^{nt_0} f_0$ and as a consequence $S_{nt_0} f_0 \in X_+ \setminus \{0\}$. We infer that necessarily $S_{T_0} f_0 \in X_+ \setminus \{0\}$, and the existence of ε_0 such that $\langle S^*_{T_0} \psi_\varepsilon, f_0\rangle = \langle \psi_\varepsilon, S_{T_0} f_0\rangle > 0$ for all $\varepsilon \in (0, \varepsilon_0)$. We thus deduce in particular $S^*_{T_0} \psi_\varepsilon \in X_+ \setminus \{0\}$, and since we also have $f \in X_+ \setminus \{0\}$, we deduce the existence of $\varepsilon' > 0$ such that $\langle S^*_{T_0} \psi_\varepsilon, g_{\varepsilon'}\rangle > 0$ and $\langle \psi_{\varepsilon'}, f\rangle > 0$. Coming back to (1.7.20), we have proved, for any $\varepsilon \in (0, \varepsilon_0)$, the existence of $\alpha_\varepsilon > 0$ such that $f \geq \alpha_\varepsilon g_\varepsilon$ and this guarantees that $f > 0$.

Symmetrically, the assumptions (1.7.6), (1.7.7) and (1.7.5) imply that the dual operator $\mathcal{L}^*$ satisfies the strong maximum principle.

In particular, we deduce that the first eigenvectors exhibited in Theorem 1.7.1 satisfies $f_1 > 0$ and $\phi_1 > 0$.

1.7.2 Application to the KFP equation: proof of Theorem 1.1.2

In this section, we consider the kinetic Fokker-Planck equation (1.1.1), (1.1.3), (1.1.4) and we prove Theorem 1.1.2 by using Theorem 1.7.1. We define $X_0 := L^2_{\omega_2}$ for a strongly confining admissible exponential weight function ω_2 and $X = X_1 := L^r_{\omega_r}$ with $r \in (2, \infty)$ given by Proposition 1.6 and an admissible exponential weight function ω_r such that $L^r_{\omega_r} \subset L^2_{\omega_2}$. We next define $Y_0 := L^{r'}_{m_r}$ with r' the conjugate exponent associated to r, $m_r := \omega_r^{-1}$ and $Y_1 := L^2_{m_2}$, with $m_2 := \omega_2^{-1}$. Many other choices are possible. This choice however contrasts with the usual $L^1 - L^\infty$ framework considered when using Doblin-Harris type arguments.

We now check that the assumptions of Theorem 1.7.1 are met.

We recall that $\mathcal{O}_\varepsilon$ has been defined in (1.1.22) and we denote $\varepsilon_0 > 0$ such that $\mathcal{O}_{\varepsilon_0} \neq \emptyset$.

• *A positive supereigenvector condition.* For a given function $0 \leq h_0 \in C^2_c(\mathcal{O})$ normalized by $\|h_0\|_{L^2_{\omega_2}} = 1$ and such that $\operatorname{supp} h_0 \subset \mathcal{O}_{\varepsilon_0}$, and for $\lambda > \omega(S_{\mathscr{L}})$ the growth bound of $S_{\mathscr{L}}$, we define $f_0 \in D(\mathscr{L})$ as the solution to

$$(\lambda - \mathscr{L}) f_0 = h_0 \ \text{ in } \ \mathcal{O}, \quad \gamma_- f_0 = \mathscr{R} \gamma_+ f_0 \ \text{ on } \ \Sigma_-.$$

The existence and uniqueness of such a solution is a classical consequence of the existence of the semigroup $S_{\mathscr{L}}$ given by Proposition 1.3. Repeating the proof of the condition (H2) in [90, Section 11.4], there exists a constructive constant $c > 0$ such that $f_0 \geq c\, h_0$. Coming back to the equation, we have

$$\mathscr{L} f_0 = \lambda f_0 - h_0 \geq (\lambda - c^{-1}) f_0 \ \text{ in } \ \mathcal{O},$$

which is a variant of (1.7.4), and in particular from [90, Remark 2.5], it implies (1.7.4) with $\lambda_0 := (\lambda - c^{-1})$.

- *Strong dissipativity conditions.* We define

$$\mathcal{B} := \mathscr{L} - \mathcal{A}, \quad \mathcal{A} f := M \chi_R f, \quad \mathbf{1}_{B_R} \leq \chi_R \leq \mathbf{1}_{B_{2R}},$$

with $B_r := \{v \in \mathbb{R}^d;\ |v| \leq r\}$ and χ_R a smooth function. We then define the semigroup $S_\mathcal{B}$ associated to $\mathcal{B}$ and the reflection condition (1.1.3) which existence is given by Proposition 1.3. We claim that for any $a^* \in \mathbb{R}$, we may choose $M, R > 0$ large enough in such a way that $S_\mathcal{B}$ satisfies

$$\|S_\mathcal{B}(t) f\|_{L^p_\omega} \leq e^{a^* t} \|f\|_{L^p_\omega}, \quad \forall\, t \geq 0,\ \forall\, f \in L^p_\omega, \tag{1.7.21}$$

for any Lebesgue space L^p_ω with admissible exponential weight function ω. Coming back indeed to the proof of Lemma 1.1, for $p = 1, 2$, and more precisely to (1.2.18), the function $f(t) := S_\mathcal{B}(t) f_0$ satisfies

$$\frac{d}{dt} \int_\mathcal{O} f^p \tilde\omega^p \ \ \leq \ \ \int_\mathcal{O} f^p \tilde\omega^p \varpi^\mathcal{B},$$

with $\varpi^\mathcal{B} := \varpi^{\mathscr{L}}_{\tilde\omega, p} - M\chi_R$. Because of (1.2.20), we may thus fix $M, R > 0$ large enough, in such a way that $\varpi^\mathcal{B} \leq a^*$. That implies (1.7.21) for $p = 1, 2$. We deduce that (1.7.21) holds for any $p \in [1, \infty]$ as we proved the similar growth estimate for $S_\mathscr{L}$, and we thus refer to Theorem 1.3.5 for more details.

On the other hand, from Theorem 1.1.3 applied to the semigroup $S_\mathcal{B}$, we know that

$$\|S_\mathcal{B}(t) f\|_{L^r_{\omega_r}} \lesssim \frac{e^{Ct}}{t^\Theta} \|f\|_{L^2_{\omega_2}}. \tag{1.7.22}$$

We finally recall that from Theorem 1.1.1, we have

$$\|S_\mathscr{L}(t) f\|_{L^q_{\omega_q}} \leq e^{Ct} \|f\|_{L^q_{\omega_q}}, \tag{1.7.23}$$

for the exponents $q = 2, r$, for any admissible weight function ω_q, for any $f \in L^q_{\omega_q}$ and any $t \geq 0$.

Iterating the Duhamel formula

$$S_\mathscr{L} = S_\mathcal{B} + S_\mathcal{B}\mathcal{A} * S_\mathscr{L},$$

we get

$$S_\mathscr{L} = \mathcal{V} + \mathcal{W} * S_\mathscr{L},$$

with

$$\mathcal{V} := S_\mathcal{B} + \cdots + (S_\mathcal{B}\mathcal{A})^{*(N-1)} * S_\mathcal{B}, \quad \mathcal{W} := (S_\mathcal{B}\mathcal{A})^{*N}.$$

Here $*$ denotes the usual convolution operation for functions defined on $\mathbb{R}_+$ and we define recursively $U^{*1} = U$, $U^{*k} = U^{*(k-1)} * U$. Combining (1.7.22) and (1.7.23), we may use [135, Proposition 2.5] (see also [102, 134]) and we deduce that

$$\|\mathcal{V}(t) f\|_{L^r_{\omega_r}} \lesssim e^{at} \|f\|_{L^r_{\omega_r}}, \qquad \|\mathcal{W}(t) f\|_{L^r_\omega} \lesssim e^{at} \|f\|_{L^2_{\omega_2}},$$

for $N \geq 1$ large enough, any $a > a^*$, any $t \geq 0$ and any $f \in L^r_{\omega_r}$, and thus

$$\|S_{\mathscr{L}}(T)f_0\|_{L^r_{\omega_r}} \leq Ce^{aT}\|f_0\|_{L^r_{\omega_r}} + Ce^{CT}\|f_0\|_{L^2_{\omega_2}}, \tag{1.7.24}$$

for any $t \geq 0$ and $f_0 \in L^r_{\omega_r}$. That is nothing but (1.7.1) for $k = 1$. The same estimate for $k = 0$ is clear, it is nothing but Lemma 1.1. The proof of (1.7.11)-(1.7.12) is similar and it is thus skipped.

• *Doblin-Harris condition.* Let us fix $0 \leq f_0 \in L^2_\omega$ and denote $f_t := S_{\mathscr{L}}(t)f_0$. For $T_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, we know from Theorem 1.6.1 that for any $T_1 > T_0 > 0$ and for every $T \geq T_1$, we have

$$\sup_{\mathcal{O}_\varepsilon} f_{T_0} \leq C \inf_{\mathcal{O}_\varepsilon} f_T,$$

for a constant C independent of f. We deduce

$$\begin{aligned} f_T &\geq (\inf_{\mathcal{O}_\varepsilon} f_T)\mathbf{1}_{\mathcal{O}_\varepsilon} \\ &\geq \frac{1}{C}(\sup_{\mathcal{O}_\varepsilon} f_{T_0})\mathbf{1}_{\mathcal{O}_\varepsilon} \\ &\geq \frac{1}{C}\frac{1}{|\mathcal{O}_\varepsilon|}\langle S_{T_0}f_0, \mathbf{1}_{\mathcal{O}_\varepsilon}\rangle \mathbf{1}_{\mathcal{O}_\varepsilon} \end{aligned}$$

what is (1.7.6) with $g_\varepsilon = \psi_\varepsilon := \mathbf{1}_{\mathcal{O}_\varepsilon}$. The proof of (1.7.7) is identical.

• *The interpolation condition.* Let us consider two exponents $p > q$ and two locally bounded weight functions ω_p, ω_q such that $\omega_p \geq \omega_q$ and $\omega_q/\omega_p \in L^{qr'}$ with $r := p/q \in (1, \infty)$, and in particular $L^p_{\omega_p} \subset L^q_{\omega_q}$. We have

$$\begin{aligned} \|f\|_{L^q_{\omega_q}} &\leq \|f\mathbf{1}_{\mathcal{O}_\varepsilon}\|_{L^q_{\omega_q}} + \|f\omega_q\mathbf{1}_{\mathcal{O}^c_\varepsilon}\|_{L^q} \\ &\leq \|f\mathbf{1}_{\mathcal{O}_\varepsilon}\|^\theta_{L^p_{\omega_q}}\|f\mathbf{1}_{\mathcal{O}_\varepsilon}\|^{1-\theta}_{L^1_{\omega_q}} + \|f\omega_p\|_{L^p}\|\omega_q/\omega_p\mathbf{1}_{\mathcal{O}^c_\varepsilon}\|_{L^{r'q}} \\ &\leq (\varepsilon^{\frac{1}{\theta}} + \|\omega_q/\omega_p\mathbf{1}_{\mathcal{O}^c_\varepsilon}\|_{L^{r'q}})\|f\|_{L^p_{\omega_p}} + \varepsilon^{\frac{1}{\theta-1}}\|f\mathbf{1}_{\mathcal{O}_\varepsilon}\|_{L^1_{\omega_q}}, \end{aligned}$$

where we have used the classical interpolation inequality (with $1/q = \theta/p + 1 - \theta$, $\theta \in (0, 1)$) and the Holder inequality in the second line, and the Young inequality in the third line. That implies both (1.7.8) and (1.7.9).

The same conditions hold for the dual problem, so that we may apply Theorem 1.7.1 in order to conclude that Theorem 1.7.1 holds in the space $X_1 = L^r_\omega$. We deduce that Theorem 1.7.1 holds in any weighted Lebesgue spaces associated to admissible weight functions by using the extension trick as developed in [102, 134] to which we refer for details. It is also worth emphasizing that the uniform estimates in (1.1.23) directly follow from the ultracontractivity estimate established in Theorem 1.1.3 for the primal and the dual semigroups and that the strict positivity properties in (1.1.23) directly follow from the discussion about the strong maximum principle just after the statement of Theorem 1.7.1. Furthermore, $f_1, \phi_1 \in C(\mathcal{O})$ as a direct consequence of Theorem 1.3.5 and [97, Theorem 3].

1.7.3 Proof of Theorem 1.7.1

This section is devoted to the proof of Theorem 1.7.1 which is split into six steps. We closely follow the material presented in [90, Section 2,3] (see also [129]) in Steps 1, 2, 3, 4 and the material presented in [38] in Steps 5 and 6.

Step 1. Existence of ϕ_1. From the fact that S^* is a positive semigroup, (1.7.5) and [90, Lemma 2.6], we know that

$$\lambda_1 := \inf\{\kappa \in \mathbb{R};\ z - \mathcal{L} \text{ is invertible for any } z \geq \kappa\} \geq \lambda_0$$

and

$$\exists\, \lambda_n \searrow \lambda_1,\ \exists\, \hat{\phi}_n \in D(\mathcal{L}^*) \cap Y_+,\ \varphi_n := \lambda_n \hat{\phi}_n - \mathcal{L}^* \hat{\phi}_n \geq 0,\ \|\hat{\phi}_n\|_0 = 1,\ \|\varphi_n\|_0 \to 0. \quad (1.7.25)$$

Because $\lambda_n > \lambda_1$, the following representation formula

$$\hat{\phi}_n := (\lambda_n - \mathcal{L}^*)^{-1} \varphi_n = \int_0^\infty S^*(t) e^{-\lambda_n t} \varphi_n dt$$

holds true. Introducing the sequences

$$v_n := \mathcal{V}_n \varphi_n, \quad \mathcal{V}_n := \int_0^\infty V(t) e^{-\lambda_n t} dt,$$

and

$$\begin{aligned} w_n &:= \int_0^\infty (W * S^*)(t) e^{-\lambda_n t} \varphi_n dt \\ &= \int_0^\infty W(t) e^{-\lambda_n t} dt \int_0^\infty S^*(t) e^{-\lambda_n t} \varphi_n dt \\ &= \mathcal{W}_n \hat{\phi}_n, \quad \mathcal{W}_n := \int_0^\infty W(t) e^{-\lambda_n t} dt, \end{aligned}$$

and using (1.7.11), we deduce that

$$\hat{\phi}_n = v_n + w_n. \quad (1.7.26)$$

By construction and (1.7.12), we have $\|v_n\|_0 \to 0$, (w_n) is bounded in Y_1 and thus weakly compact in Y_0. That implies that $(\hat{\phi}_n)$ is weakly compact in Y_0. There thus exist a subsequence $(\hat{\phi}_{n_\ell})$ and $\phi_1 \in Y_0$ such that $\hat{\phi}_{n_\ell} \rightharpoonup \phi_1$ weakly in Y_0. In particular, $\phi_1 \geq 0$ and $\mathcal{L}^* \phi_1 = \lambda_1 \phi_1$. On the other hand, from (1.7.26), we have

$$1 = \|\hat{\phi}_n\|_0 \leq \|v_n\|_0 + \|w_n\|_0,$$

with $\|v_n\|_0 \to 0$ again and

$$\begin{aligned} \|w_n\|_0 &\leq \xi_\varepsilon \|\mathcal{W}_n \hat{\phi}_n\|_1 + \Xi_\varepsilon [\mathcal{W}_n \hat{\phi}_n]_{g_\varepsilon} \\ &\leq \xi_\varepsilon C \|\hat{\phi}_n\|_0 + \Xi_\varepsilon [\mathcal{W}_n \hat{\phi}_n]_{g_\varepsilon}, \end{aligned}$$

where we have used (1.7.9) in the first line and (1.7.12) in the second line. Choosing $n \geq 1$ large enough so that $\|v_n\|_0 \leq 1/4$ and $\varepsilon > 0$ small enough so that $\xi_\varepsilon C \leq 1/4$, we deduce from the two above estimates and the fact that $\mathcal{W}_n \geq 0$ that

$$\frac{1}{2} \leq \Xi_\varepsilon [\mathcal{W}_n \hat{\phi}_n]_{g_\varepsilon} = \Xi_\varepsilon \langle w_n, g_\varepsilon \rangle, \quad \forall\, n \geq 0.$$

Using that $g_\varepsilon \in Y_0'$, we may pass to the limit in the above inequality and we deduce

$$\frac{1}{2\Xi_\varepsilon} \leq \langle \phi_1, g_\varepsilon \rangle, \quad (1.7.27)$$

in particular $\phi_1 \not\equiv 0$ and that concludes the proof of the existence of a dual eigenelement. We have thus established the existence of a first dual eigenelement, that is $(\lambda_1, \phi_1) \in \mathbb{R} \times Y$ such that

$$\mathcal{L}^* \phi_1 = \lambda_1 \phi_1, \quad \phi_1 \geq 0, \quad \phi_1 \neq 0, \quad \lambda_1 \geq \lambda_0. \tag{1.7.28}$$

It is worth emphasizing that we only have $\|\phi_1\|_0 \leq 1$ from the lsc property of the norm $\|\cdot\|_0$.

Step 2. More about the dual eigenfunction. From (1.7.2) applied to ϕ_1, we have

$$e^{\lambda_1 t} \|\phi_1\|_1 = \|S^*(t)\phi_1\|_1 \leq C_0 e^{\lambda t} \|\phi_1\|_1 + C_1 \int_0^t e^{\lambda(t-s)+\lambda_1 s} ds \|\phi_1\|_0,$$

so that

$$(1 - C_0 e^{(\lambda-\lambda_1)t}) \|\phi_1\|_1 \leq C_1 \int_0^t e^{(\lambda-\lambda_1)\tau} d\tau \|\phi_1\|_0.$$

We recall that $\lambda_1 \geq \lambda_0$ from (1.7.28), thus

$$\frac{\lambda_0 - \lambda}{C_1} (1 - C_0 e^{(\lambda-\lambda_0)t}) \|\phi_1\|_1 \leq \|\phi_1\|_0,$$

and finally, passing to the limit $t \to \infty$, we deduce that

$$\|\phi_1\|_1 \leq \frac{C_1}{\lambda_0 - \lambda} \|\phi_1\|_0. \tag{1.7.29}$$

We normalize the dual eigenfunction with the norm $\|\cdot\|_0$. Note that since for the eigenvector ϕ_1 built in the step 1 we had $\|\phi_1\|_0 \leq 1$, the lower bound (1.7.27) remains valid for the new normalization $\|\phi_1\|_0 = 1$. Using (1.7.7), we thus deduce

$$\phi_1 \geq e^{\lambda_1(T_0 - T_1)} [\phi_1]_{g_\varepsilon} \psi_\varepsilon \geq \frac{e^{\lambda_1(T_0-T_1)}}{2\Xi_\varepsilon} \psi_\varepsilon. \tag{1.7.30}$$

Step 3. The Lyapunov condition. We define $\widetilde{S}(t) := e^{-\lambda_1 t} S(t)$. From (1.7.1), we have

$$\begin{aligned} \|\widetilde{S}(t) f\|_1 &\leq C_0 e^{(\lambda-\lambda_0)t} \|f\|_1 + C_1 \int_0^t e^{(\lambda-\lambda_0)(t-s)} \|\widetilde{S}(s) f\|_0 ds \\ &\leq \gamma_L' \|f\|_1 + K' \|f\|_0, \end{aligned}$$

for any $t \geq 0$, and with $\gamma_L' := C_0 e^{(\lambda-\lambda_0)t}$, $K' := C_0 e^{(\lambda-\lambda_0)t}(e^{C_1 t} - 1)$. From (1.7.8) and (1.7.30), we have

$$\|f\|_0 \leq \xi_\varepsilon \|f\|_1 + \Xi_\varepsilon [f]_{\psi_\varepsilon} \leq \xi_\varepsilon \|f\|_1 + 2\Xi_\varepsilon^2 e^{\lambda_1(T_1 - T_0)} [f]_{\phi_1}.$$

We then fix $T \geq T_1 > 0$ such that $\gamma_L' < 1$ and next $\varepsilon > 0$ such that $\gamma_L := \gamma_L' + \xi_\varepsilon K' < 1$ and we deduce that S satisfies the Lyapunov condition

$$\|\widetilde{S}_T f\|_1 \leq \gamma_L \|f\|_1 + K [f]_{\phi_1}, \tag{1.7.31}$$

with $K := 2\Xi_\varepsilon^2 e^{\lambda_1(T_1-T_0)} K'$.

Step 4. Take $f \geq 0$ such that $\|f\|_1 \leq A[f]_{\phi_1}$ with $A > K/(1-\gamma_L)$. We have

$$\begin{aligned} \|\widetilde{S}_{T_0} f\|_1 &\leq C_0' e^{(\lambda'-\lambda_0)T_0} \|f\|_1 \\ &\leq C_0' e^{(\lambda'-\lambda_0)T_0} A [f]_{\phi_1} \\ &= C_0' e^{(\lambda'-\lambda_0)T_0} A [\widetilde{S}_{T_0} f]_{\phi_1} \\ &\leq C_0' e^{(\lambda'-\lambda_0)T_0} A \|\phi_1\|_1 \|\widetilde{S}_{T_0} f\|_0 \\ &\leq C_0' e^{(\lambda'-\lambda_0)T_0} A \|\phi_1\|_1 (\xi_\varepsilon \|\widetilde{S}_{T_0} f\|_1 + \Xi_\varepsilon [\widetilde{S}_{T_0} f]_{\psi_\varepsilon}) \end{aligned}$$

for any $\varepsilon > 0$, where we have used successively the growth estimate (1.7.10) in the first line, the condition on f in the second line, the eigenfunction property of ϕ_1 in the third line, the duality bracket estimate (1.7.3) in the fourth line and the interpolation inequality (1.7.8) in the last line. Choosing $\varepsilon > 0$ small enough, we immediately obtain

$$\|\widetilde{S}_{T_0} f\|_1 \le 2C_0' e^{(\lambda'-\lambda_0)T_0} A \|\phi_1\|_1 \Xi_\varepsilon [\widetilde{S}_{T_0} f]_{\psi_\varepsilon}.$$

Together with

$$[f]_{\phi_1} = [\widetilde{S}_{T_0} f]_{\phi_1} \le \|\widetilde{S}_{T_0} f\|_1$$

and the relaxed Doblin-Harris positivity condition (1.7.6), we conclude to the conditional Doblin-Harris positivity estimate

$$\widetilde{S}_T f \ge c g_\varepsilon [f]_{\phi_1} \tag{1.7.32}$$

for all $T \ge T_1$, with $c^{-1} = 2C_0' e^{(\lambda'-\lambda_0)T_0} A \|\phi_1\|_1 \Xi_\varepsilon e^{\lambda'(T-T_0)} \eta_{\varepsilon,T}^{-1}$.

Step 5. We define $\mathcal{N} := \{f \in X_1; \langle f, \phi_1 \rangle = 0\}$. As a consequence of the last conditional Doblin-Harris positivity estimate, we show that there exists $\gamma_H \in (0,1)$ such that holds the following local coupling condition

$$(f \in \mathcal{N}, \ \|f\|_1 \le A[f]_{\phi_1}) \quad \text{implies} \quad [\widetilde{S}_T f]_{\phi_1} \le \gamma_H [f]_{\phi_1}, \tag{1.7.33}$$

still under the same condition $A > K/(1-\gamma_L)$. Take indeed $f \in \mathcal{N}$. Because $\langle f, \phi_1 \rangle = 0$, the Doblin-Harris condition (1.7.32) tells us that

$$\widetilde{S}_T f_\pm \ge c g_\varepsilon [f_\pm]_{\phi_1} = r g_\varepsilon, \quad r := c[f]_{\phi_1}/2, \tag{1.7.34}$$

and we may thus write

$$\begin{aligned} |\widetilde{S}_T f| &= |\widetilde{S}_T f_+ - r g_\varepsilon - \widetilde{S}_T f_- + r g_\varepsilon| \\ &\le |\widetilde{S}_T f_+ - r g_\varepsilon| + |\widetilde{S}_T f_- - r g_\varepsilon| \\ &= \widetilde{S}_T f_+ - r g_\varepsilon + \widetilde{S}_T f_- - r g_\varepsilon \ = \ \widetilde{S}_T |f| - 2 r g_\varepsilon, \end{aligned}$$

where we have used the inequality (1.7.34) in the third line. We deduce

$$[\widetilde{S}_T f]_{\phi_1} \le [\widetilde{S}_T |f|]_{\phi_1} - 2r[g_\varepsilon]_{\phi_1} = (1 - c[g_\varepsilon]_{\phi_1})[f]_{\phi_1},$$

where we have used $\widetilde{S}_T^* \phi_1 = \phi_1$, and that ends the proof of (1.7.33).

We now introduce a new norm $|||\cdot|||$ on X_1 defined by

$$|||f||| := [f]_{\phi_1} + \beta \|f\|_1, \tag{1.7.35}$$

and we claim that there exist $\beta > 0$ small enough and $\gamma \in (0,1)$ such that

$$|||\widetilde{S}_T f||| \le \gamma |||f|||, \quad \text{for any } f \in \mathcal{N}. \tag{1.7.36}$$

Note that $|||\cdot|||$ and $\|\cdot\|_1$ are equivalent norms, with

$$(1+\beta)^{-1} |||f||| \le \|f\|_1 \le \beta^{-1} |||f|||.$$

In order to establish the contraction estimate (1.7.36), we fix $f \in \mathcal{N}$ and estimate the norm $|||\widetilde{S}_T f|||$ in two alternative cases:

First case. *Contractivity for small X_1 norm.* When

$$\|f\|_1 < A[f]_{\phi_1}, \tag{1.7.37}$$

the local coupling condition (1.7.33) implies

$$[\widetilde{S}_T f]_{\phi_1} \le \gamma_H [f]_{\phi_1}.$$

Together with the Lyapunov condition (1.7.31), we have

$$|||\widetilde{S}_T f||| \le (\gamma_H + \beta K)[f]_{\phi_1} + \beta \gamma_L \|f\|_1 \le \gamma_1 |||f|||,$$

with

$$\gamma_1 := \max\{\gamma_H + \beta K, \gamma_L\}.$$

Choosing $\beta > 0$ small enough such that $\beta K < 1 - \gamma_H$, we get $\gamma_1 < 1$ and that gives the contractivity property (1.7.36) in this case.

Second case. *Contractivity for large X_1 norm.* Assume on the contrary that

$$\|f\|_1 \ge A[f]_{\phi_1}. \tag{1.7.38}$$

Directly from (1.7.31) we deduce that

$$\|\widetilde{S}_T f\|_1 \le \gamma_L \|f\|_1 + K[f]_{\phi_1} \le (\gamma_L + K/A)\|f\|_1,$$

with $\gamma_L + K/A < 1$ by assumption. On the other hand, we have

$$[\widetilde{S}_T f]_{\phi_1} \le \langle \widetilde{S}_T |f|, \phi_1 \rangle = \langle |f|, \phi_1 \rangle,$$

by using the positivity property of $\widetilde{S}_T$ and the eigenvector property of ϕ_1. Using both last estimates together, we deduce

$$\begin{aligned} |||\widetilde{S}_T f||| &= [\widetilde{S}_T f]_{\phi_1} + \beta \|\widetilde{S}_T f\|_1 \\ &\le [f]_{\phi_1} + \beta(\gamma_L + K/A)\|f\|_1 \\ &\le (1 - \beta\delta_0)[f]_{\phi_1} + \beta(\gamma_L + K/A + \delta_0)\|f\|_1, \end{aligned}$$

for any $\delta_0 \ge 0$. We thus get

$$|||\widetilde{S}_T f||| \le \gamma_2 |||\widetilde{S}_T f|||,$$

with $\gamma_2 := \max(1 - \beta\delta_0, \gamma_L + K/A + \delta_0)$. We get the contractivity property (1.7.36) in this case by choosing $\delta_0 > 0$ small enough (and keeping the choice of $\beta > 0$ made in the previous case) so that $\gamma_2 \in (0,1)$. The proof of (1.7.36) is completed by setting $\gamma := \max\{\gamma_1, \gamma_2\}$.

Step 6. In order to prove the existence and uniqueness of the eigenvector $f_1 \in X_1$, we fix $g_0 \in \mathcal{M} := \{g \in X_1, g \ge 0, \langle g, \phi_1 \rangle = 1\}$, and we define recursively $g_k := \widetilde{S}_T g_{k-1}$ for any $k \ge 1$. Thanks to (1.7.36), we get

$$\sum_{k=1}^{\infty} |||g_k - g_{k-1}||| \le \sum_{k=0}^{\infty} \gamma^k |||g_1 - g_0||| < \infty,$$

so that (g_k) is a Cauchy sequence in $\mathcal{M}$. We set $f_1 := \lim g_k \in \mathcal{M}$ which is a stationary state for the mapping $\widetilde{S}_T$, as seen by passing to the limit in the recursive equations defining (g_k). From (1.7.36) again, this is the unique stationary state for this mapping in $\mathcal{M}$. From the semigroup property, we have $\widetilde{S}_t f_1 = \widetilde{S}_t \widetilde{S}_T f_1 = \widetilde{S}_T(\widetilde{S}_t f_1)$ for any $t > 0$, so that $\widetilde{S}_t f_1$ is also a stationary state in $\mathcal{M}$, and thus $\widetilde{S}_t f_1 = f_1$ for any $t > 0$, by uniqueness. That precisely means that f_1 is a positive eigenvector associated to λ_1 for the original problem.

For $f \in X$, we see that $h := f - \langle f, \phi_1 \rangle \phi_1 \in \mathcal{N}$, and using recursively (1.7.36), we deduce

$$|||\widetilde{S}_{nT} h||| \le \gamma^n |||h|||, \quad \forall\, n \ge 0,$$

from what (1.7.13) follows by standard arguments. □

Chapter 2

NESS of a non-linear KFP equation in a domain

We study a weakly non-linear Fokker-Planck equation with BGK heat thermostats in a spatially bounded domain with conservative Maxwell boundary conditions, presenting a space-dependent accommodation coefficient and a space-dependent temperature on the spatial boundary. The model is based from a problem introduced in [43] where the authors studied the properties of the non-equilibrium steady states for non-linear kinetic Fokker-Planck equations with BGK thermostats in the torus. We generalize those results for bounded domains using the recent results presented in [45] for the study of general kinetic Fokker-Planck equations with Maxwell boundary conditions. More precisely, in a weakly non-linear regime, we obtain the existence of a non-equilibrium steady state and its stability in the perturbative regime.

This paper is based on the preprint [84], in collaboration with Josephine Evans.

2.1 Introduction

The study of non-equilibrium steady states (NESS) remains one of the central problems in statistical mechanics. There are only a few models where fundamental questions, such as whether they exist, whether they are unique, and whether they are stable, can be answered, even if partially (see Subsection 2.1.3 for a detailed review of known results). This chapter aims to contribute to the study of NESS within the context of kinetic theory.

This field of mathematics study equations coming from statistical physics modeling multi-particle systems by taking a statistical viewpoint. The unknown quantity is $f(t, x, v)$ which is the probability density function of a single "typical" particle which at any time t is located at position x and is moving with velocity v. In such a setting the system is described as being in *(local) thermodynamic equilibrium* when the particles velocities have a Maxwellian (Gaussian outside the world of kinetic theory) distribution of the form

$$\rho(x)(2\pi T(x))^{-d/2} \exp\left(-\frac{|v-u(x)|^2}{2T(x)}\right),$$

for some functions ρ, u, T depending on the spatial variable x. Most of the time kinetic equations model systems which are outisde thermodynamic equilibrium, however as these systems approach a global steady state, as $t \to \infty$, they move towards an equilibrium state. In this chapter however, we are interested in situations where the steady state does not have a Gaussian velocity distribution.

Physically, non-equilibrium steady states occur in open systems where there exist flows of macroscopic variables such as heat, see for instance [94, 162]. In particular, in the context of gas dynamics this can arise when gasses are in contact with thermostats, which is the physical situation we study in this chapter. Furthermore it is worth remarking that the study of non-equilibrium steady states in kinetic theory is fundamental as it can help us understand how non-equilibrium behaviors involving the flow of macroscopic quantities can emerge from particle systems.

More precisely, in this chapter we study a nonlinear kinetic Fokker-Planck (KFP) equation with several thermostat terms. The specific nonlinearity in the KFP equation we take comes from [43, Equation (1.27)] where they study NESSs for such an equation on the torus. In contrast, our work is on a bounded domain and contains a different and more challenging set of "thermostat" terms which describe how heat flows in and out of the system.

We are able to attack this more complicated situation using tools from [45], where they develop powerful techniques to study the existence, uniqueness, and stability of steady states of a linear KFP equation in a bounded domain, presenting Maxwell boundary conditions with a space-dependent temperature. As we rely strongly on these linear tools we work in a regime where the nonlinear term is small.

Finally, it is important to note that, even in a weakly non-linear regime, boundary thermostats make the study of the NESS more challenging. For instance, and in contrast with [43], we lose the access to an explicit formula for the NESS, we no longer have reasons to believe that the steady states are spatially uniform, and we cannot rule out the existence of steady states with infinite energy.

2.1.1 Framework

We consider $\alpha \in (0, 1/2)$, dimension $d \geq 3$, and we study the following non-linear equation

$$\partial_t f = -v\cdot\nabla_x f + (\alpha\mathcal{E}_f + (1-\alpha)\tau)\,\Delta_v f + \mathrm{div}_v(vf) + \mathscr{G} f \quad \text{in } \mathcal{U} := (0,\infty)\times\Omega\times\mathbb{R}^d, \tag{2.1.1}$$

describing the evolution of the density function $f := f(t, x, v)$, depending on the variables associated to time $t \in (0, \infty)$, space $x \in \Omega \subset \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. We have considered the function $\tau = \tau(x) : \Omega \to \mathbb{R}$ such that

$$\tau_0 \leq \tau(x) \leq \tau_1 \qquad \forall x \in \Omega,$$

for some constants $\tau_0, \tau_1 > 0$, and we have defined the *total energy* functional

$$\mathcal{E} = \mathcal{E}_f := \frac{1}{d}\int_{\Omega\times\mathbb{R}^d} |v|^2 f \mathrm{d}x\mathrm{d}v,$$

and the BGK *heat thermostat*

$$\mathscr{G} f = \sum_{n=1}^{\mathcal{N}} \eta_n \mathscr{G}_n f \quad \text{ with } \quad \mathscr{G}_n f = \mathbf{1}_{\Omega_n}\left(\varrho_f \mathcal{M}_{T_n} - f\right),$$

for some $\mathcal{N} \in \mathbb{N}$, some parameters $\eta_n \geq 0$, $T_n > 0$, the subsets $\Omega_n \subset \Omega$, and where we have defined

$$\varrho_f = \int_{\mathbb{R}^d} f dv, \quad \text{ and } \quad \mathcal{M}_{\mathcal{T}} = \mathcal{M}_{\mathcal{T}}(v) := \frac{1}{(2\pi\mathcal{T})^{d/2}} \exp\left(-\frac{|v|^2}{2\mathcal{T}}\right) \quad \text{ for } \mathcal{T} : \Omega \to \mathbb{R}_+^*.$$

We present to the reader a discussion on the physical interpretation of the different operators involved in Equation (2.1.1) in Subsection 2.1.3 (see also [43] for further details on the modeling).

We take Ω to be a bounded domain and, without loss of generality, we impose $|\Omega| = 1$. Moreover we assume Ω to be a C^1 domain, more precisely we assume that

$$\Omega := \{x \in \mathbb{R}^d;\ \delta(x) > 0\}$$

for a function $\delta \in W^{2,\infty}(\mathbb{R}^d)$ such that $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ on a neighborhood of the boundary set $\partial\Omega$, thus $n_x = n(x) := -\nabla\delta(x)$ coincides with the unit normal outward vector field on $\partial\Omega$. We define the boundary set $\Sigma = \partial\Omega \times \mathbb{R}^d$ and we differentiate between the sets of *outgoing* velocities (Σ_+) and incoming velocities (Σ_-) on the boundary by

$$\Sigma_\pm = \{(x, v) \in \Sigma,\ \pm n_x \cdot v > 0\}.$$

Furthermore we set $\Gamma = (0, \infty) \times \Sigma$ and accordingly $\Gamma_\pm = (0, \infty) \times \Sigma_\pm$. We define γf as the trace function associated with f over Γ and $\gamma_\pm f = \mathbf{1}_{\Gamma_\pm}\gamma f$. We then consider the *accommodation coefficient* $\iota \in C(\partial\Omega, [0, 1])$ and we complement Equation (2.1.1) with the *Maxwell boundary condition*

$$\gamma_- f = \mathscr{R}\gamma_+ f := (1 - \iota)\mathscr{S}\gamma_+ f + \iota\mathscr{D}\gamma_+ f, \qquad \text{on } \Gamma_-, \tag{2.1.2}$$

where we have defined the *specular reflection* operator

$$\mathscr{S} g(t, x, v) := g(t, x, \mathcal{V}_x v), \qquad \text{where} \quad \mathcal{V}_x v := v - 2n_x(n_x \cdot v),$$

and the *diffussive reflection* operator

$$\mathscr{D} g(t, x, v) = \mathscr{M}_\Theta(v) \int_{\mathbb{R}^d} g(t, x, u)(n_x \cdot u)_+ du, \qquad \text{where} \quad \mathscr{M}_\Theta := \sqrt{2\pi/\Theta}\mathcal{M}_\Theta,$$

and we have introduced the boundary temperature $\Theta \in W^{1,\infty}(\overline{\Omega}, \mathbb{R})$ such that

$$\Theta_* \le \Theta(x) \le \Theta^*, \quad \text{for some constants } 0 < \Theta_* \le \Theta^* < \infty. \tag{2.1.3}$$

Finally we complement the nonlinear Equation (2.1.1)-(2.1.2) with the initial condition

$$f(t = 0, x, v) = f_0(x, v) \quad \text{on } \mathcal{O}, \tag{2.1.4}$$

and we assume, without loss of generality, that $\langle\!\langle f_0 \rangle\!\rangle_{\mathcal{O}} := \int_{\mathcal{O}} f_0 \,\mathrm{d}v\,\mathrm{d}x = 1$.

2.1.2 Main results

In order to state our main results we introduce the class of the so-called *admissible weight functions* $\omega : \mathbb{R}^d \to (0, \infty)$ defined by

$$\omega = \langle v\rangle^k \exp(\zeta\langle v\rangle^s),$$

and such that either

$$s = 0,\ \zeta \ge 0,\ \text{and } k > k_* \text{ with } k_* = d{+}1, \quad \text{or} \quad s \in (0, 1] \text{ with } \zeta \in (0, \infty),\ \text{and any } k \ge 0.$$

We refer the reader towards [45] for an explanation on how this choice of admissible weight functions provides, in particular, a control on the behavior of local Kolmogorov equations. Moreover, we emphasize that due to the fact that $s \in [0, 1]$ we will also have the control

$$\nabla_v\, \omega/\omega \in L^\infty(\mathbb{R}^d), \tag{2.1.5}$$

which will be necessary in order to appropriately control the non-linearity of the equation.

We also need to introduce some functional spaces. For a given measure space $(Z, \mathscr{Z}, \mu)$, a weight function $\sigma : Z \to (0, \infty)$ and an exponent $p \in [1, \infty]$, we define the weighted Lebesgue spaces $L^p_\sigma(Z)$ associated to the norm

$$\|g\|_{L^p_\sigma(Z)} = \|\sigma g\|_{L^p(Z)}. \tag{2.1.6}$$

Furthermore, we define the space of continuous functions in Y as $C(Z)$. We then have the following results.

We first present a well-posedness and stability theorem in the linear framework when $\alpha = 0$.

Theorem 2.1.1. *We assume $\alpha = 0$. There exists $\mathfrak{F}^0 \in L^2(\Omega, H^1(\mathbb{R}^d)) \cap L^\infty(\mathcal{O})$ unique steady solution to the linear Equation* (2.1.1)-(2.1.2)-(2.1.4). *Moreover, $\langle\!\langle \mathfrak{F}^0 \rangle\!\rangle_{\mathcal{O}} = 1$ and for any admissible weight function ς there holds*

$$\|\nabla_v \mathfrak{F}^0\|_{L^2_\varsigma(\mathcal{O})} < \infty \qquad \text{and} \qquad \mathfrak{F}^0(x,v) \lesssim (\varsigma(v))^{-1}. \tag{2.1.7}$$

Furthermore, let ω be an admissible weight function, for any initial data $f_0 \in L^2_\omega(\mathcal{O})$ there is a unique global renormalized solution $f \in C(\mathbb{R}_+, L^2_\omega(\mathcal{O}))$ to the linear Equation (2.1.1)-(2.1.2)-(2.1.4) *and there is $\lambda > 0$ such that*

$$\|f_t - \mathfrak{F}^0\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-\lambda t}\|f_0 - \mathfrak{F}^0\|_{L^2_\omega(\mathcal{O})} \qquad \forall t \geq 0. \tag{2.1.8}$$

The precise sense of the global solutions provided in Theorem 2.1.1 is given by Theorem 2.3.3 with the choice of $\Lambda = \tau$. We also remark that Theorem 2.3.3 is but a direct application of [49, Theorem 2.11] and the trace theory from [49, Theorem 2.8].

The existence and uniqueness of a stationary solution for the linear problem, which we remark is also in the sense of Theorem 2.3.3, as well as its stability are obtained as a direct application of the Krein-Rutmann-Doblin-Harris theory developed in [160, Theorem 6.1] but we also refer towards [45, Theorem 7.1] for a similar result in a non-conservative setting and to [90] for the study of a general Krein-Rutmann-Doblin-Harris result in a general theoretical framework.

We futher note that Theorem 2.1.1 is a slight generalization of [45, Theorems 1.1 and 1.2].

In the non-linear framework we have first the existence of a steady state for $\alpha > 0$ sufficiently small.

Theorem 2.1.2. *There exists $\alpha^\star \in (0, 1/2)$ such that for every $\alpha \in (0, \alpha^\star)$, there is a positive function $\mathfrak{F}^\alpha \in L^2(\Omega, H^1(\mathbb{R}^d)) \cap L^\infty(\mathcal{O})$, steady solution of Equation* (2.1.1)-(2.1.2)-(2.1.4). *Moreover, $\langle\!\langle \mathfrak{F}^\alpha \rangle\!\rangle_{\mathcal{O}} = 1$ and for every admissible weight function ω there holds*

$$\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} < \infty, \quad \mathfrak{F}^\alpha(x,v) \lesssim (\omega(v))^{-1}, \quad \text{and} \quad \mathcal{E}_{\mathfrak{F}^\alpha} \leq 2\mathcal{E}_{\mathfrak{F}^0}, \tag{2.1.9}$$

uniformly in α, and where $\mathfrak{F}^0$ is given by Theorem 2.1.1.

The main consequence of Theorem 2.1.2 is the existence of a NESS for the non-linear Equation (2.1.1)-(2.1.2), as well as some of its qualitative properties regarding regularity and decay tail in velocity. We remark that the proof of Theorem 2.1.2 is based in the application of a fixed point argument in the spirit of the proof of [85, Theorem 1]. Additionally $\mathfrak{F}^\alpha$ is a stationary state in the sense of Theorem 2.3.3 by taking $\Lambda = \alpha \mathcal{E}_{\mathfrak{F}^\alpha} + (1-\alpha)\tau$.

We remark that Theorem 2.1.2 generalizes [43, Theorem 1.2] and that, in contrast with this work, we observe major differences on the behavior and properties of the NESS in the absence of periodic boundary conditions: we have no reasons to believe that the NESS will be independent of the spatial variable x, the bounds on the energy functional $\mathcal{E}$ cannot be obtained as during [43, Lemma 1.1] (see Subsection 2.1.4), we lack the information to rule out the existence of steady states with unbounded total energy, and we don't have access to an explicit representation of the NESS.

Finally, we state the following stability result for the previous NESS.

Theorem 2.1.3. *We consider an admissible weight function ω. There are $\alpha^{\star\star} \in (0, \alpha^\star)$ and $\delta > 0$, where $\alpha^\star$ is given by Theorem 2.1.2, such that for every $\alpha \in (0, \alpha^{\star\star})$ and for any initial datum $f_0 \in L^2_\omega(\mathcal{O})$ satisfying*

$$\|f_0 - \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \leq \delta,$$

there is $f \in L^2_\omega(\mathcal{U})$ global weak solution of Equation (2.1.1)-(2.1.2)-(2.1.4). *Moreover, there is $\eta > 0$ for which there holds*

$$\|f_t - \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-\eta t}\|f_0 - \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \qquad \forall t \geq 0. \tag{2.1.10}$$

The global solutions provided by Theorem 2.1.3 are constructed in the sense that the function $h := f - \langle\!\langle f_0 \rangle\!\rangle_{\mathcal{O}} \mathfrak{F}^\alpha$ satisfies Equation (2.8.1), in the sense of Proposition 2.4.

It is worth remarking that Proposition 2.4 is mainly an application of the Lion's variant of the Lax-Milgram theorem [126, Chap III, §1] as used in [49], see also [45, 48, 49, 90] for similar arguments on the existence of solutions of kinetic equations. The trace theory was taken mainly from [45, 49] but we also refer to [90, 137, 138] for further references on the trace theory for kinetic equations.

Furthermore, we emphasize that to obtain the a priori estimates leading to the proof of Proposition 2.4 we have used the modified weight functions from [45, 48, 90] to control the Maxwell boundary condition.

In addition, the decay estimate was obtained by defining a new norm in the spirit of [46, Proposition 3.6], [50, Proposition 3.2] and [139, Proposition 4.1]. It is worth remarking that we are not able to construct of a *hypocoercivity* theory in the spirit of [21, 48, 79, 166] due to the lack of extra information on the steady state, namely positivity bounds and regularity.

We note that Theorem 2.1.3 generalizes [43, Theorem 1.3] and we remark that the techniques used for the obtention of these results are different from those developed during the proof of the main theorems from [43]. In particular, we do not need to study the underlying ergodic process associated with the linearized operator to obtain our results.

2.1.3 Physical motivation and state of the art

The study of non-equilibrium systems within statistical physics has been motivated by the investigation of many physical and biological models (see [23, 24, 140]). We refer the reader towards [94, 117] and [93, Chapter 9] for an exposition on non-equilibrium statistical mechanics and towards [162] for a characterization of the entropy for NESS, including the case of the KFP equation.

We expose now why Equation (2.1.1)-(2.1.2)-(2.1.4) represents a system which is not in *thermal equilibrium* and we explain the physical process that each term models.

We define the Fokker-Planck type operators

$$\mathcal{C}_{\mathcal{T}} f := \mathcal{T} \operatorname{div}_v \left(\mathcal{M}_{\mathcal{T}} \nabla_v \left(\frac{f}{\mathcal{M}_{\mathcal{T}}} \right) \right) = \mathcal{T} \Delta_v f + \operatorname{div}_v (vf), \tag{2.1.11}$$

associated to the temperature $\mathcal{T} : \Omega \to \mathbb{R}_+^*$. We observe that the operator described in (2.1.11) can be interpreted as having a *thermostat* effect, as it represents the gasses interaction with another gas which is present throughout the domain and has density $\mathcal{M}_{\mathcal{T}}$. We refer towards [57, 70, 154, 165] and the references therein for more on the modeling and properties of Fokker-Planck type operators.

On the other hand, for each $n \in [\![1, \mathcal{N}]\!]$, the BGK thermostat $\mathscr{G}_n$ model the gasses interaction with other gasses with density $\mathcal{M}_{T_n}$, which are only present in some parts of the domain Ω_n. We refer towards [43, 85, 165] and the references therein for more on the modeling and properties of BGK operators.

We can then rewrite the right hand side of Equation (2.1.1) as

$$-v \cdot \nabla_x f + \alpha \mathcal{C}_{\mathcal{E}_f} f + (1 - \alpha) \mathcal{C}_\tau f + \sum_{n=1}^{\mathcal{N}} \eta_n \mathscr{G}_n f,$$

which highlights the different forces acting simultaneously upon the particles. On the one hand, we have the effect of the transport operator, followed by a non-linear interaction where the gasses interact with themselves via a Fokker-Planck operator whose temperature is the gasses total energy, and acting with intensity α. On the other hand, we observe the presence of several energy exchange mechanisms:

- a linear Fokker-Planck thermostat $\mathcal{C}_\tau$ whose associated temperature is the function τ defined in Subsection 2.1.1, and acting with intensity $1 - \alpha$,
- $\mathcal{N}$ BGK thermostats as described above, each with an associated intensity η_n, with $n \in [\![1, \mathcal{N}]\!]$,
- and the boundary thermostat at temperature Θ, given by the diffusive boundary condition.

From the mathematical point of view, the study of NESS poses several challenges: they are generally not explicit, and it is often difficult to prove that they satisfy Poincaré or logarithmic Sobolev inequalities which in turn makes it challenging to apply the standard machinery of *hypocoercivity* (see [21, 79, 166]) to investigate their stability. Nonetheless, we highlight some results regarding non-equilibrium systems for various kinetic models.

- For a collisionless transport equation in a bounded domain with a non-isothermal boundary condition, we cite [19], where A. Bernou shows the existence and stability of a steady state.
- The study of NESS for BGK equations has received relatively more attention over the last years. We can cite for instance [41, 43] where E. Carlen, R. Esposito, J. Lebowitz, R. Marra and C. Mouhot study a non-linear BGK equation in the torus with the presence of BGK thermostats at different temperatures. In each of this articles the authors construct a NESS and they prove their exponential stability. Moreover, they also prove uniqueness of such steady state in [41] whereas in [43] they obtain local uniqueness.

We also mention the works of J. Evans and A. Menegaki on BGK equations. In [87] the authors study a non-linear BGK equation in a 1-dimensional torus and obtain existence, local uniqueness and stability of a NESS. It is also worth remaking that the decay estimates were obtained by the used of a hypocoercivity technique in the spirit of [79]. Furthermore, in [85] these authors study a non-linear BGK equation in a real interval presenting diffusive boundary conditions with different temperatures at each side. In this model they prove existence of a NESS under weak conditions on the boundary temperatures.

Finally, within the framework of BGK equation we have [20], where A. Bernou study a non-isothermal problem, for which he constructs a NESS, and provides its uniqueness and stability under suitable conditions in dimensions 2 and 3.

• We take a glance now at KFP equation. Regarding linear problems we cite two main papers: on the one hand, we have [40] where C. Cao studied a problem in the whole $\mathbb{R}^d$ as spatial domain, and proved the existence and uniqueness of a steady solution (non necessarily Maxwellian) as well as its stability. On the other hand, for linear KFP equations with bounded domains we cite [45] where K. Carrapatoso, P. Gabriel, R. Medina and S. Mischler obtained the existence, uniqueness and stability of a NESS. It is worth remarking that the equation studied in [45] was non-conservative and it exhibited Maxwell boundary conditions with a space-dependent temperature.

Furthermore, we cite again [43] where E. Carlen, R. Esposito, J. Lebowitz, R. Marra and C. Mouhot also studied a non-linear KFP equations with BGK thermostats in the torus and they obtain the existence, local uniqueness, stability, and an explicit formula of a NESS.

• For the Boltzmann equation with hard spheres near the hydrodynamic limit, we have the papers [8, 10, 82, 83] exploring the effects of a heat thermostat on the boundary.

Whiting the previously mentioned papers, we highlight [82] where R. Esposito, Y. Guo, C. Kim and R. Marra construct a NESS and provide its uniqueness and stability in a perturbative regime. Such perturbative regime consisted, besides the fact of being close to the hydrodynamical limit which is a perturbative regime in itself, on the imposition of boundary temperatures that don't fluctuate too much, some smallness condition on the initial data and a small *Knudsen* number.

Similarly, in [8] L. Arkeryd, R. Esposito, R. Marra and A. Nouri construct a NESS which is locally unique, and stable, under the conditions of a small *Knudsen* number and some smallness on the initial condition.

In the study of NESS for Boltzmann equations not necessarily near its hydrodynamic limit we have [10, 42, 44]. More precisely, in [10] the authors construct L^1 solutions to the stationary Boltzmann equation in a 1d slab. In [44], E. Carlen, J. Lebowitz and C. Mouhot investigate a space homogeneous Boltzmann equation with pseudo-Maxwellian molecules and the non-equilibrium effects come from a Boltzmann-type thermostat. In this setting, the authors obtain existence, uniqueness and stability of a NESS. Furthermore, in [42], E. Carlen, R. Esposito, J. Lebowitz, R. Marra and A. Rokhlenko studied a homogeneous Boltzmann equation with KFP and BGK thermostats.

2.1.4 First mathematical observations and strategy of the proof of the main results

We first remark that Equation (2.1.1)-(2.1.2) is mass conservative. Indeed, we observe that at least at a formal level, there holds

$$\frac{d}{dt}\int_{\mathcal{O}} f_t = \int_{\mathcal{O}} -v\cdot\nabla_x f + (\alpha\mathcal{E}_f + (1-\alpha)\tau)\,\Delta_v f + \mathrm{div}_v(vf) + \mathscr{G}f = 0, \tag{2.1.12}$$

where we have used the fact that the Maxwell boundary condition satisfies

$$\mathscr{R} : L^1(\Sigma_+, d\xi_1^1) \to L^1(\Sigma_-, d\xi_1^1), \quad \|\mathscr{R}\|_{L^1(\Sigma_+, d\xi_1^1)} \le 1.$$

This together with our assumptions on f_0 imply in particular that $\langle\langle f_t \rangle\rangle_{\mathcal{O}} = 1$ for every $t \geq 0$.

Furthermore, due to the structure of Equation (2.1.1) it would be of interest to establish a priori bounds on the behavior of the energy operator $\mathcal{E}$ in time, however, and in difference with [43], this is challenging due to the presence of a diffusion condition at the boundary. Indeed at a formal level, using the conservation of mass, we observe that

$$\frac{d}{dt}\mathcal{E}_{f_t} \leq 2d(1-\alpha)\left(1 - \frac{\mathcal{E}_{f_t}}{\tau_1}\right) + \sum_{n=1}^{\mathcal{N}} \eta_n T_n + \int_{\Sigma_+} \iota\, \gamma_+ f \left(-|v|^2 + \frac{d+1}{2}\sqrt{\frac{2}{\pi}}\Theta^{3/2}\right)(n_x \cdot v)_+,$$

and using the Grönwall lemma we further deduce that

$$\begin{aligned} \mathcal{E}_{f_t} \leq 2d(1-\alpha)\, e^{-\frac{1}{\tau_1}t}\mathcal{E}_{f_0} + \tau_1 \left(2d(1-\alpha) + \sum_{n=1}^{\mathcal{N}} \eta_n T_n\right)\left(1 - e^{-\frac{1}{\tau_1}t}\right) \\ + \int_0^t e^{-\frac{1}{\tau_1}(t-s)} \int_{\Sigma_+} \iota(x)\, \gamma_+ f_s \left(-|v|^2 + \frac{d+1}{2}\sqrt{\frac{2}{\pi}}\Theta^{3/2}\right)(n_x \cdot v)_+. \end{aligned} \tag{2.1.13}$$

The previous formula has several consequences, on the one hand if $\iota \equiv 0$, i.e there is only specular reflection at the boundary, we can immediately deduce the existence of a ball enclosing the functional $\mathcal{E}$ for all $t \geq 0$.

On the other hand however, when $\iota \not\equiv 0$ and there is a heat source anywhere at the boundary, we cannot guarantee that the energy functional won't explode with time. Nonetheless, it is worth remarking that the term

$$-|v|^2 + \frac{d+1}{2}\sqrt{\frac{2}{\pi}}\Theta^{3/2},$$

in (2.1.13) helps in ensuring that the velocity of the particles doesn't grow *too much* at the boundary, and this makes us believe that there should be a mechanism in which this fact helps in bounding the total energy for all time.

We are unable however, to exploit the previous idea and our strategy then consists on studying first the linear equation

$$\left\{ \begin{array}{rcll} \partial_t f &=& \mathcal{L} f & \text{in } \mathcal{U}, \\ \gamma_- f &=& \mathscr{R}\gamma_+ f & \text{on } \Gamma_-, \\ f_{t=0} &=& f_0 & \text{in } \mathcal{O}. \end{array} \right. \tag{2.1.14}$$

where $\mathcal{L}f := -v \cdot \nabla_x f + \mathcal{C}_\Lambda f + \mathscr{G} f$, for a function $\Lambda : \Omega \to \mathbb{R}$ such that

$$\Lambda_0 \leq \Lambda(x) \leq \Lambda_1 \quad \text{for every } x \in \Omega,$$

for some constants $\Lambda_0, \Lambda_1 > 0$, and we recall that $\mathcal{C}_\Lambda$ is defined in (2.1.11). We then proceed as follows:

① *Well-posedness of Equation* (2.1.14). By using the well-posedness and trace theory from [49] we deduce that there is a strongly continuous semigroup $S_{\mathcal{L}}$ associated with the solutions of Equation (2.1.14). We will also study extensively the properties of the semigroup $S_{\mathcal{L}}$ which will be obtained by using the results from [45] on the local part of the operator $\mathcal{L}$ and extending them to the full operator by using the Duhamel formula.

② *Krein-Rutmann-Doeblin-Harris theory.* By using [160, Theorem 6.1] we obtain the existence, uniqueness and stability of a steady solution for Equation (2.1.14). In particular, we will deduce that such a steady state has finite total energy.

③ *Proof of Theorem 2.1.1.* We observe that by taking $\Lambda = \tau$, Equation (2.1.14) coincides with Equation (2.1.1)-(2.1.2)-(2.1.4) with $\alpha = 0$. Therefore the previous results immediately imply Theorem 2.1.1.

④ *Existence of the NESS for the non-linear problem.* Using the existence of a steady state of Equation (2.1.14) with bounded total energy we deduce, by the use of a fixed point argument in the spirit of the proof of [85, Theorem 1], that for small values of $\alpha > 0$ we can construct a NESS for Equation (2.1.1)-(2.1.2) satisfying the energy bound (2.1.9).

After having proved Theorem 2.1.2 we perturb Equation (2.1.1)-(2.1.2)-(2.1.4) around the NESS: we take $h = f - \mathfrak{F}^\alpha$ and we study the resulting linearized perturbed problem

$$\begin{cases} \partial_t h &=& -v \cdot \nabla_x h + \mathcal{C}_{\Lambda^\star} h + \mathscr{G} h + \alpha \mathcal{E}_g \Delta_v h + \alpha \mathcal{E}_h \Delta_v \mathfrak{F}^\alpha & \text{in } \mathcal{U} \\ \gamma_- h &=& \mathscr{R} \gamma_+ h & \text{on } \Gamma_- \\ h_{t=0} &=& f_0 - \mathfrak{F}^\alpha & \text{in } \mathcal{O}, \end{cases} \tag{2.1.15}$$

where we have taken $g : \mathcal{U} \to \mathbb{R}$ and we have defined $\Lambda^\star := \alpha \mathcal{E}_{\mathfrak{F}^\alpha} + (1-\alpha)\tau$. Then to prove Theorem 2.1.3 we proceed as follows:

⑤ *Well-posedness of Equation* (2.1.15)*.* We use again the theory developed in [49] to obtain the well-posedness of Equation (2.1.15) under a smallness condition on g. It is worth remarking that the arguments leading to this result are more delicate than the proof of the well-posedness for Equation (2.1.14), due to the presence of the bilinear term $\mathcal{E}_g \Delta_v h$ and of the H^{-1} term $\mathcal{E}_h \Delta_v \mathfrak{F}^\alpha$.

⑥ *Hypodissipativity.* We remark now that we have access to a decay estimate in a weighted L^2 space for the solutions of the following equation

$$\partial_t h = -v \cdot \nabla_x h + \mathcal{C}_{\Lambda^\star} h + \mathscr{G} h, \qquad \gamma_- h = \mathscr{R} \gamma_+ h, \qquad h_{t=0} = f_0 - \mathfrak{F}^\alpha. \tag{2.1.16}$$

This is nothing but a consequence of the fact that it coincides with Equation (2.1.14) by taking $\Lambda = \Lambda^\star$. We then prove that, for α and g *small* we can construct a new norm $|\!|\!|\cdot|\!|\!|$, equivalent to the aforementioned weighted L^2 norm, in which we can extend the dissipativity properties of Equation (2.1.16) to the solutions of Equation (2.1.15) by treating the term $\alpha \mathcal{E}_g \Delta_v h + \alpha \mathcal{E}_h \Delta_v \mathfrak{F}^\alpha$ as a *small* perturbation in some sense.

⑦ *Fixed point argument on the perturbed setting.* Finally, using the previous informations we will prove that the map that to g associates h a solution of Equation (2.1.15), leaves invariant a ball in a weighted L^2 space, and that it is continuous for the weak L^2 topology. This, together with the hypodissipativity result above, lead to the proof of Theorem 2.1.3 by using the Schauder fixed point theorem.

2.1.5 Structure of this chapter

The chapter is organized as follows.

In Section 2.2 we introduce some elementary lemmas to control the non-local terms from the BGK operators, and we present some powerful results developed in [45] for the control of local KFP equations. During Section 2.3 we develop a priori estimates and the well-posedness of Equation (2.1.14). We derive the existence of the semigroup $S_{\mathscr{L}}$ and we provide its ultracontractive properties. We also study the existence and properties of the backwards equation dual to Equation (2.1.14). In Section 2.4 we present the Krein-Rutman-Doblin-Harris theorem from [160, Theorem 6.1] and we use it to prove Theorem 2.1.1. We devote Section 2.5 to prove Theorem 2.1.2 by using the results from the previous sections and arguing by using a fixed point argument in the spirit of the proof of [85, Theorem

1]. In Section 2.6 we study the perturbed Equation (2.6.2) and we obtain some a priori estimates as well as its well-posedness under suitable assumptions. Additionally, during Section 2.7 we prove a hypodissipativity result for the solutions of Equation (2.6.2). Finally, in Section 2.8 we obtain an equivalent version of Theorem 2.1.3 by proving the existence of solutions of the non-linear perturbed problem. The proof is based on the application of the Schauder fixed point theorem and by using the results obtained from Sections 2.6 and 2.7.

2.1.6 Notation

We state now some of the notations we will be using during this chapter.

• Given two admissible weight functions ω, ς we say that $\omega \prec \varsigma$ when $\omega\varsigma^{-1} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

• Consider a measure space $(Z, \mathscr{Z}, \mu)$ and a weight function $\sigma : Z \to (0, \infty)$, we observe that $L^2_\sigma(Z)$ is a Hilbert space with the scalar product

$$\langle \phi, \psi \rangle_{L^2_\sigma(Z)} := \int_Z \phi\, \psi\, \sigma^2.$$

Furthermore, we also define the weighted Sobolev space $H^1_\sigma(Z)$ as the functions $\psi \in H^1(Z)$ such that the norm

$$\|\psi\|_{H^1_\sigma(Z)} := \|\psi\|_{L^2_\sigma(Z)} + \|\nabla\psi\|_{L^2_\sigma(Z)} < \infty.$$

• For two Banach spaces Z_1, Z_2 we define $\mathscr{B}(Z_1, Z_2)$ as the space of the linear bounded operators from Z_1 to Z_2. In particular, we will denote it only as $\mathscr{B}(Z_1)$ when $Z_1 = Z_2$.

2.2 Toolbox

We define the local ultraparabolic operator

$$\mathcal{B}f := -v \cdot \nabla_x f + \Lambda(x)\Delta_v f + v \cdot \nabla_v f + \left(d - \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n} \right) f. \tag{2.2.1}$$

and the non-local operator

$$\mathcal{A}f := \varrho_f \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n} \mathcal{M}_{T_n}. \tag{2.2.2}$$

and we observe that $\mathcal{L} = \mathcal{B} + \mathcal{A}$, where we recall that $\mathcal{L}$ is given in Subsection 2.1.4. This section is devoted to provide the necessary tools in order to study Equation (2.1.14). More precisely, in Subsection 2.2.1 we prove the boundedness properties of the operator $\mathcal{A}$ in every suitable Lebesgue space followed by Subsection 2.2.2 where we summarize the results from [45] on KFP equations with Maxwell boundary conditions.

2.2.1 Properties of the non-local operator $\mathcal{A}$

We have the following proposition on the properties of the non-local term of the BGK thermostat.

Proposition 2.1. *For any two admissible weight functions ω and $\omega_\star$, and for any $p \in [1, \infty]$, there is a constant $C > 0$ such that*

$$\|\mathcal{A}f\|_{L^p_\omega(\mathcal{O})} \le C \sum_{n=1}^{\mathcal{N}} \|f\|_{L^p_{\omega_\star}(\mathcal{O}_n)} \lesssim \|f\|_{L^p_{\omega_\star}(\mathcal{O})}. \tag{2.2.3}$$

where we have defined $\mathcal{O}_n = \Omega_n \times \mathbb{R}^d$.

Proof. We observe that due to the very definition of admissible weight functions there holds $\omega \prec \mathcal{M}_{T_n}^{-1}$, for every $n \in [\![1, \mathcal{N}]\!]$. We remark that for the case $p = \infty$ the result is obvious, furthermore if $p \in [1, \infty)$ we have that

$$\sum_{n=1}^{\mathcal{N}} \int_{\mathcal{O}_n} (\varrho_f \, \eta_n \mathcal{M}_{T_n})^p \, \omega^p \lesssim \sum_{n=1}^{\mathcal{N}} \int_{\Omega_n} \left(\int_{\mathbb{R}^d} f \right)^p \lesssim \sum_{n=1}^{\mathcal{N}} \int_{\mathcal{O}_n} f^p,$$

where we have used the Hölder inequality to obtain the second inequality. □

2.2.2 Well-posedness and properties of the local ultraparabolic part

We look now at the local KFP equation

$$\begin{cases} \partial_t f &= \mathcal{B} f & \text{in } \mathcal{U}, \\ \gamma_- f &= \mathscr{R} \gamma_+ f & \text{on } \Gamma_-, \\ f_{t=0} &= f_0 & \text{in } \mathcal{O}. \end{cases} \tag{2.2.4}$$

where we recall that $\mathcal{B}$ is given by (2.2.1). Furthermore, we also present during this subsection the first eigenproblem associated to the previous equation, i.e the existence and properties of the eigentriplet (λ_1, f_1, ϕ_1) satisfying

$$\lambda_1 \in \mathbb{R}, \quad \mathcal{B} f_1 = \lambda_1 f_1, \quad \gamma_- f_1 = \mathscr{R} \gamma_+ f_1, \quad \mathcal{B}^* \phi_1 = \lambda_1 \phi_1, \quad \gamma_+ \phi_1 = \mathscr{R}^* \gamma_- \phi_1.$$

We observe then that, by taking $b(x,v) = v$, $c(x,v) = d - \sum_{n=1}^{N} \eta_n \mathbf{1}_{\Omega_n}(x)$ and since we have that $\Lambda(\Omega) \subset [\Lambda_0, \Lambda_1]$, this equation fits the framework developed in [45] (see also [49]) with $\gamma = 2$, $b_0 = b_1 = 1$ and $k_p = d$ for all $p \in [1, \infty]$, thus by repeating its arguments we have the following theorem summarizing the results from [45, Theorems 1.1, 1.2, 5.2, 6.1 and Proposition 4.10].

Theorem 2.2.2. *Let ω be an admissible weight function, then for any $f_0 \in L^p_\omega(\mathcal{O})$ with $p \in [1, \infty]$, there exists $f \in C(\mathbb{R}_+, L^p_\omega(\mathcal{O}))$, unique global weak solution to the local KFP Equation* (2.2.4) *in the sense of distributions (see also [45, Proposition 3.3]). Moreover, there are constants $\kappa \geq 0$ and $C > 0$ such that for all $p \in [1, \infty]$ there holds*

$$\|f_t\|_{L^p_\omega(\mathcal{O})} \leq C e^{\kappa t} \|f_0\|_{L^p_\omega(\mathcal{O})} \qquad \forall t \geq 0. \tag{2.2.5}$$

Additionally, the following statements hold.

(1) Let $\omega^\star$ be an admissible weight function such that either $s > 0$ or $k > K + k^*$ if $s = 0$ with $K := 4(3d+1)(2d+3)$. Define $\omega^\star_\infty := (\omega^\star)^{1/2}$ if $s > 0$ or $\omega^\star_\infty := \omega^\star \langle v \rangle^{-K}$ if $s = 0$. There exist $\kappa, \eta > 0$ such that any solution f to the local KFP Equation (2.2.4) *satisfies*

$$\|f(T, \cdot)\|_{L^\infty_{\omega^\star_\infty}(\mathcal{O})} \lesssim e^{\kappa T} T^{-\eta} \|f_0\|_{L^1_{\omega^\star}(\mathcal{O})}. \tag{2.2.6}$$

(2) Consider a weak solution $0 \leq f \in L^2((0,T) \times \mathcal{O}) \cap L^2((0,T) \times \Omega; H^1(\mathbb{R}^d))$ to the local KFP Equation (2.2.4). *For any $0 < T_0 < T_1 < T$ and $\varepsilon > 0$, there holds*

$$\sup_{\mathcal{O}_\varepsilon} f_{T_0} \leq C \inf_{\mathcal{O}_\varepsilon} f_{T_1}, \tag{2.2.7}$$

for some constant $C = C(T_0, T_1, \varepsilon) > 0$ and where we have defined

$$\mathcal{O}_\varepsilon := \Omega_\varepsilon \times B_{\varepsilon^{-1}}, \quad \Omega_\varepsilon := \{x \in \Omega, \, \delta(x) > \varepsilon\}. \tag{2.2.8}$$

(3) There exist two weight functions ω_1, m_1 and an exponent $r > 2$ with $L^r_{\omega_1} \subset (L^2_{m_1})'$ such that there exists a unique eigentriplet $(\lambda_1^{\mathcal{B}}, f_1^{\mathcal{B}}, \phi_1^{\mathcal{B}}) \in \mathbb{R} \times L^r_{\omega_1} \times L^2_{m_1}$ satisfying the first eigenproblem (2.2.2)*. These eigenfunctions are continuous functions and they also satisfy*

$$0 < f_1^{\mathcal{B}} \lesssim \varsigma^{-1} \quad \text{and} \quad 0 < \phi_1^{\mathcal{B}} \lesssim \varsigma \quad \text{on} \quad \mathcal{O}, \tag{2.2.9}$$

for any admissible weight function ς.

Remark 2.3. Theorem 2.2.2 implies, in particular, that the family of mappings $S_{\mathcal{B}}(t) : L^p_\omega \to L^p_\omega$, defined by $S_{\mathcal{B}}(t) f_0 := f(t, \cdot)$ for $t \geq 0$, $f_0 \in L^p_\omega$ and f_t given by Theorem 2.2.2, is a positive semigroup of linear and bounded operators.

2.3 Study of the semigroup $S_{\mathcal{L}}$

In this section we study the properties of the semigroup generated by the linear operator $\mathcal{L}$, namely, well-posedness of the associated evolution equation, control on the growth on the semigroup in weighted Lebesgue spaces, ultracontractivity and existence and properties of the associated dual semigroup.

2.3.1 Growth estimates on a weighted L^2 framework

We define

$$\mathcal{B}_0 f := \mathcal{C}_\Lambda f - f \sum_{n=1}^{N} \eta_n \mathbf{1}_{\Omega_n} = \Lambda(x) \Delta_v f + v \cdot \nabla_v f + f \left(d - \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n} \right),$$

and by following the ideas presented in [45, Subsection 2.1] for the study of local Kolmogorov equations we first observe that for two functions $h, \omega : \mathbb{R}^d \to \mathbb{R}_+$ and any $p \in [1, \infty)$, we have

$$\int_{\mathbb{R}^d} (\mathcal{B}_0 h)\, h^{p-1} \omega^p = -\frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} |\nabla_v (h\omega)^{p/2}|^2 + \int_{\mathbb{R}^d} h^p \omega^p \varpi^{\mathcal{B}_0}_{\omega,p}, \tag{2.3.1}$$

where

$$\varpi^{\mathcal{B}_0}_{\omega,p}(x,v) := 2\Lambda \left(1 - \frac{1}{p}\right) \frac{|\nabla_v \omega|^2}{\omega^2} + \left(\frac{2}{p} - 1\right) \Lambda \frac{\Delta_v\, \omega}{\omega} - v \cdot \frac{\nabla_v \omega}{\omega} + \left(d - \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n} \right) - \frac{d}{p}. \tag{2.3.2}$$

By choosing ω to be an admissible weight function, our assumptions during Subsection 2.1.2 imply that $(\varpi^{\mathcal{B}_0}_{\omega,p})_+ \in L^\infty(\mathcal{O})$ and moreover there holds

$$\limsup_{|v| \to \infty} (\sup_{\Omega} \varpi^{\mathcal{B}_0}_{\omega,p} - \varpi^\sharp_{\omega,p}) \leq 0, \quad \text{where} \quad \varpi^\sharp_{\omega,p} := -b^\sharp_p \langle v \rangle^s,$$

with $b^\sharp_p > 0$ given by

$$\begin{array}{ll} b^\sharp_p := k - d\,(1 - 1/p) + \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n} & \text{if } s = 0, \\ b^\sharp_p := s\zeta & \text{if } s \in (0,1]. \end{array} \tag{2.3.3}$$

In a more quantitative way, for any $\vartheta \in (0,1)$, there exists $\kappa', R' > 0$ such that

$$\vartheta \chi^c_{R'} \varpi^\sharp_{\omega,p} \leq \sup_{\Omega} \varpi^{\mathcal{B}_0}_{\omega,p} \leq \kappa' \chi_{R'} + \vartheta \chi^c_{R'} \varpi^\sharp_{\omega,p}, \tag{2.3.4}$$

where $\chi_R(v) := \chi(|v|/R)$, $\chi \in C^2(\mathbb{R}_+)$, $\mathbf{1}_{[0,1]} \leq \chi \leq \mathbf{1}_{[0,2]}$, and $\chi^c_R : 1 - \chi_R$. Then we have the following lemma.

Lemma 2.1. *Consider an admissible weight function ω. For every $p \in \{1, 2\}$ there are constants $\kappa \geq 0$ and $C \geq 1$ such that for every solution $f(t, x, v) \geq 0$ to Equation* (2.1.14) *there holds*

$$\|f_t\|_{L^p_\omega(\mathcal{O})} \leq C\, e^{\kappa t} \|f_0\|_{L^p_\omega(\mathcal{O})} \qquad \forall t \geq 0, \tag{2.3.5}$$

together with the energy estimate on the gradient

$$\int_0^t \|\nabla_v f_s\|^2_{L^2_\omega(\mathcal{O})} ds \lesssim_C \|f_0\|^2_{L^2_\omega(\mathcal{O})} + \int_0^t \|f_s\|^2_{L^2_\omega(\mathcal{O})} ds \qquad \forall t > 0. \tag{2.3.6}$$

Remark 2.2. The proof consists on the use of a modified weight function, introduced in [48, Lemma 2.3], in order to control the terms coming from the Maxwell boundary conditions. We will proceed as during the proof of [45, Lemma 2.1] to control the local the operator $\mathcal{B}$ and we will use Proposition 2.1 to control the non-local part $\mathcal{A}$.

Proof of Lemma 2.1. We introduce, as during the proof of [45, Lemma 2.1], the modified weight functions ω_A and $\widetilde{\omega}$ defined by

$$\omega_A^p := \mathscr{M}_\Theta^{1-p} \chi_A + \omega^p \chi_A^c, \qquad \widetilde{\omega}^p := \left(1 + \frac{1}{2} \frac{n_x \cdot v}{\langle v \rangle^4}\right) \omega_A^p, \tag{2.3.7}$$

with $A \geq 1$ to be chosen later, $\hat{v} := v/\langle v \rangle$, $\widetilde{v} := \hat{v}/\langle v \rangle$, and we recall that n_x is the normal vector on $\partial\Omega$ defined in Subsection 2.1.1. It is worth emphasizing that

$$c_A^{-1} \omega \leq \tfrac{1}{2} \omega_A \leq \widetilde{\omega} \leq \tfrac{3}{2} \omega_A \leq c_A \omega, \tag{2.3.8}$$

for some constant $c_A \in (0, \infty)$ depending only on A. We recall that $\mathcal{L} f = -v \cdot \nabla_x f + \mathcal{B}_0 f + \mathcal{A} f$, thus we have that

$$\frac{1}{p} \frac{d}{dt} \int_{\mathcal{O}} f^p\, \widetilde{\omega}^p = \int_{\mathcal{O}} (\mathcal{B}_0 f) f^{p-1} \widetilde{\omega}^p + \frac{1}{p} \int_{\mathcal{O}} f^p\, (v \cdot \nabla_x \widetilde{\omega}^p) - \frac{1}{p} \int_\Sigma (\gamma f)^p\, \widetilde{\omega}^p (n_x \cdot v) + \sum_{n=1}^{\mathcal{N}} \eta_n \int_{\Omega_n} \varrho_f \int_{\mathbb{R}^d} f^{p-1} \widetilde{\omega}^p \mathcal{M}_{T_n}. \tag{2.3.9}$$

We divide then the proof into 6 Steps and we emphasize that Steps 1, 2 and 3 are a repetition of Steps 1, 2 and 3 of the proof of [45, Lemma 2.1] thus we only sketch them.

Step 1. By repeating exactly the arguments from the Step 1 of the proof of [45, Lemma 2.1] we immediately have that there is $A > 0$ large enough, such that

$$-\int_\Sigma (\gamma f)^p\, \widetilde{\omega}^p (n_x \cdot v) \leq 0.$$

Step 2. We now deal with the first term at the right-hand side of (2.3.9). On the one hand from (2.3.1), we have that

$$\int_{\mathbb{R}^d} (\mathcal{B}_0 f) f^{p-1} \widetilde{\omega}^p = -\frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} \Lambda(x) |\nabla_v (f^{p/2} \widetilde{\omega}^{p/2})|^2 + \int_{\mathbb{R}^d} f^p \widetilde{\omega}^p \varpi^{\mathcal{B}_0}_{\widetilde{\omega}, p},$$

with

$$\varpi^{\mathcal{B}_0}_{\widetilde{\omega}, p} := 2\Lambda \left(1 - \frac{1}{p}\right) \frac{|\nabla_v \widetilde{\omega}|^2}{\widetilde{\omega}^2} + \Lambda \left(\frac{2}{p} - 1\right) \frac{\Delta_v \widetilde{\omega}}{\widetilde{\omega}} - v \cdot \frac{\nabla_v \widetilde{\omega}}{\widetilde{\omega}} + \left(d - \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n}\right) - \frac{d}{p}.$$

Following exactly the computations from the Step 2 of the proof of [45, Lemma 2.1] we have that

$$\varpi^{\mathcal{B}_0}_{\widetilde{\omega},p} = \varpi^{\mathcal{B}_0}_{\omega,p} + \mathfrak{W}_p \qquad \text{with} \qquad \mathfrak{W}_p = o(\varpi^\sharp_{\omega,p}).$$

Combining the above estimates together with (2.3.4), we deduce that for any $\vartheta \in (0,1)$, there exists $\tilde\kappa, \tilde R > 0$ such that

$$\sup_\Omega \varpi^{\mathcal{B}_0}_{\tilde\omega,p} \le \tilde\kappa \chi_{\tilde R} + \vartheta \chi^c_{\tilde R} \varpi^\sharp_{\omega,p}.$$

Step 3. We control then the second term at the right-hand side of (2.3.9). We first compute

$$v \cdot \nabla_x(\widetilde{\omega}^p) = \frac{1}{2}(\hat v \cdot \nabla_x(n_x \cdot \hat v)) \frac{\omega_A^p}{\langle v\rangle^2} + \left(1 + \frac{1}{2}\frac{n_x \cdot v}{\langle v\rangle^4}\right) v \cdot \nabla_x(\omega_A^p),$$

and since

$$\nabla_x(\omega_A^p) = (p-1)\chi_A \mathscr{M}_\Theta^{1-p}\left[\frac{(d-1)}{2}\frac{\nabla_x\Theta}{\Theta} - \frac{|v|^2}{2}\frac{\nabla_x\Theta}{\Theta^2}\right],$$

assumption (2.1.3) together with the fact that χ_A is compactly supported and the regularity assumption on Ω imply that

$$v \cdot \nabla_x(\widetilde{\omega}^p) \lesssim \frac{1}{\langle v\rangle^2}\widetilde{\omega}^p \lesssim \frac{|\varpi^\sharp_{\omega,p}|}{\langle v\rangle^{s+2}}\widetilde{\omega}^p.$$

Step 4. We now control the terms coming from the non-local terms. We compute by using the Hölder inequality

$$\begin{aligned} \int_{\mathbb{R}^d} f &\le \left(\int_{\mathbb{R}^d} f^p \widetilde{\omega}^p\right)^{1/p}\left(\int_{\mathbb{R}^d}\widetilde{\omega}^{-p'}\right)^{1/p'} \le c_A^{-1}\|\omega^{-1}\|_{L^{p'}(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d} f^p\widetilde{\omega}^p\right)^{1/p} \\ \int_{\mathbb{R}^d} f^{p-1}\widetilde{\omega}^p \mathcal{M}_{T_n} &\le \left(\int_{\mathbb{R}^d} f^p\widetilde{\omega}^p\right)^{1-1/p}\left(\int_{\mathbb{R}^d}\widetilde{\omega}^p\mathcal{M}^p_{T_n}\right)^{1/p} \le c_A\|\omega\mathcal{M}_{T_n}\|_{L^p(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d} f^p\widetilde{\omega}^p\right)^{1-1/p} \end{aligned}$$

where $p' = p/(p-1)$, with the convention $1/0 = \infty$, is the conjugate of p, and we have used (2.3.8) to obtain the previous estimates. We then define the constant

$$\varpi_{\mathscr{G},p} = \|\omega^{-1}\|_{L^{p'}(\mathbb{R}^d)}\sum_{n=1}^{\mathcal{N}}\eta_n\|\omega\mathcal{M}_{T_n}\|_{L^p(\mathbb{R}^d)} < \infty,$$

and we have that

$$\begin{aligned} \sum_{n=1}^{\mathcal{N}}\eta_n\int_{\Omega_n}\varrho_f\left(\int_{\mathbb{R}^d} f^{p-1}\widetilde{\omega}^p\mathcal{M}_{T_n}\right) &\le \sum_{n=1}^{\mathcal{N}}\eta_n\int_\Omega\left(\int_{\mathbb{R}^d} f\right)\left(\int_{\mathbb{R}^d} f^{p-1}\widetilde{\omega}^p\mathcal{M}_{T_n}\right) \\ &\le \varpi_{\mathscr{G},p}\int_{\mathcal{O}} f^p\widetilde{\omega}^p, \end{aligned}$$

where we have used the above estimates to deduce the second line.

Step 5. Coming back to (2.3.9) and using Steps 1, 2, 3 and 4, we deduce that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathcal{O}} f^p\widetilde{\omega}^p \le -\frac{4(p-1)}{p^2}\Lambda_0\int_{\mathcal{O}}|\nabla_v(f\widetilde{\omega})^{p/2}|^2 + \int_{\mathcal{O}} f^p\widetilde{\omega}^p\varpi^{\mathcal{L}}_{\tilde\omega,p} \tag{2.3.10}$$

with

$$\varpi^{\mathcal{L}}_{\tilde\omega,p} := \varpi^{\mathcal{B}_0}_{\tilde\omega,p} + \frac{1}{p}\frac{1}{\widetilde{\omega}^p}v\cdot\nabla_x\widetilde{\omega}^p + \varpi_{\mathscr{G},p}.$$

Gathering the estimate from (2.3.4) and those established in Step 2 and Step 3, we deduce that for any $\vartheta \in (0,1)$, there are $\kappa, R > 0$ such that

$$\varpi^{\mathcal{L}}_{\widetilde{\omega},p} \leq \kappa \chi_R + \vartheta \chi_R^c \varpi^{\sharp}_{\omega,p}. \tag{2.3.11}$$

In particular, choosing $\vartheta = 1/2$ we have that $\varpi^{\mathcal{L}}_{\widetilde{\omega},p} \leq \kappa$ and we immediately conclude (2.3.5), thanks to Grönwall's lemma and the equivalence between ω and $\widetilde{\omega}$ given by (2.3.8).

Step 6. Coming back now to (2.3.10) in the case $p = 2$ and integrating in the time interval $(0,t)$ we have that

$$\frac{1}{2}\int_{\mathcal{O}} f_t^2 \widetilde{\omega}^2 \mathrm{d}v\mathrm{d}x + \Lambda_0 \int_0^t \int_{\mathcal{O}} |\nabla_v (f_s \widetilde{\omega})|^2 \mathrm{d}v\mathrm{d}x\mathrm{d}s \leq \frac{1}{2}\int_{\mathcal{O}} f_0^2 \widetilde{\omega}^2 \mathrm{d}v\mathrm{d}x + \kappa \int_0^t \int_{\mathcal{O}} f_s^2\, \widetilde{\omega}^2 \mathrm{d}v\mathrm{d}x\mathrm{d}s, \tag{2.3.12}$$

where we recall that $\kappa > 0$ is given by (2.3.11). Using the triangular inequality we observe now that

$$|\nabla_v (f\widetilde{\omega})|^2 \geq |\nabla_v f|^2 \widetilde{\omega}^2 - \left|\frac{\nabla_v \widetilde{\omega}}{\widetilde{\omega}}\right|^2 f^2 \widetilde{\omega}^2. \tag{2.3.13}$$

Putting together (2.3.12) and (2.3.13) we deduce that

$$\Lambda_0 \int_0^t \int_{\mathcal{O}} |\nabla_v f_s|^2 \widetilde{\omega}^2 \leq \frac{1}{2}\int_{\mathcal{O}} f_0^2 \widetilde{\omega}^2 + \left(\kappa + \Lambda_0 \left\|\frac{\nabla_v \widetilde{\omega}}{\widetilde{\omega}}\right\|^2_{L^\infty(\mathcal{O})}\right) \int_0^t \int_{\mathcal{O}} f_s^2\, \widetilde{\omega}^2. \tag{2.3.14}$$

Defining, $\wp^2 := 1 + (n_x \cdot v)/(2\langle v\rangle^4)$ and $\wp_A^2 := 1 + \chi_A(\mathscr{M}_\Theta^{-1}\omega^{-2} - 1)$ so that $\widetilde{\omega} = \wp\omega_A$ and $\omega_A = \wp_A \omega$, we observe that

$$\left|\frac{\nabla_v \widetilde{\omega}}{\widetilde{\omega}}\right| \leq \left|\frac{\nabla_v \omega}{\omega}\right| + \left|\frac{\nabla_v \wp}{\wp}\right| + \left|\frac{\nabla_v \wp_A}{\wp_A}\right| \lesssim 1 \tag{2.3.15}$$

where we have used (2.1.5) and the fact that χ_A has compact support, see for instance the proof of [45, Lemma 2.1]. Putting together the previous computations with (2.3.14) and using again (2.3.8) we conclude (2.3.6). □

2.3.2 Well-posedness of the kinetic Fokker-Planck equation with BGK thermostats

We obtain in this subsection the well-posedness of Equation (2.1.14). It is worth remarking that existence results for Kolmogorov type equations presenting Maxwell boundary conditions in the context of kinetic equations have been deeply studied in recent years. In particular we construct our solutions using the existence results from [49, Section 2], but we also refer to [90, Section 11], [45, Section 3], [138, Section 4.1] and [133] for further references on the subject.

We denote the boundary measures

$$\mathrm{d}\xi^1_\omega := \omega^2 |n_x \cdot v| \mathrm{d}\sigma_x \mathrm{d}v, \quad \mathrm{d}\xi^2_\omega := \omega^2 \langle v\rangle^{-2} (n_x \cdot v)^2 \mathrm{d}\sigma_x \mathrm{d}v, \tag{2.3.16}$$

where $\mathrm{d}\sigma_x$ represents the Lebesgue measure on the boundary set $\partial\Omega$, we define $\mathfrak{B}$ as the set of renormalizing functions $\beta \in W^{2,\infty}_{loc}(\mathbb{R})$ such that $\beta'' \in L^\infty(\mathbb{R})$, and we consider the Hilbert space $\mathscr{H}_\omega$ associated to the Hilbert norm $\|\cdot\|_{\mathscr{H}_\omega}$ defined by

$$\|g\|^2_{\mathscr{H}_\omega} := \|g\|^2_{L^2_\omega} + \|g\|^2_{H^{1,\dagger}_\omega}, \quad \text{with} \quad \|g\|^2_{H^{1,\dagger}_\omega} := \int \left\{\Lambda |\nabla_v (f\omega)|^2 + \langle \varpi^\sharp_{\omega,2}\rangle\, \omega^2 g^2\right\}. \tag{2.3.17}$$

Then we have the following well-posedness result.

Theorem 2.3.3. *For any admissible weight function ω and for any $f_0 \in L^2_\omega(\mathcal{O})$, there exists a unique global renormalized solution $f \in C(\mathbb{R}_+, L^2_\omega(\mathcal{O})) \cap \mathscr{H}_\omega(\mathcal{U})$ to Equation* (2.1.14) *associated to the initial datum f_0. More precisely, f satisifies* (2.1.14) *in the renormalized sense, i.e there holds*

$$\int_{\mathcal{O}} \beta(f_t)\,\varphi(t,\cdot)\,\mathrm{d}v\mathrm{d}x + \int_0^t \int_{\mathcal{O}} \beta(f)\,[-\partial_t\varphi - \mathcal{B}^*\varphi] - \beta'(f)\,\varphi\mathscr{G}f\,\mathrm{d}v\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^t \int_{\mathcal{O}} \Lambda\beta''(f)|\nabla_v f|^2\varphi\,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_0^t \int_\Sigma \beta(\gamma f)\,\varphi(n_x\cdot v)\,\mathrm{d}v\mathrm{d}\sigma_x\mathrm{d}t = \int_{\mathcal{O}} \beta(f_0)\,\varphi(0,\cdot)\,\mathrm{d}v\mathrm{d}x, \tag{2.3.18}$$

for any $t > 0$, any test function $\varphi \in \mathcal{D}(\bar{\mathcal{U}})$, any $\beta \in \mathfrak{B}$ and where we have defined the formal dual operator $\mathcal{B}^*\varphi := v\cdot\nabla_x\varphi + \Lambda\Delta_v\varphi - v\cdot\nabla_v\varphi$. Moreover, we emphasize that γf is given by [49, Theorem 2.8] and it satisfies $\gamma f \in L^2(\Gamma, \mathrm{d}\xi^2_\omega\mathrm{d}t)$, as well as the Maxwell boundary condition (2.1.2) *point-wisely. Likewise, it is worth remarking that $f(0,\cdot) = f_0$ also holds point-wisely. Furthermore, f satisfies the conclusions of Lemma 2.1.*

Remark 2.4. Theorem 2.3.3 in particular implies that we may associate a strongly continuous in time semigroup, that we denote during the sequel by $S_{\mathcal{L}} : L^2_\omega(\mathcal{O}) \to L^2_\omega(\mathcal{O})$, to the solutions of the Equation (2.1.14).

Proof of Theorem 2.3.3. We split the proof into three steps.

Step 1. (Modified weight function) We take $\widetilde{\omega}$ as defined in (2.3.7) and we remark that $\widetilde{\omega} = \theta\omega$ with

$$\theta(x,v) := \left(1 + \frac{1}{2}\frac{n_x\cdot v}{\langle v\rangle^4}\right)\left(\mathscr{M}_\Theta^{-1}\omega^{-2}\chi_A + (1-\chi_A)\right),$$

and we readily observe that there holds

$$\frac{1}{2} \le \left(1 - \frac{1}{2}\right)\left(\left[\mathscr{M}_\Theta^{-1}\omega^{-2} - 1\right]\chi_A + 1\right) \le \theta \le \frac{3}{2}\left(\mathscr{M}_\Theta^{-1}(A) + 1\right). \tag{2.3.19}$$

where we have used that $\mathscr{M}_\Theta^{-1}\omega^{-2} \ge 1$ due to our hypothesis on ω. Moreover, recalling that $\omega(v) = \langle v\rangle^k e^{\zeta\langle v\rangle^s}$ we compute

$$\begin{aligned}
\nabla_x\theta &= \frac{1}{2}\frac{\nabla_x(n_x\cdot v)}{\langle v\rangle^4}\left(\mathscr{M}_\Theta^{-1}\omega^{-2}\chi_A + (1-\chi_A)\right) \\
&\quad + \left(1 + \frac{1}{2}\frac{n_x\cdot v}{\langle v\rangle^4}\right)\omega^{-2}\chi_A\mathscr{M}_\Theta^{-1}\left(-\frac{(1+d)}{2}\frac{\nabla_x\Theta}{\Theta} - \frac{|v|^2\nabla_x\Theta}{2\Theta^2}\right), \\
\nabla_v\theta &= \frac{1}{2}\frac{n_x\langle v\rangle^2 - 4v\,(n_x\cdot v)}{\langle v\rangle^6}\left(\mathscr{M}_\Theta^{-1}\omega^{-2}\chi_A + (1-\chi_A)\right) \\
&\quad + \left(1 + \frac{1}{2}\frac{n_x\cdot v}{\langle v\rangle^4}\right)\left[\omega^{-2}\mathscr{M}_\Theta^{-1}\left(\chi_A\frac{v}{2\Theta} + \nabla_v\chi_A - 2\chi_A\left(\frac{k}{\langle v\rangle^2} + \zeta s\langle v\rangle^{s-2}\right)\right) - \nabla_v\chi_A\right].
\end{aligned}$$

From the previous computations and recalling that χ_A has compact support in the ball of radius $2A$, we deduce that

$$\begin{aligned}
|\nabla_x\theta| &\le \langle v\rangle^{-1}\left(\frac{1}{2}\|\delta\|_{W^{2,\infty}}\left(\mathscr{M}_\Theta^{-1}(2A) + 1\right) + \frac{3}{4}\langle 2A\rangle^3\mathscr{M}_\Theta^{-1}(2A)\|\Theta\|_{W^{1,\infty}}\left(\frac{(1+d)}{\Theta_*} + \frac{1}{\Theta_*^2}\right)\right) \\
|\nabla_x\theta| &\le \langle v\rangle^{-1}\left(\frac{5}{2}\left(\mathscr{M}_\Theta^{-1}(2A) + 1\right)\right. \\
&\quad \left. + \frac{3\langle 2A\rangle}{2}\left(\mathscr{M}_\Theta^{-1}(2A)\|\chi_A\|_{W^{1,\infty}}\left(\frac{\langle A\rangle}{2\Theta_*} + 1 + 2\,(k+\zeta s)\right) + 1\right)\right)
\end{aligned}$$

where we have used that $\delta \in W^{2,\infty}$ as defined during Subsection 2.1.1. Finally, from the above computations together with (2.3.19) and our assumptions on the regularity of Ω, we deduce that

$$|\nabla_x \theta| + |\nabla_v \theta| \lesssim \theta \langle v \rangle^{-1}. \tag{2.3.20}$$

This concludes Step 1.

Step 2. (Well-posedness) We recall now that from the Step 2 of the proof of Lemma 2.1 we have that the function $\varpi^{\mathcal{B}_0}_{\widetilde{\omega},2}$, defined in (2.3.2), satisfies

$$\varpi^{\sharp}_{\omega,2} \le \varpi^{\mathcal{B}_0}_{\widetilde{\omega},2} \le \kappa + \varpi^{\sharp}_{\omega,2},$$

for some constant $\kappa > 0$. Furthermore, Proposition 2.1 implies that

$$\sup_{(0,T)\times\Omega} \|\mathcal{A}\|_{\mathscr{B}(L^1_v(\omega))} < \infty, \qquad \sup_{(0,T)\times\Omega} \|\mathcal{A}\|_{\mathscr{B}(L^2_v(\omega))} < \infty,$$

and from the Step 1 of the proof of Lemma 2.1 we further have that the boundary collision operator $\mathscr{R}$ satisfies the bound

$$\mathscr{R} : L^2(\Sigma_+, \mathrm{d}\xi^1_{\widetilde{\omega}}) \to L^2(\Sigma_-, d\xi^1_{\widetilde{\omega}}), \quad \|\mathscr{R}\|_{L^2(\Sigma_+, \mathrm{d}\xi^1_{\widetilde{\omega}})} \le 1. \tag{2.3.21}$$

We remark that, from its very definition there holds $\varpi^{\mathcal{B}_0}_{1,1} \le 0$, and also due to the hypothesis on admissible weight functions we have that

$$\left(\Lambda + |v| + 2d + \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n} \right) \omega^{-1} \in L^2(\mathcal{U}).$$

The above informations together with (2.3.19) and (2.3.20) imply that we may use [49, Theorem 2.11] and [49, Theorem 2.8], and we deduce that there exists $f \in C(\mathbb{R}_+, L^2_\omega(\mathcal{O})) \cap \mathscr{H}_\omega(\mathcal{U})$ unique renormalized solution to Equation (2.1.14) with an associated trace function $\gamma f \in L^2(\Gamma, \mathrm{d}\xi^2_\omega \mathrm{d}t)$ satisfying (2.3.18).

Step 3. (Energy estimates) We obtain the validity of (2.3.5) with $p = 2$ and (2.3.6) as a consequence of (2.3.18) with $\beta(s) = s^2$ and $\varphi = \widetilde{\omega}^2 \chi_R$, for any $R > 0$, repeating the computations performed during the proof of Lemma 2.1, passing $R \to \infty$ and using the integral version of the Grönwall lemma instead. Moreover, (2.3.5) for $p = 1$ is obtained in a similar fashion by taking instead $\beta(s) = s$ and $\varphi = \widetilde{\omega}\chi_R$. □

2.3.3 Ultracontractivity

We establish now the ultracontractive properties of the semigroup $S_{\mathcal{L}}$, which will be obtained from the interplay, via the Duhamel formulation, between the ultracontractive properties of the semigroup $S_{\mathcal{B}}$ and the boundedness of the BGK non-local term $\mathcal{A}$.

Proposition 2.5. *Let ω be an admissible weight function such that either $s > 0$ or $s = 0$ and $k > K + k^*$ for $K := 4(3d+1)(2d+3)$. Define ω_∞ as during Theorem 2.2.2-(1), i.e $\omega_\infty := (\omega)^{1/2}$ if $s > 0$ or $\omega_\infty := \omega \langle v \rangle^{-K}$ if $s = 0$. There are constants $\kappa, \eta > 0$ such that for any solution $f \ge 0$ to Equation* (2.1.14) *there holds*

$$\|f(T,\cdot)\|_{L^\infty_{\omega_\infty}(\mathcal{O})} \lesssim e^{\kappa T} T^{-\eta} \|f_0\|_{L^1_\omega(\mathcal{O})} \qquad \forall T > 0. \tag{2.3.22}$$

Proof. We recall the splitting $\mathcal{L} = \mathcal{B} + \mathcal{A}$ and using the Duhamel formula we have

$$S_{\mathcal{L}} = S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}).$$

Iterating this formula we further deduce that there holds

$$S_{\mathcal{L}} = \mathcal{V} + \mathcal{W} * S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}),$$

with

$$\mathcal{V} := S_{\mathcal{B}} + \cdots + (S_{\mathcal{B}}\mathcal{A})^{*(N-1)} * S_{\mathcal{B}}, \quad \mathcal{W} := (S_{\mathcal{B}}\mathcal{A})^{*N},$$

and where we define recursively $U^{*k} = U^{*(k-1)} * U$, with the convention $U^{*1} = U$, and some $N \in \mathbb{N}$ to be fixed later. We proceed now in three steps.

Step 1. We set $\widetilde{S}_{\mathcal{B}} := S_{\mathcal{B}}\mathcal{A}$ and from (2.2.5) and Proposition 2.1 we have that there are constants $\kappa_1 > 0$ and $C_1 \geq 1$ such that for every $p \in \{1, \infty\}$ there holds

$$\|\widetilde{S}_{\mathcal{B}}(t) f_0\|_{L^p_{\omega_\infty}(\mathcal{O})} \leq C_1 e^{\kappa_1 t} \|f_0\|_{L^p_{\omega_\infty}(\mathcal{O})} \qquad \forall t \geq 0. \tag{2.3.23}$$

Furthermore, Theorem 2.2.2-(1) implies that there are constants $\kappa_2, \eta, C_2^0 > 0$ for which there holds

$$\|\widetilde{S}_{\mathcal{B}}(t) f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})} \leq C_2^0 t^{-\eta} e^{\kappa_2 t} \|\mathcal{A} f_0\|_{L^1_\omega(\mathcal{O})} \leq C_2 t^{-\eta} e^{\kappa_2 t} \|f_0\|_{L^1_{\omega_\infty}(\mathcal{O})} \qquad \forall t > 0, \tag{2.3.24}$$

for some constant $C_2 > 0$ and where we remark that we have used Proposition 2.1 to obtain the second inequality. Remarking now that $\mathcal{W} = (\widetilde{S}_{\mathcal{B}})^{*N}$, and using (2.3.23) and (2.3.24) we may apply [135, Proposition 2.5] and we have that we can set $N \in \mathbb{N}$ such that

$$\|(\widetilde{S}_{\mathcal{B}})^{*N}(t) f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})} \leq C_3 e^{\kappa_3 t} \|f_0\|_{L^1_{\omega_\infty}(\mathcal{O})} \qquad \forall t \geq 0, \tag{2.3.25}$$

for some constant $C_3 > 0$ and any $\kappa_3 > \max(\kappa_1, \kappa_2)$. Using now (2.3.25) and once again Proposition 2.1 we further deduce that

$$\begin{aligned} \|\mathcal{W} * S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})(t) f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})} &\leq \int_0^t \int_0^s \|\mathcal{W}\|_{\mathscr{B}(L^1_{\omega_\infty}, L^\infty_{\omega_\infty})}(t-s) \|S_{\mathcal{L}}\|_{\mathscr{B}(L^1_{\omega_\infty})}(s-r) \\ &\qquad \|\mathcal{A}\|_{\mathscr{B}(L^1_\omega, L^1_{\omega_\infty})} \|S_{\mathcal{B}}\|_{\mathscr{B}(L^1_\omega)}(r) \|f_0\|_{L^1_\omega} drds \\ &\lesssim e^{\kappa_3 t} \|f_0\|_{L^1_\omega}, \end{aligned}$$

where we have successively used (2.3.25), Lemma 2.1, Proposition 2.1 and Theorem 2.2.2, and we have chosen $\kappa_3 = \max(2\kappa_1, 2\kappa_2, \kappa_4)$ where κ_4 is given by Lemma 2.1.

Step 2. Now we analyze the rest of the terms, on the one hand Theorem 2.2.2-(1) immediately gives that there is a constant $C_4 > 0$ for which there holds

$$\|S_{\mathcal{B}}(t) f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})} \leq C_4 t^{-\eta} e^{\kappa_2 t} \|f\|_{L^1_\omega(\mathcal{O})} \qquad \forall t > 0. \tag{2.3.26}$$

On the other hand we set $\mathcal{V}_j := (\widetilde{S}_{\mathcal{B}})^{*(j-1)} * S_{\mathcal{B}}$ for $j \in [\![2, N]\!]$ and, from [135, proof of Proposition 2.5, Equation (2.9)] we have that for every $p \in \{1, \infty\}$ there holds

$$\|\mathcal{V}_j(t) f\|_{L^p_{\omega_\infty}(\mathcal{O})} \leq \frac{C^j t^{j-1}}{(j-1)!} e^{\kappa_3 t} \|f\|_{L^p_{\omega_\infty}(\mathcal{O})} \qquad \forall t \geq 0, \tag{2.3.27}$$

where we have defined $C = \max(C_1, C_2, C_4, C_5)$ with $C_5 > 0$ given by (2.2.5) in Theorem 2.2.2.

Furthermore, following then the same ideas as in the proof of [135, Proposition 2.5], we will prove now by induction that

$$\|\mathcal{V}_j(t)f\|_{L^\infty_{\omega_\infty}(\mathcal{O})} \leq p_j(t)\, t^{-\eta} e^{\kappa_2 t}\|f\|_{L^1_\omega(\mathcal{O})} \qquad \forall t \geq 0, \tag{2.3.28}$$

for some polynomial function p_j.

Step 2.1 (Base case). In particular, for $j = 2$ we have that $\mathcal{V}_2 = \widetilde{S}_{\mathcal{B}} * S_{\mathcal{B}}$ and we compute

$$\begin{aligned}\|\widetilde{S}_{\mathcal{B}} * S_{\mathcal{B}}(t)f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})} &\leq \int_0^{t/2}\|\widetilde{S}_{\mathcal{B}}(t-s)S_{\mathcal{B}}(s)f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})} + \int_{t/2}^t\|\widetilde{S}_{\mathcal{B}}(t-s)S_{\mathcal{B}}(s)f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})}\\ &=: I_1 + I_2,\end{aligned}$$

and we bound then each integral separately. We compute for the first one

$$\begin{aligned}I_1 &\leq C_2\int_0^{t/2}(t-s)^{-\eta}e^{\kappa_2(t-s)}\|S_{\mathcal{B}}(s)f_0\|_{L^1_\omega(\mathcal{O})}\mathrm{d}s \leq (C_1C_2)e^{\kappa_3 t}\|f_0\|_{L^1_\omega(\mathcal{O})}\int_0^{t/2}(t-s)^{-\eta}\mathrm{d}s\\ &\leq \frac{C^2}{2^{1-\eta}(1-\eta)}\, t^{1-\eta}e^{\kappa_3 t}\|f_0\|_{L^1_\omega(\mathcal{O})},\end{aligned}$$

where we have successively used (2.3.23), Lemma 2.1 and the very definitions of κ_3 and C. Moreover we similarly compute for the second integral as follows

$$\begin{aligned}I_2 &\leq C_1\int_{t/2}^t e^{\kappa_1(t-s)}\|S_{\mathcal{B}}(s)f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})}\mathrm{d}s \leq C^2 e^{\kappa_3 t}\|f_0\|_{L^1_\omega(\mathcal{O})}\int_{t/2}^t s^{-\eta}\mathrm{d}s\\ &\leq \frac{C^2}{2^{1-\eta}(1-\eta)}\, t^{1-\eta}e^{\kappa_3 t}\|f_0\|_{L^1_\omega(\mathcal{O})},\end{aligned}$$

where we have successively used (2.3.24), (2.3.26) and the very definitions of κ_3 and C. Altogether this implies that

$$\|\widetilde{\mathcal{V}}_2 f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})} \leq \frac{C^2}{2^{-\eta}(1-\eta)}\, t^{1-\eta}e^{\kappa_3 t}\|f_0\|_{L^1_\omega(\mathcal{O})}.$$

and we observe in particular that $p_2(t) = t\, C^2/(2^{-\eta}(1-\eta))$.

Step 2.2 (Induction step). We then assume (2.3.28) holds for some $j \in [\![2, N-2]\!]$ and we will prove it for $j+1$ following an argument similar as above. We observe that $\mathcal{V}_{j+1} = \widetilde{S}_{\mathcal{B}} * \mathcal{V}_j$, we compute

$$\begin{aligned}\|\widetilde{S}_{\mathcal{B}} * \mathcal{V}_j\|_{L^\infty_{\omega_\infty}(\mathcal{O})} &\leq \int_0^{t/2}\|\widetilde{S}_{\mathcal{B}}(t-s)\mathcal{V}_j(s)f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})} + \int_{t/2}^t\|\widetilde{S}_{\mathcal{B}}(t-s)\mathcal{V}_j(s)f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})}\\ &=: I_1^j + I_2^j,\end{aligned}$$

and we bound then each integral separately. We compute for the first one

$$\begin{aligned}I_1^j &\leq C_2\int_0^{t/2}(t-s)^{-\eta}e^{\kappa_2(t-s)}\|\mathcal{V}_j(s)f_0\|_{L^1_\omega(\mathcal{O})}\mathrm{d}s \leq \frac{C^{j+1}}{(j-1)!}e^{\kappa_3 t}\|f_0\|_{L^1_\omega(\mathcal{O})}\int_0^{t/2}(t-s)^{-\eta}s^{j-1}\mathrm{d}s\\ &\leq \frac{C^{j+1}t^{j-1}}{2^{1-\eta}(j-1)!(1-\eta)}\, t^{1-\eta}e^{\kappa_3 t}\|f_0\|_{L^1_\omega(\mathcal{O})},\end{aligned}$$

where we have successively used (2.3.23), Lemma 2.1 and the very definitions of κ_3 and C. Moreover we similarly compute for the second integral as follows

$$\begin{aligned}I_2^j &\leq C_1\int_{t/2}^t e^{\kappa_1(t-s)}\|\mathcal{V}_j(s)f_0\|_{L^\infty_{\omega_\infty}(\mathcal{O})}\mathrm{d}s \leq p_n(t)e^{\kappa_3 t}\|f_0\|_{L^1_\omega(\mathcal{O})}\int_{t/2}^t s^{-\eta}\mathrm{d}s\\ &\leq \frac{p_j(t)}{2^{1-\eta}(1-\eta)}\, t^{1-\eta}e^{\kappa_3 t}\|f_0\|_{L^1_\omega(\mathcal{O})},\end{aligned}$$

and we have that p_j is defined recurrently by the formula

$$p_{j+1}(t) = \frac{C^{j+1}}{2^{1-\eta}(j-1)!(1-\eta)}\, t^j + \frac{p_j(t)}{2^{1-\eta}(1-\eta)}\, t, \qquad \text{for every } j \geq 2, \text{ with } p_2(t) \text{ as defined above.}$$

This concludes the induction argument and the validity of (2.3.28).

Step 3. (Conclussion) We conclude by putting together the results from Steps 1 and 2 and choosing any $\kappa > \kappa_3$. □

2.3.4 Weak Maximum Principle

Combining the positivity of the semigroup $S_{\mathcal{B}}$ given by Theorem 2.2.2, and the fact that the non-local operator $\mathcal{A}$ is also positive we will prove the following proposition.

Proposition 2.6 (Weak maximum principle). *Let ω be an admissible weight function and let $f \in L^\infty((0,\infty); L^2_\omega(\mathcal{O})) \cap \mathscr{H}_\omega(\mathcal{U})$ be a solution to Equation* (2.1.14) *associated to the initial data $0 \leq f_0 \in L^2_\omega(\mathcal{O})$. There holds $f_t \geq 0$, for every $t > 0$.*

Proof. We consider $f^0 = 0$ and then we define recurrently $f^k \in L^\infty((0,\infty); L^2_\omega(\mathcal{O})) \cap \mathscr{H}_\omega(\mathcal{U})$ as the solution of the following equation

$$\left\{ \begin{array}{rcll} \partial_t f^{k+1} & = & \mathcal{B} f^{k+1} + \mathcal{A} f^k & \text{in } \mathcal{U} \\ \gamma_- f^{k+1} & = & \mathscr{R}\gamma_+ f^{k+1} & \text{on } \Gamma_- \\ f^{k+1}_{t=0} & = & f_0 & \text{in } \mathcal{O}, \end{array} \right. \tag{2.3.29}$$

which is given by Theorem 2.3.3. We assume then that $f^k \geq 0$ and by using the Duhamel formula together with Theorem 2.2.2 -(2) and Proposition 2.1, we immediately deduce that

$$f^{k+1}_t = S_{\mathcal{B}}(t) f_0 + \int_0^t S_{\mathcal{B}}(t-s) \mathcal{A} f^k_s\, ds \geq 0,$$

due to the fact that $\mathcal{A}$ is a positive operator. Moreover, from the linearity of the operators $\mathcal{A}$ and $\mathcal{B}$ we deduce that, in the weak sense, there holds

$$\left\{ \begin{array}{rcll} \partial_t (f^{k+1} - f^k) & = & \mathcal{B}(f^{k+1} - f^k) + \mathcal{A}(f^k - f^{k-1}) & \text{in } \mathcal{U} \\ \gamma_-(f^{k+1} - f^k) & = & \mathscr{R}\gamma_+(f^{k+1} - f^k) & \text{on } \Gamma_- \\ (f^{k+1} - f^k)_{t=0} & = & 0 & \text{in } \mathcal{O}. \end{array} \right. \tag{2.3.30}$$

We define $\psi^{k+1} := f^{k+1} - f^k \in L^\infty((0,\infty); L^2_\omega(\mathcal{O})) \cap \mathscr{H}_\omega(\mathcal{U})$ and, at the level of a priori estimates, by arguing then as during the proof of Lemma 2.1 we have that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_{\mathcal{O}} (\psi^{k+1})^2\, \widetilde{\omega}^2 = \int_{\mathcal{O}} (\mathcal{B}_0 \psi^{k+1}) \psi^{k+1} \widetilde{\omega}^2 + \frac{1}{2} \int_{\mathcal{O}} (\psi^{k+1})^2\, (v \cdot \nabla_x \widetilde{\omega}^2) \\ - \frac{1}{2} \int_{\Sigma} (\gamma \psi^{k+1})^2\, \widetilde{\omega}^2 (n_x \cdot v) - \int_{\mathcal{O}} \mathcal{A}\psi^k\, \psi^{k+1} \widetilde{\omega}^2. \end{aligned} \tag{2.3.31}$$

where we recall that $\widetilde{\omega}$ is defined in (2.3.7). Using the Cauchy-Schwartz inequality and the Young inequality we deduce that

$$\int_{\mathcal{O}} \mathcal{A}\psi^k\, \psi^{k+1} \widetilde{\omega}^2 \lesssim \|\mathcal{A}\psi^k\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \|\psi^{k+1}\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} \lesssim \|\psi^k\|^2_{L^2_\omega(\mathcal{O})} + \|\psi^{k+1}\|^2_{L^2_\omega(\mathcal{O})}, \tag{2.3.32}$$

where we have used (2.3.8) and (2.2.3) to obtain the second inequality. Coming back now to (2.3.31), arguing similarly as during the proof of Lemma 2.1 and using (2.3.32) we have that there is a constant $\kappa > 0$ such that

$$\|\psi_t^{k+1}\|^2_{L^2_\omega(\mathcal{O})} + \|\psi^{k+1}\|^2_{H^{1,\dagger}_\omega(\mathcal{U})} \lesssim \int_0^t e^{\kappa(t-s)}\|\psi_s^k\|^2_{L^2_\omega(\mathcal{O})} \lesssim e^{\kappa t}\|\psi_s^k\|^2_{L^\infty((0,t);L^2_\omega(\mathcal{O}))} \qquad \forall t \geq 0.$$

We remark now that the previous estimate is valid for weak solutions either by arguing as during the proof of [45, Proposition 3.3] or by using the weak formulation provided by Theorem 2.3.3 taking $\beta(s) = s^2$ and $\varphi = \widetilde{\omega}^2\chi_R$, repeating the previous analysis, letting $R \to \infty$, and using the integral version of the Grönwall lemma.

Moreover, by arguing as during the Step 2 of the proof of [45, Proposition 3.3] or Step 3 of the proof of [49, Theorem 2.11] (see also Step 3 of the proof of Proposition 2.4) we have that

$$\|\gamma\psi^{k+1}\|^2_{L^2(\Gamma,\mathrm{d}\xi^2_\omega \mathrm{d}t)} \lesssim e^{\kappa t}\|\psi_s^k\|^2_{L^\infty((0,t);L^2_\omega(\mathcal{O}))} \qquad \forall t \geq 0$$

Choosing then $T > 0$ small enough we deduce that $f^k, \gamma f^k$ are a Cauchy sequences in the Banach spaces $L^\infty((0,T);L^2_\omega(\mathcal{O})) \cap \mathscr{H}_\omega(\mathcal{U})$ and $L^2(\Gamma, \mathrm{d}\xi^2_\omega \mathrm{d}t)$ respectively, thus there are functions $\mathfrak{f} \in L^\infty((0,T;L^2(\mathcal{O})) \cap \mathscr{H}_\omega(\mathcal{U})$ and $\gamma\mathfrak{f} \in L^2(\Gamma, \mathrm{d}\xi^2_\omega \mathrm{d}t)$ such that

$$f^k \to \mathfrak{f} \quad \text{strongly in} \quad L^\infty((0,T);L^2_\omega(\mathcal{O})) \cap \mathscr{H}_\omega(\mathcal{U}) \quad \text{as} \quad k \to \infty.$$

and

$$\gamma f^k \to \gamma\mathfrak{f} \quad \text{strongly in} \quad L^2(\Gamma, \mathrm{d}\xi^2_\omega \mathrm{d}t) \quad \text{as} \quad k \to \infty.$$

Furthermore, we remark that as the limit of positive functions we have that $\mathfrak{f} \geq 0$, and by using the previous convergences we may pass to the limit in the weak formulation associated to Equation (2.3.29) and we deduce that $\mathfrak{f}$ is a weak solution to Equation (2.1.14). Finally [49, Theorem 2.8-(3)] implies that $\mathfrak{f}$ is a renormalized solution of Equation (2.1.14), thus a solution in the sense of Theorem 2.3.3. By using the uniqueness provided by Theorem 2.3.3 we deduce that $\mathfrak{f} = f$ a.e and repeating this argument in the time intervals $[jT,(j+1)T]$ for $j \in \mathbb{N}$, we deduce that $S_{\mathcal{L}}$ is a positive semigroup. □

2.3.5 Study of the dual backwards KFP equation with BGK thermostats in a weighted L^2 framework

For any finite $T > 0$ and a final datum g_T, we study in this subsection the dual backwards equation associated to the linear Equation 2.1.14,

$$\begin{cases} -\partial_t g &= \mathcal{L}^* g & \text{in } \mathcal{U}_T := (0,T) \times \mathcal{O} \\ \gamma_+ g &= \mathscr{R}^*\gamma_- g & \text{on } \Gamma_{T,+} := (0,T) \times \Sigma_+ \\ g_{t=T} &= g_T & \text{in } \mathcal{O}, \end{cases} \tag{2.3.33}$$

where we have defined the formal dual operator $\mathcal{L}^* g = v \cdot \nabla_x g + \mathcal{B}_0^* g + \mathcal{A}^* g$ with

$$\mathcal{B}_0^* g := \Lambda(x)\Delta_v g - v \cdot \nabla_v g - \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n} g, \quad \text{and} \quad \mathcal{A}^* g = \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n} \int_{\mathbb{R}^d} g \mathcal{M}_{T_n} \mathrm{d}v.$$

Moreover, we have defined the dual Maxwell boundary condition operator as follows

$$\mathscr{R}^* g(x,v) = (1-\iota(x))\mathscr{S} g(x,v) + \iota(x)\mathscr{D}^* g(x), \tag{2.3.34}$$

with

$$\mathscr{D}^* g(x) := \int_{\mathbb{R}^d} g(x,w)\mathscr{M}_\Theta(w)(n_x \cdot w)_- \, dw,$$

for any function g with support on Σ_-. We observe that the dual equation is defined in such a way that two solutions f to the forward Cauchy problem (2.1.14) and g to the dual problem (2.3.33) satisfy, at least formally, the usual dual identity

$$\int_{\mathcal{O}} f(T) g_T = \int_{\mathcal{O}} f_0 g(0). \tag{2.3.35}$$

Properties of the dual non-local operator

We now have a result analogue to Proposition 2.1 for the dual operator $\mathcal{A}^*$.

Proposition 2.7. *We consider two admissible weight functions ω, $\omega_\star$ and we define $m = \omega^{-1}$ and $m_\star = (\omega_\star)^{-1}$. For any $q \in [1, \infty]$ there holds*

$$\|\mathcal{A}^* g\|_{L^q_m(\mathcal{O})} \leq C \sum_{n=1}^{\mathcal{N}} \|g\|_{L^p_{m_\star}(\mathcal{O}_n)} \lesssim \|g\|_{L^q_{m_\star}(\mathcal{O})}. \tag{2.3.36}$$

for some constant $C \geq 0$.

Proof. The proof follows similar as that of Proposition 2.1, thus we skip it. □

A priori estimates for the dual kinetic Fokker-Planck equation with BGK thermostats

Lemma 2.8. *For any admissible weight function ω there are constants $\kappa > 0$ and $C \geq 1$ such that for every $T > 0$ and any initial final datum $g_T \in L^2_m(\mathcal{O})$ with $m = \omega^{-1}$, the associated solution g to the backwards dual Equation* (2.3.33) *satisfies*

$$\|g(0)\|_{L^2_m(\mathcal{O})} \leq C\, e^{\kappa T} \|g_T\|_{L^2_m(\mathcal{O})}, \tag{2.3.37}$$

together with the following energy estimate on the gradient

$$\int_0^T \|\nabla_v g_s\|^2_{L^2_m(\mathcal{O})} ds \lesssim_C \|g_T\|^2_{L^2_m(\mathcal{O})} + \int_0^T \|g_s\|^2_{L^2_m(\mathcal{O})} ds. \tag{2.3.38}$$

Remark 2.9. As for Lemma 2.1, we emphasize that the proof of Lemma 2.8 consists on the introduction of a modified weight function from [45, Lemma 2.2], in order to control the terms coming from the Maxwell boundary conditions. As for the rest of the terms, we repeat the computations from [45, Lemma 2.2] to control the terms coming from the local Kolmogorov part of the equation and we use Proposition 2.7 to control the non-local terms.

Proof of Lemma 2.8. Without loss of generality we may suppose that $m \geq \mathscr{M}_\Theta$, otherwise we replace m by cm where $c > 0$ is such that $m \geq c^{-1}\mathscr{M}_\Theta$. For $A \geq 1$, we introduce the modified weight functions

$$m_A^2 := \chi_A \mathscr{M}_\Theta + \chi_A^c m^2, \quad \widetilde{m}^2 := \left(1 - \frac{1}{2}\frac{n_x \cdot v}{\langle v \rangle^4}\right) m_A^2, \tag{2.3.39}$$

and we emphasize that

$$\mathscr{M}_\Theta \leq m_A \leq m \quad \text{and} \quad c_A^{-1} m \leq \frac{1}{2} m_A \leq \widetilde{m} \leq \frac{3}{2} m_A \leq \frac{3}{2} m, \tag{2.3.40}$$

for some constant $c_A \in (0,\infty)$ depending on A. We then compute

$$-\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} g^2\,\widetilde{m}^2 = \int_{\mathcal{O}} g\,(\mathcal{B}_0^* g)\,\widetilde{m}^2 - \frac{1}{2}\int_{\mathcal{O}} g^2\,\left(v\cdot\nabla_x \widetilde{m}^2\right) + \int_{\mathcal{O}} g\,(\mathcal{A}^* g)\,\widetilde{m}^2 + \frac{1}{2}\int_{\Sigma} (\gamma g)^2\,\widetilde{m}^2\,(n_x\cdot v), \quad (2.3.41)$$

and we proceed to control each of this terms.

Step 1. On the one hand, arguing exactly as during the Step 1 of the proof of [45, Lemma 2.2] we immediately have that there is $A>0$ large enough for which there holds

$$\frac{1}{2}\int_{\Sigma}(\gamma g)^2\,\widetilde{m}^2\,(n_x\cdot v)\le 0.$$

Step 2. On the other hand, from (2.3.1)-(2.3.2), we have

$$\int_{\mathbb{R}^d}(\mathcal{B}_0^* g) g\,\widetilde{m}^2 = -\Lambda(x)\int_{\mathbb{R}^d}|\nabla_v(g\widetilde{m})|^2 + \int g^2\;\widetilde{m}^2 \varpi^{\mathcal{B}_0^*}_{\widetilde{m},2},$$

with

$$\varpi^{\mathcal{B}_0^*}_{\widetilde{m},2} = \Lambda\frac{|\nabla_v\varphi|^2}{\varphi^2} + v\cdot\frac{\nabla_v\widetilde{m}}{\widetilde{m}} - \sum_{n=1}^{\mathcal{N}}\eta_n \mathbf{1}_{\Omega_n} + \frac{d}{2}.$$

Arguing exactly as in Step 2 of the proof of [45, Lemma 2.2], we can write

$$\varpi^{\mathcal{B}_0^*}_{\widetilde{m},2} = \varpi^{\mathcal{B}_0^*}_{m,2} + \mathfrak{M}, \quad \text{with} \quad \varpi^{\mathcal{B}_0^*}_{m,2} = \varpi^{\mathcal{B}_0}_{\omega,2}, \quad \text{and} \quad \mathfrak{M} = o(|\varpi^{\sharp}_{\omega,2}|).$$

Step 3. We then deal with the third term on the right side of (2.3.41). We compute by using the Cauchy-Schwartz inequality

$$\begin{aligned}\int_{\mathbb{R}^d} g\,\widetilde{m}^2 &\le \|\widetilde{m}\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2_{\widetilde{m}}(\mathbb{R}^d)} \le \frac{3}{2}\|m\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2_{\widetilde{m}}(\mathbb{R}^d)}\\ \int_{\mathbb{R}^d} g\,\mathcal{M}_{T_n} &\le \|\mathcal{M}_{T_n}\widetilde{m}^{-1}\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2_{\widetilde{m}}(\mathbb{R}^d)} \le c_A^{-1}\|\mathcal{M}_{T_n} m^{-1}\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2_{\widetilde{m}}(\mathbb{R}^d)}\end{aligned}$$

where we have used (2.3.40) to obtain the second inequalities in each estimate and we recall that c_A is given by (2.3.40). We define the constant

$$\varpi^*_{\mathscr{G}} = \frac{3}{2}c_A^{-1}\|m\|_{L^2(\mathbb{R}^d)}\|\mathcal{M}_{T_n} m^{-1}\|_{L^2(\mathbb{R}^d)},$$

and using the above estimates we then obtain

$$\int_{\mathcal{O}} g\,(\mathcal{A}^* g)\,\widetilde{m}^2 = \sum_{n=1}^{\mathcal{N}}\eta_n\int_{\Omega_n}\left(\int_{\mathbb{R}^d} g\,\widetilde{m}^2\right)\left(\int_{\mathbb{R}^d} g\,\mathcal{M}_{T_n}\right) \le \varpi^*_{\mathscr{G}}\|g\|^2_{L^2_{\widetilde{m}}(\mathbb{R}^d)}.$$

Step 4. Coming back to (2.3.41) and using Steps 1, 2 and 3, we deduce that

$$-\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} g^2\widetilde{m}^2 \le -\Lambda_0\int_{\mathcal{O}}|\nabla_v(g\,\widetilde{m})|^2 + \int_{\mathcal{O}} g^2\widetilde{m}^2\varpi^{\mathcal{L}^*}_{\tilde{m},2} \quad (2.3.42)$$

with

$$\varpi^{\mathcal{L}^*}_{\tilde{m},2} := \varpi^{\mathcal{B}_0^*}_{\tilde{m},2} + \frac{1}{\widetilde{m}^2}v\cdot\nabla_x\widetilde{m}^2 + \varpi^*_{\mathscr{G}}.$$

Gathering the estimates (2.3.4) and those established in Step 2, 3 and Step 4, we deduce that for any $\vartheta\in(0,1)$, there are $\kappa, R>0$ such that

$$\varpi^{\mathcal{L}^*}_{\tilde{m},2} \le \kappa\chi_R + \chi_R^c\vartheta\varpi^{\sharp}_{\omega,2}. \quad (2.3.43)$$

In particular, $\varpi^{\mathcal{L}^*}_{\tilde{m},2}\le\kappa$ and we obtain (2.3.37) by using the Grönwall lemma. We conclude by integrating (2.3.42) in the time interval $(0,T)$ to obtain (2.3.38) as during the Step 6 of the proof of Lemma 2.1. □

Well-posedness of the dual backwards KFP equation with BGK thermostats

We provide now a well-posedness result for the backwards dual Equation (2.3.33).

Proposition 2.10. *We consider an admissible weight function ω, a finite time $T > 0$, and a final datum $g_T \in L^2_m$ with $m = \omega^{-1}$. There is $g \in C([0,T], L^2_m(\mathcal{O})) \cap \mathscr{H}_m(\mathcal{U})$ unique global renormalized solution to the dual backwards Equation* (2.3.33) *associated to the final datum g_T. Furthermore, there is a trace function $\gamma g \in L^2(\Gamma_T; \mathrm{d}\xi^2_m \mathrm{d}t)$, where $\Gamma_T := (0,T) \times \Sigma$, associated to g which is given by [49, Theorem 2.8]. Furthermore g satisfies the conclusions of Lemma 2.8.*

Remark 2.11. In particular we may associate a strongly continuous in time semigroup denoted by $S_{\mathcal{L}^*} : L^2_m(\mathcal{O}) \to L^2_m(\mathcal{O})$ to the solutions of the dual Equation (2.3.33).

Proof of Proposition 2.10. The proof for the well-posedness follows similar lines as in [45, Proposition 3.4] and [49, Theorem 2.11] (see also [17], [133], [90, Sec. 8 & Sec. 11]) and it is thus only sketched.

Step 1. Given $\mathfrak{g} \in L^2(\Gamma_{T,-}; \mathrm{d}\xi^1_m \mathrm{d}t)$, where $\Gamma_{T,-} := (0,T) \times \Sigma_-$ and we recall that $\mathrm{d}\xi^1_m$ is defined in (2.3.16), we consider the backwards inflow problem

$$\begin{cases} -\partial_t g &= v \cdot \nabla_x g + \Lambda \Delta_v g - v \cdot \nabla_v g + \mathscr{G}^* g & \text{in } \mathcal{U}_T \\ \gamma_+ g &= \mathfrak{g} & \text{on } \Gamma_{T,+} \\ g_{t=T} &= g_T & \text{in } \mathcal{O}, \end{cases} \tag{2.3.44}$$

where we have defined

$$\mathscr{G}^* g := \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n} \left(\int_{\mathbb{R}^d} \mathcal{M}_{T_n} \, g \right) - g \sum_{n=1}^{\mathcal{N}} \eta_n \mathbf{1}_{\Omega_n}, \tag{2.3.45}$$

the formal dual operator of $\mathscr{G}$.

We introduce the bilinear form $\mathscr{E} : \mathscr{H}_m(\mathcal{U}_T) \times C^1_c((0,T] \times \mathcal{O} \cup \Gamma_+) \to \mathbb{R}$, defined by

$$\mathscr{E}(g, \varphi) \ := \ \int_{\mathcal{U}_T} g (\partial_t + v \cdot \nabla_x)(\varphi \widetilde{m}^2) + \int_{\mathcal{U}_T} g \, (\lambda - \mathcal{B}_0 - \mathcal{A}) \, (\varphi \widetilde{m}^2),$$

which is coercive for λ large enough thanks to Lemma 2.8 (and more precisely (2.3.42)-(2.3.43)). Using Lions' variant of the Lax-Milgram theorem [126, Chap III, §1], we have that there is a variational solution $g \in \mathscr{H}_m(\mathcal{U}_T)$ to (2.3.44), and more precisely g satisfies

$$\mathscr{E}(g, \varphi) = \int_{\Gamma_{T,+}} \mathfrak{g} \varphi \widetilde{m}^2 \, \mathrm{d}\xi^1_1 + \int_{\mathcal{O}} g_T \, \varphi(T, \cdot) \widetilde{m}^2 \, \mathrm{d}v \, \mathrm{d}x, \quad \forall \, \varphi \in C^1_c((0,T] \times \mathcal{O} \cup \Gamma_+).$$

We then remark that $\mathscr{G}^* g \in L^2$, and applying the trace theorem [49, Theorem 2.8] we deduce that $g \in C([0,T]; L^2_m(\mathcal{O})) \cap \mathscr{H}_m(\mathcal{U}_T)$ and that g is a renormalized solution of Equation (2.3.44).

Step 2. For a sequence $\alpha_k \in (0,1)$, such that $\alpha_k \nearrow 1$, we consider a sequence (g_k) of solutions to the modified Maxwell boundary condition problem

$$\begin{cases} -\partial_t g_k &= v \cdot \nabla_x g_k + \Lambda \Delta_v g_k + v \cdot \nabla_v g_k + \mathscr{G}^* g_k & \text{in } \mathcal{U}_T \\ \gamma_+ g_k &= \alpha_k \mathscr{R}^* \gamma_- g_k & \text{on } \Gamma_{T,+} \\ g_k(t = T, \cdot) &= g_T & \text{in } \mathcal{O}. \end{cases} \tag{2.3.46}$$

We observe that the Step 1 of the proof of Lemma 2.8 implies that

$$\mathscr{R}^* : L^2(\Sigma_-; \mathrm{d}\xi^1_{\widetilde{m}}) \to L^2(\Sigma_+; \mathrm{d}\xi^1_{\widetilde{m}}), \quad \|\mathscr{R}^*\|_{L^2(\Sigma_-, \mathrm{d}\xi^1_{\widetilde{m}})} \leq 1,$$

and using the Step 1 and a Banach fixed point argument as during the Step 2 of the proof of [49, Theorem 2.11] (see also the Step 2 of the proof of Proposition 2.4) we deduce the existence of a function $g_k \in C([0,T], L^2_m(\mathcal{O})) \cap \mathscr{H}_m(\mathcal{U}_T)$, unique weak solution of Equation (2.3.46) with an associated trace $\gamma g \in L^2(\Gamma_T, \mathrm{d}\xi^2_m \mathrm{d}t)$ given by [49, Theorem 2.8]. Furthermore, arguing as during the Step 1, we may apply [49, Theorem 2.8] and we further have that g_k is a renormalized solution of Equation (2.3.46).

By taking then the renormalized formulation associated to Equation (2.3.46) with $\beta(s) = s^2$ and $\varphi = \tilde{m}^2 \chi_R$ for any $R > 0$, we may repeat the computations performed during the proof of Lemma 2.8 and we obtain, taking $R \to \infty$ and using the integral version of the Grönwall lemma that there is $\kappa_0 > 0$ such that this sequence satisfies

$$\|g_{k0}\|^2_{L^2_{\tilde{m}}(\mathcal{O})} + \int_0^T \left\{ \|\gamma g_{ks}\|^2_{L^2(\Sigma_+; \mathrm{d}\xi^2_{\tilde{m}})} + \|g_{ks}\|^2_{H^{1,\dagger}_{\tilde{m}}(\mathcal{O})} \right\} \mathrm{d}s \le e^{\kappa_0 T} \|g_T\|^2_{L^2_{\tilde{m}}(\mathcal{O})}, \tag{2.3.47}$$

and the gradient energy estimate

$$\int_0^T \|\nabla_v g_{ks}\|^2_{L^2_m(\mathcal{O})} ds \lesssim_C \|g_T\|^2_{L^2_m(\mathcal{O})} + \int_0^T \|g_{ks}\|^2_{L^2_m(\mathcal{O})} ds, \tag{2.3.48}$$

for any $k \ge 1$ and a constant $C > 0$ independent of k. We may then extract converging subsequences $(g_{k'})$ and $(\gamma g_{k'})$ with associated limits g and $\bar{\gamma}$ satisfying (2.3.47) and (2.3.48), and passing to the limit in the weak formulation associated to Equation (2.3.46), with the help of the stability result [49, Proposition 2.9], we deduce that $\bar{\gamma} = \gamma g$ and that g is a weak solution to Equation (2.3.33). Additionally we emphasize that this implies the validity of Lemma 2.8.

Step 3. We consider now two solutions g_1 and $g_2 \in C([0,T]; L^2_m(\mathcal{O})) \cap \mathscr{H}_m(\mathcal{U}_T)$ to the backwards dual Equation (2.3.33) associated to the same final datum g_T. We remark then, by the linearity of the problem, that the function $G := g_2 - g_1 \in C([0,T]; L^2_m(\mathcal{O})) \cap \mathscr{H}_m(\mathcal{U}_T)$ is a solution to the backwards Equation (2.3.33) associated to the initial datum $G(T) = 0$, thus (2.3.37), which was obtained as part of Step 2, implies that $G \equiv 0$. ☐

Rigourous dual relation between the semigroups $S_{\mathscr{L}}$ and $S_{\mathscr{L}^*}$

We now prove that the semigroups $S_{\mathscr{L}}$ and $S_{\mathscr{L}^*}$ are dual toward one another.

Proposition 2.12. *The semigroups $S_{\mathscr{L}}$ and $S_{\mathscr{L}^*}$ are dual one toward the other. In other words, for any admissible weight function ω and any finite $T > 0$, there holds* (2.3.35) *for any $f_0 \in L^2_\omega(\mathcal{O})$ and $g_T \in L^2_m(\mathcal{O})$, where $m = \omega^{-1}$.*

Proof. The proof follows by repeating the ideas of the Step 3 of the proof of [45, Theorem 3.5] thus we only sketch it.

We denote f_n the solution to Equation (2.1.14) by replacing the boundary condition by the modified Maxwell boundary conditions

$$\gamma_- f_n = \mathscr{R}_n \gamma_+ f := \left(1 - \frac{1}{n}\right)(1-\iota)\mathscr{S}\gamma_+ f_n + \left(1 - \frac{1}{n}\right)\iota \mathscr{D}\gamma_+ f_n \qquad \text{on } \Gamma_-,$$

we remark that f_n is given by Theorem 2.3.3 (see also Step 2 of Proposition 2.4 or [49, Step 2 of Theorem 2.11]). Moreover, arguing as during the Step 2 of Proposition 2.4 (see also [49, Step 2 of Theorem 2.11]) we further have that $\gamma f_n \in L^2(\Gamma; \mathrm{d}\xi^1_\omega \mathrm{d}t)$. We also define g_n as the solution of the backwards dual equation

$$\begin{cases} -\partial_t g_n = \mathcal{L}^* g_n & \text{in } \mathcal{U}_T \\ \gamma_+ g_n = \mathscr{R}^*_n \gamma_- g_n := \left(1 - \frac{1}{n}\right)(1-\iota)\mathscr{S}\gamma_- g_n + \left(1 - \frac{1}{n}\right)\iota \mathscr{D}^* \gamma_- g_n & \text{on } \Gamma_{T,+} \\ g_{t=T} = g_T & \text{in } \mathcal{O}, \end{cases}$$

which is given by Proposition 2.10. Furthermore, from the Step 2 of the proof of Proposition 2.10, we have that $\gamma g_n \in L^2(\Gamma_T; \mathrm{d}\xi^1_m \mathrm{d}t)$.

Using then the extra boundary estimates and a density argument we have that

$$\begin{aligned}\partial_t(f_n g_n) = -v\cdot\nabla_x(f_n g_n) + \Lambda\Delta_v(f_n)g_n - \Lambda\Delta(g_n)f_n + \mathrm{div}_v(vf_n)g_n \\ + v\cdot\nabla_v(g_n)f_n + \mathscr{G}(f_n)g_n - \mathscr{G}^*(g_n)f_n \qquad \text{in } \mathcal{D}'(\mathcal{U}), \quad (2.3.49)\end{aligned}$$

where we recall that $\mathscr{G}^*$ is given by (2.3.45). Integrating against χ_R for any $R > 0$, and taking $R \to \infty$, we obtain that

$$\int_{\mathcal{O}} f_n(T) g_T = \int_{\mathcal{O}} f_0 g_n(0), \quad \forall\, n \geq 1. \qquad (2.3.50)$$

Since (f_n) is bounded in $L^\infty((0,T); L^2_\omega(\mathcal{O})) \cap W^{1,\infty}((0,T); \mathcal{D}'(\mathcal{O}))$, we deduce that

$$f_n(T) \rightharpoonup f(T) := S_{\mathcal{L}}(T) f_0 \quad \text{weakly in } L^2_\omega(\mathcal{O}).$$

Similarly, we have that $g_n(0) \rightharpoonup g(0) := S_{\mathscr{L}^*}(T) g_T$ weakly in $L^2_m(\mathcal{O})$. We may thus pass to the limit $n \to \infty$ in (2.3.50) and we deduce that (2.3.35) holds, which exactly means that $(S_{\mathscr{L}})^* = S_{\mathscr{L}^*}$. □

2.4 Proof of Theorem 2.1.1

2.4.1 Krein-Rutmann-Doblin-Harris Theorem in a Banach lattice

We now present a general Krein-Rutman-Doblin-Harris result taken from [160, Section 6] obtained for a conservative setting, but we also refer to [38, 45, 90, 111] and the references therein for more general results and their applications.

We consider a Banach lattice X, which means that X is a Banach space endowed with a closed positive cone X_+ (we write $f \geq 0$ if $f \in X_+$ and we recall that $f = f_+ - f_-$ with $f_\pm \in X_+$ for any $f \in X$. We also denote $|f| := f_+ + f_-$). We assume that X is in duality with another Banach lattice Y, with closed positive cone Y_+, so that the bracket $\langle \phi, f\rangle$ is well defined for any $f \in X$, $\phi \in Y$, and that $f \in X_+$ (resp. $\phi \geq 0$) iff $\langle \psi, f\rangle \geq 0$ for any $\psi \in Y_+$ (resp. iff $\langle \phi, g\rangle \geq 0$ for any $g \in X_+$), typically $X = Y'$ or $Y = X'$. We write $\psi \in Y_{++}$ if $\psi \in Y$ satisfies $\langle \psi, f\rangle > 0$ for any $f \in X_+\backslash\{0\}$.

We consider a positive and conservative (or stochastic) semigroup $S = (S_t) = (S(t))$ on X, that means that S_t is a bounded linear mapping on X such that

$$S_t : X_+ \to X_+ \quad \text{for any } t \geq 0,$$

and there exist $\phi_1 \in Y_{++}$, $\|\phi_1\| = 1$, and a dual semigroup $S^* = S^*_t = S^*(t)$ on Y such that $S^*_t \phi_1 = \phi_1$ for any $t \geq 0$. More precisely, we assume that S^*_t is a bounded linear mapping on Y such that $\langle S(t) f, \phi\rangle = \langle f, S^*(t)\phi\rangle$, for any $f \in X$, $\phi \in Y$ and $t \geq 0$, and in particular $S^*_t : Y_+ \to Y_+$ for any $t \geq 0$.

We denote by $\mathfrak{L}$ the generator of S with domain $D(\mathfrak{L})$. For $\psi \in Y_+$, we define the seminorm

$$[f]_\psi := \langle |f|, \psi\rangle, \quad \forall\, f \in X,$$

and we introduce now some assumptions on the semigroup S.

• First we introduce the strong dissipativity condition

$$\|S(t)f\| \;\leq\; C_0 e^{\lambda t}\|f\| + C_1 \int_0^t e^{\lambda(t-s)} [S(s)f]_{\phi_1} ds, \qquad (2.4.1)$$

for any $f \in X$ and $t \geq 0$, where $\lambda < 0$ and $C_i \in (0, \infty)$, $i = 0, 1$.

• Next, we make the slightly relaxed Doblin-Harris positivity assumption

$$S_T f \geq \eta_{\varepsilon,T} g_\varepsilon [S_{T_0} f]_{\psi_\varepsilon}, \quad \forall f \in X_+, \tag{2.4.2}$$

for any $T \geq T_1 > T_0 \geq 0$ and $\varepsilon > 0$, where $\eta_{\varepsilon,T} > 0$, and (g_ε) and (ψ_ε) are bounded decreasing families of functions of $X_+ \backslash \{0\}$ and $Y_+ \backslash \{0\}$ respectively.

• We finally assume the compatibility interpolation like condition

$$[f]_{\phi_1} \leq \xi_\varepsilon \|f\| + \Xi_\varepsilon [f]_{\psi_\varepsilon}, \ \forall f \in X, \ \varepsilon \in (0, 1], \tag{2.4.3}$$

for two families of positive real numbers (ξ_ε) and (Ξ_ε) such that $\xi_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$.

We refer to [45, Section 7] for a detailed discussion about these assumptions. Then we have the following theorem from [160, Theorem 6.1], see also [45, Theorem 7.1].

Theorem 2.4.1. *Consider a semigroup S on a Banach lattice X which satisfies* (2.4.1)-(2.4.2)-(2.4.3). *There exists a unique normalized positive stationary state $\mathfrak{f}_1 \in D(\mathfrak{L})$, that is*

$$\mathfrak{L}\mathfrak{f}_1 = 0, \quad \mathfrak{f}_1 \geq 0, \quad \langle \phi_1, \mathfrak{f}_1 \rangle = 1. \tag{2.4.4}$$

Furthermore, there exist some constructive constants $C \geq 1$ and $\lambda_2 > 0$ such that

$$\|S(t)f - \langle f, \phi_1 \rangle \mathfrak{f}_1\| \leq C e^{-\lambda_2 t} \|f - \langle f, \phi_1 \rangle \mathfrak{f}_1\| \tag{2.4.5}$$

for any $f \in X$ and $t \geq 0$.

2.4.2 Proof of Theorem 2.1.1

We will now prove our first main result, Theorem 2.1.1. The proof will be done by verifying that the semigroup $S_{\mathcal{L}}$ defined by Theorem 2.3.3, satisfies the necessary hypothesis to use Theorem 2.4.1. It is worth remarking that the verification of the dissipativity and the interpolation conditions will be done similarly as in the Steps 1 and 2 of the proof of [160, Theorem 1.2] and the proof of [45, Theorem 1.2].

During the rest of this subsection we consider any fixed admissible weight function ω and we define $X = L^2_\omega(\mathcal{O})$. Moreover, we remark that since Equation (2.1.14) conserves mass then we immediately deduce that $\phi_1 = 1$.

Strong dissipativity condition

We consider two real constants $M, R > 0$ to be specified later and we define the operators

$$\mathscr{A} f = \mathcal{A} f + M\chi_R f \qquad \text{and} \qquad \mathscr{B} f = \mathcal{B} f - M\chi_R f,$$

where we recall that χ_R is defined in Section 2.3. We study then the local equation

$$\partial_t f = \mathscr{B} f \qquad \text{in } \mathcal{U}, \tag{2.4.6}$$

complemented with the Maxwell boundary conditions (2.1.2) and the initial datum (2.1.4). We observe that Equation (2.4.6)-(2.1.2)-(2.1.4) fits the framework developed in [45], thus by repeating the arguments leading to the proof of [45, Theorem 1.1] we deduce the existence of a strongly continuous semigroup $S_{\mathscr{B}}$ associated to the solutions of Equation (2.4.6)-(2.1.2)-(2.1.4) and with generator $\mathscr{B}$.

Moreover, by exactly repeating the arguments leading to the proof of Lemma 2.1 (see also the proof of [45, Lemma 2.1]) we deduce that for any $p \in \{1,2\}$ and for every $a > 0$ there are $M, R > 0$ such that

$$\|S_{\mathscr{B}}(t) f_0\|_{L^p_\omega(\mathcal{O})} \leq C_1 e^{-at} \|f_0\|_{L^p_\omega(\mathcal{O})} \qquad \forall t \geq 0, \tag{2.4.7}$$

for some constant $C_1 > 0$. Indeed, by taking $\widetilde{\omega}$ as defined in (2.3.7) we have by using integration by parts

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_{\mathcal{O}} f^p \widetilde{\omega}^p &= \int_{\mathcal{O}} (\mathcal{B}_0 f)\, f^{p-1}\, \widetilde{\omega}^p + \frac{1}{p}\int_{\mathcal{O}} f^p \widetilde{\omega}^p \left(\frac{v\cdot\nabla_x \widetilde{\omega}}{\widetilde{\omega}}\right) - \frac{1}{p}\int_\Sigma (\gamma f)^p\, \widetilde{\omega}^p (n_x\cdot v) - \int_{\mathcal{O}} M\chi_R f^p \\ &\leq \int_{\mathcal{O}} f^p\, \widetilde{\omega}^p\, \varpi^{\mathscr{B}}_{\widetilde{\omega},p}, \end{aligned}$$

where we have defined

$$\varpi^{\mathscr{B}}_{\widetilde{\omega},p} = \varpi^{\mathcal{B}_0}_{\widetilde{\omega},p} + \frac{1}{\widetilde{\omega}}\, v\cdot\nabla_x \widetilde{\omega} - M\chi_R.$$

Then by proceeding as during the Steps 2 and 3 of the proof of Lemma 2.1 we deduce that for every $\vartheta \in (0,1)$ there are $\kappa, A > 0$ such that

$$\varpi^{\mathscr{B}}_{\widetilde{\omega},p} = \kappa\chi_A + \vartheta\chi_A^c \varpi^\sharp_{\omega,p} - M\chi_R,$$

and we obtain (2.4.7) by choosing $\vartheta = 1/2$, $M = -\kappa - a$, $R = A$, and using the Gronwall lemma together with (2.3.8).

Furthermore, using (2.4.7) and following the arguments leading to the proof of [45, Proposition 4.10] we deduce that, up to possibly choosing $M, R > 0$ larger, we also have that there are constants $\nu > 0$ and $C_2 \geq 1$ such that

$$\|S_{\mathscr{B}}(t) f_0\|_{L^2_\omega(\mathcal{O})} \leq C_2^0 \|S_{\mathscr{B}}(t) f_0\|_{L^\infty_{\omega_1}(\mathcal{O})} \leq C_2^0 C_2 \frac{e^{-at}}{t^\nu} \|f_0\|_{L^1_{\omega_2}(\mathcal{O})} \quad \forall\, t > 0, \tag{2.4.8}$$

where we have defined $\omega_1 = (\omega)^2$ and $\omega_2 = (\omega_1)^2$ if $s > 0$ or $\omega_2 = \omega_1 \langle v\rangle^K$ if $s = 0$ with $K = 4(3d+1)(2d+3)$, and we have taken the constant $C_2^0 = \|\omega(\omega_1)^{-1}\|_{L^2(\mathbb{R}^d)}$.

We define $\widetilde{S}_{\mathscr{B}} = S_{\mathscr{B}}\mathscr{A}$ and arguing as during (2.3.24) we observe that (2.4.7) and (2.4.8) together with Proposition 2.1 imply that for every $p \in \{1,2\}$ there holds

$$\|\widetilde{S}_{\mathscr{B}}(t) f_0\|_{L^p_\omega(\mathcal{O})} \leq C_2 e^{-at} \|f_0\|_{L^p_\omega(\mathcal{O})} \quad \text{and} \quad \|\widetilde{S}_{\mathscr{B}}(t) f_0\|_{L^2_\omega(\mathcal{O})} \leq C_2^0 C_2 \frac{e^{-at}}{t^\nu} \|f_0\|_{L^1_\omega}. \tag{2.4.9}$$

We then remark that (2.4.9) implies that we may use [135, Proposition 2.5] and we obtain that there is $N \in \mathbb{N}$ such that

$$\|(\widetilde{S}_{\mathscr{B}})^{*N}(t) f_0\|_{L^2_\omega(\mathcal{O})} \leq C_3 e^{-a't} \|f_0\|_{L^1_\omega},$$

for any $a' < a$ and some constant $C_3 > 0$. Using then the iterated Duhamel formula as during the proof of Proposition 2.5 we have that

$$S_{\mathcal{L}} = \mathscr{V} + \mathscr{W} * S_{\mathcal{L}},$$

with

$$\mathscr{V} := S_{\mathscr{B}} + \cdots + (S_{\mathscr{B}}\mathscr{A})^{*(N)} * S_{\mathscr{B}}, \quad \text{and} \quad \mathscr{W} := (S_{\mathscr{B}}\mathscr{A})^{*(N+1)}.$$

On the one hand, using (2.4.7) and Proposition 2.1 we immediately deduce that

$$\|\mathscr{V}(t) f_0\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-at} \|f_0\|_{L^2_\omega(\mathcal{O})}.$$

On the other hand, we observe that

$$\begin{aligned}\|\mathscr{W}(t)f\|_{L^2_\omega(\mathcal{O})} &\le \int_0^t \|(\widetilde{S}_{\mathscr{B}}\mathscr{A})^{*N}(t-s)S_{\mathscr{B}}(s)\mathscr{A}f\|_{L^2_\omega(\mathcal{O})}\mathrm{d}s \\ &\lesssim \int_0^t e^{-a'(t-s)}e^{-as}\|\mathcal{A}f\|_{L^1_\omega(\mathcal{O})} \lesssim e^{-a't}\|f\|_{L^1(\mathcal{O})}\end{aligned}$$

where we have successively used the very definition of $\mathscr{W}$, (2.4.9) and Proposition 2.1. Finally choosing $a = 2$ and $a' = 1$ and putting together the previous computations we obtain that

$$\|S_{\mathcal{L}}f_0\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-t}\|f_0\|_{L^2_\omega(\mathcal{O})} + \int_0^t e^{-(t-s)}\|S_{\mathcal{L}}(s)f_0\|_{L^1(\mathcal{O})},$$

which is nothing but the strong dissipativity condition (2.4.1).

Relaxed Doblin-Harris positivity condition.

We fix $0 \le f_0 \in L^2_\omega$ and denote $f_t := S_{\mathcal{L}}(t)f_0$, and $\phi_1^{\mathcal{B}}$ the steady solution of the dual backwards equation of Equation (2.2.4) given by Theorem 2.2.2-(3). For any $T > 0$ we set $0 < T_0 < T_1 \le T$, we fix $\varepsilon > 0$ and an admissible weight function ς such that $\omega \prec \varsigma$, and we compute

$$\begin{aligned}\varsigma f_{T_1} \ge \phi_1^{\mathcal{B}} f_{T_1} &= \phi_1^{\mathcal{B}}\left(S_{\mathcal{B}}(T_1-T_0)f_{T_0} + \int_0^{T_1-T_0} S_{\mathcal{B}}(T_1-T_0-s)\mathcal{A}f_{T_0+s}\,ds\right) \\ &\ge \phi_1^{\mathcal{B}}S_{\mathcal{B}}(T_1-T_0)f_{T_0} \ge \left(\inf_{\mathcal{O}_\varepsilon}\phi_1^{\mathcal{B}}\right)\left(\inf_{\mathcal{O}_\varepsilon} S_{\mathcal{B}}(T_1-T_0)f_{T_0}\right)\mathbf{1}_{\mathcal{O}_\varepsilon},\end{aligned}$$

where we have used (2.2.9) and the Duhamel formula to obtain the first line and we have used the fact that $f_{T_0+s} \ge 0$ for every $s \in [0, T_1 - T_0]$, due to the weak maximum principle established in Proposition 2.6, the fact that $\mathcal{A}$ is a positive operator, and the fact that $S_{\mathcal{B}}$ is a positive semigroup (see Theorem 2.2.2-(2)) to obtain the second line.

Then using the Harnack inequality from Theorem 2.2.2-(2) and the fact that $0 < \phi_1^{\mathcal{B}}$ is a continuous function in $\mathcal{O}$, we further deduce that there are constants $C_1 := C_1(T_0, T_1, \varepsilon) > 0$ and $C_2 > 0$ for which there holds

$$\begin{aligned}\varsigma f_{T_1} &\ge \frac{1}{C_2}\left(\sup_{\mathcal{O}_\varepsilon}\phi_1^{\mathcal{B}}\right)\frac{1}{C_1}\left(\sup_{\mathcal{O}_\varepsilon} S_{\mathcal{B}}((T_1-T_0)/2)f_{T_0}\right)\mathbf{1}_{\mathcal{O}_\varepsilon} \\ &\ge \frac{1}{C_1C_2}\mathbf{1}_{\mathcal{O}_\varepsilon}\sup_{\mathcal{O}_\varepsilon}\left(\phi_1^{\mathcal{B}}S_{\mathcal{B}}((T_1-T_0)/2)f_{T_0}\right) \\ &\ge \frac{1}{C_1C_2}\mathbf{1}_{\mathcal{O}_\varepsilon}\frac{1}{|\mathcal{O}_\varepsilon|}\int_{\mathcal{O}_\varepsilon}\phi_1^{\mathcal{B}}\,S_{\mathcal{B}}((T_1-T_0)/2)f_{T_0} \\ &\ge \frac{1}{C_1C_2}\mathbf{1}_{\mathcal{O}_\varepsilon}\frac{1}{|\mathcal{O}_\varepsilon|}\int_{\mathcal{O}_\varepsilon} f_{T_0}\,S_{\mathcal{B}^*}((T_1-T_0)/2)\phi_1^{\mathcal{B}} = \frac{e^{-\lambda_1(T_1-T_0)/2}}{C_1C_2}\mathbf{1}_{\mathcal{O}_\varepsilon}\frac{1}{|\mathcal{O}_\varepsilon|}\langle f_{T_0}, \phi_1^{\mathcal{B}}\mathbf{1}_{\mathcal{O}_\varepsilon}\rangle,\end{aligned}$$

where λ_1 is given by Theorem 2.2.2-(3), $S_{\mathcal{B}^*}$ is the dual semigroup of $S_{\mathcal{B}}$, which existence is given by [45, Proposition 3.4] and the duality relation is given by [45, Theorem 3.5-(3)]. Moreover, to obtain the final equality we have used the fact that $g(t,\cdot) := e^{-\lambda_1 t}\phi_1^{\mathcal{B}}(\cdot)$ is a solution to the backwards in time problem

$$-\partial_t g = \mathcal{B}^* g \qquad \text{in } (0,T)\times\mathcal{O},$$

complemented with the dual Maxwell boundary conditions (2.3.34) and associated to the final datum $g_T = \phi_1^{\mathcal{B}}$. Altogether, we have obtained that

$$f_{T_1} \geq \frac{e^{-\lambda_1(T_1-T_0)/2}}{C_1C_2} \frac{1}{|\mathcal{O}_\varepsilon|} \varsigma^{-1} \mathbf{1}_{\mathcal{O}_\varepsilon} \langle S_{T_0} f_0, \phi_1^{\mathcal{B}} \mathbf{1}_{\mathcal{O}_\varepsilon} \rangle$$

which is (2.4.2) with the constant $\eta_{\varepsilon,T} = e^{-\lambda_1(T_1-T_0)/2}/(C_1C_2|\mathcal{O}_\varepsilon|)$, and the families of functions $g_\varepsilon = \varsigma^{-1}\mathbf{1}_{\mathcal{O}_\varepsilon} \in L^2_\omega(\mathcal{O})$ and $\psi_\varepsilon = \phi_1^{\mathcal{B}} \mathbf{1}_{\mathcal{O}_\varepsilon} \in L^2_m(\mathcal{O})$, with $m = \omega^{-1}$.

Interpolation condition.

For any $f \in L^2_\omega$, we have

$$\int_{\mathcal{O}} |f| = \int_{\mathcal{O}_\varepsilon^c} |f| + \int_{\mathcal{O}_\varepsilon} |f| \leq \Big(\int_{\mathcal{O}_\varepsilon^c} \omega^{-2}\Big)^{1/2} \|f\|_{L^2_\omega} + \int |f| \mathbf{1}_{\mathcal{O}_\varepsilon},$$

so that (2.4.3) holds true with $\Xi_\varepsilon := 1$, $\xi_\varepsilon := \|\mathbf{1}_{\mathcal{O}_\varepsilon^c}\|_{L^2_\omega} \to 0$ because $\omega^{-1} \in L^2(\mathcal{O})$.

Krein-Rutmann-Doblin-Harris result for Equation (2.1.14)

Due to the results in sub-subsections 2.4.2, 2.4.2 and 2.4.2 we deduce the following result.

Proposition 2.2. *There exists a unique normalized positive stationary state $\mathfrak{C}$ to Equation* (2.1.14) *such that $\langle\!\langle \mathfrak{C} \rangle\!\rangle_{\mathcal{O}} = 1$. Moreover, for any admissible weight function ς there holds*

$$\|\nabla_v \mathfrak{C}\|_{L^2_\varsigma(\mathcal{O})} < \infty \qquad \text{and} \qquad \mathfrak{C}(x,v) \lesssim (\varsigma(v))^{-1}. \tag{2.4.10}$$

Furthermore, let ω be an admissible weight function, for any initial data $f_0 \in L^2_\omega(\mathcal{O})$ there is a unique global renormalized solution $f \in C(\mathbb{R}_+, L^2_\omega(\mathcal{O}))$ to Equation (2.1.14) *and there is $\lambda > 0$ such that*

$$\|S(t)f - \langle\!\langle f_0 \rangle\!\rangle_{\mathcal{O}}\, \mathfrak{C}\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-\lambda t} \|f - \langle\!\langle f_0 \rangle\!\rangle_{\mathcal{O}}\, \mathfrak{C}\|_{L^2_\omega(\mathcal{O})} \tag{2.4.11}$$

for any $t \geq 0$.

Proof. The existence of $\mathfrak{C}$ is a direct application of Theorem 2.4.1 to Equation (2.1.14) by using the results from sub-subsections 2.4.2, 2.4.2 and 2.4.2. The estimates (2.4.10) are a consequence of Lemma 2.1 and Proposition 2.5. Furthermore, the global existence is given by Theorem 2.3.3 and the decay estimate (2.4.11) is a consequence of Theorem 2.4.1. □

Proof of Theorem 2.1.1

We remark that since $\alpha = 0$, Equation (2.1.1) coincides with Equation (2.1.14) by taking $\Lambda = \tau$, thus Proposition 2.2 implies the existence of $\mathfrak{F}^0 \in L^2_\omega(\mathcal{U})$ unique stationary solution to this linear equation. □

2.4.3 Decay for the dual semigroup $S_{\mathcal{L}^*}$

In this subsection we deduce a decay estimate for the solutions of the dual Equation (2.3.33) in the spirit of Proposition 2.2 that we present in the following proposition.

Proposition 2.3. *We consider a finite $T > 0$, an admissible weight function ω and we define $m = \omega^{-1}$. There are constants $\lambda, C > 0$ such that for any g solution to Equation* (2.3.33) *there holds*

$$\|g_0 - \langle g_T, \mathfrak{C}\rangle_{L^2(\mathcal{O})}\|_{L^2_m(\mathcal{O})} \leq Ce^{-\lambda T}\|g_T - \langle g_T, \mathfrak{C}\rangle_{L^2(\mathcal{O})}\|_{L^2_m(\mathcal{O})} \qquad \forall t \geq 0, \tag{2.4.12}$$

where $\mathfrak{C}$ is given by Proposition 2.2.

Proof. We remark that

$$\langle g(0), \mathfrak{C}\rangle_{L^2(\mathcal{O})} = \langle S_{\mathcal{L}^*} g_T, \mathfrak{C}\rangle_{L^2(\mathcal{O})} = \langle g_T, S_{\mathcal{L}}\mathfrak{C}\rangle_{L^2(\mathcal{O})} = \langle g_T, \mathfrak{C}\rangle_{L^2(\mathcal{O})},$$

where we have used the duality relation (2.3.35) together with the fact that 1 and $\mathfrak{C}$ are steady states of Equation (2.3.33) and Equation (2.1.14) respectively. This implies in particular that the function $g(0) - \langle g_T, \mathfrak{C}\rangle_{L^2(\mathcal{O})}$ is also a solution of Equation (2.3.33) and

$$\langle g(0) - \langle g_T, \mathfrak{C}\rangle_{L^2(\mathcal{O})}, \mathfrak{C}\rangle_{L^2(\mathcal{O})} = \langle g(0), \mathfrak{C}\rangle_{L^2(\mathcal{O})} + \langle\!\langle \mathfrak{C}\rangle\!\rangle_{\mathcal{O}} \langle g_T, \mathfrak{C}\rangle_{L^2(\mathcal{O})} = 0,$$

due to the fact that $\langle\!\langle \mathfrak{C}\rangle\!\rangle_{\mathcal{O}} = 1$ as provided by Proposition 2.2. In particular, arguing exactly in the same way we also have that

$$\langle g_T - \langle g_T, \mathfrak{C}\rangle_{L^2(\mathcal{O})}, \mathfrak{C}\rangle_{L^2(\mathcal{O})} = 0. \tag{2.4.13}$$

We then compute

$$\begin{aligned}
\|g(0) - \langle g_T, \mathfrak{C}\rangle\|_{L^2_m(\mathcal{O})} &= \sup_{f_0 \in L^2_\omega, \|f_0\|_{L^2_\omega(\mathcal{O})} \leq 1} \int_{\mathcal{O}} f_0 \left(g(0) - \langle g_T, \mathfrak{C}\rangle\right) \\
&= \sup_{f_0 \in L^2_\omega, \|f_0\|_{L^2_\omega(\mathcal{O})} \leq 1} \int_{\mathcal{O}} f(T) \; \left(g_T - \langle g_T, \mathfrak{C}\rangle\right) \\
&= \sup_{f_0 \in L^2_\omega, \|f_0\|_{L^2_\omega(\mathcal{O})} \leq 1} \int_{\mathcal{O}} \left(f(T) - \langle\!\langle f_0\rangle\!\rangle \mathfrak{C}\right) \; \left(g_T - \langle g_T, \mathfrak{C}\rangle\right) \\
&\leq \|g_T - \langle g_T, \mathfrak{C}\rangle\|_{L^2_m(\mathcal{O})} \sup_{f_0 \in L^2_\omega, \|f_0\|_{L^2_\omega(\mathcal{O})} \leq 1} \|f(T) - \langle\!\langle f_0\rangle\!\rangle \mathfrak{C}\|_{L^2_\omega(\mathcal{O})} \\
&\lesssim \|g_T - \langle g_T, \mathfrak{C}\rangle\|_{L^2_m(\mathcal{O})} \sup_{f_0 \in L^2_\omega, \|f_0\|_{L^2_\omega(\mathcal{O})} \leq 1} e^{-\lambda T}\|f_0 - \langle\!\langle f_0\rangle\!\rangle \mathfrak{C}\|_{L^2_\omega(\mathcal{O})} \\
&\lesssim e^{-\lambda T}\|g_T - \langle g_T, \mathfrak{C}\rangle\|_{L^2_m(\mathcal{O})},
\end{aligned}$$

where we have defined $f_T = S_{\mathcal{L}}(T) f_0$, and we have successively used the Riesz representation theorem, the duality identity (2.3.35), the conservation of mass, (2.4.13), the Cauchy-Schwartz inequality, and the decay estimate (2.4.11). ☐

2.5 Proof of Theorem 2.1.2

We now consider a constant $\nu \in [0, 2\mathcal{E}_{\mathfrak{F}^0}]$, where $\mathfrak{F}^0$ is given by Theorem 2.1.1, and we study the problem

$$\left\{\begin{array}{lcll}
\partial_t f &=& -v \cdot \nabla_x f + \alpha \mathcal{C}_\nu f + (1-\alpha)\mathcal{C}_\tau f + \mathscr{G} f & \text{on } \mathcal{U}, \\
\gamma_- f &=& \mathscr{R} \gamma_+ f & \text{on } \Gamma_-, \\
f_{t=0} &=& f_0 & \text{on } \mathcal{O},
\end{array}\right. \tag{2.5.1}$$

where $\alpha \in (0, 1/2)$ and we remark that

$$\alpha \mathcal{C}_\nu f + (1-\alpha)\mathcal{C}_\tau f = (\alpha\nu + (1-\alpha)\tau)\Delta_v f + \mathrm{div}_v(vf) \quad \text{with} \quad \alpha\nu + (1-\alpha)\tau \in [\tau_0/2, \tau_1 + 2\mathscr{E}_{\mathfrak{F}^0}].$$

We observe that Equation (2.5.1) fits the framework developed in Sections 2.3 and 2.4, with the choice $\Lambda(x) = \alpha\nu + (1-\alpha)\tau(x)$ and we remark that, as as pointed out above, the bounds on Λ are independent of α and ν. We then define the map $\mathscr{F} : [0, 2\mathcal{E}_{\mathfrak{F}^0}] \to \mathbb{R}$ by

$$\mathscr{F}(\nu) = \mathcal{E}_{\mathfrak{F}^\alpha_\nu}, \tag{2.5.2}$$

where $\mathfrak{F}^\alpha_\nu$ is the steady solution of Equation (2.5.1) given by Proposition 2.2. We will then prove Theorem 2.1.2 by proving that there is $\alpha^\star > 0$ small enough such that for every $\alpha \in (0, \alpha^\star)$ the map $\mathscr{F}$ has a fixed point $\nu^\star$, which in particular will imply that $\mathfrak{F}^\alpha_{\nu^\star}$ is a steady solution of the non-linear Equation (2.1.1)-(2.1.2).

2.5.1 Proof of Theorem 2.1.2

In this subsection we will proceed to verify the necessary hypothesis to deduce the existence of the fixed point for $\mathscr{F}$ by using a fixed point theorem.

Continuity of the map $\mathscr{F}$

We prove the continuity of the map defined on (2.5.2) for which we will first prove the following lemma.

Lemma 2.1. *We consider f_1, f_2 solutions of Equation* (2.5.1) *associated with $\nu_1, \nu_2 \in [0, 2\mathcal{E}_{\mathfrak{F}^0}]$ respectively. For any admissible weight function ω there are constants $\kappa \geq 0$ and $C \geq 1$ such that there holds*

$$\|f_{2,t} - f_{1,t}\|_{L^2_\omega} \leq \alpha^2 |\nu_1 - \nu_2|^2 C e^{\kappa t} \|f_0\|_{L^2_\omega} \qquad \forall t \geq 0. \tag{2.5.3}$$

Proof. We take $F = f_1 - f_2$ and we observe that F solves, in the weak sense, the following equation

$$\begin{cases} \partial_t F &=& -v \cdot \nabla_x F + \mathcal{B}_1 F + \mathcal{A} F + \alpha(\nu_1 - \nu_2)\Delta_v f_2 & \text{in } \mathcal{U} \\ \gamma_- F &=& \mathscr{R}\gamma_+ F & \text{on } \Gamma_- \\ F_{t=0} &=& 0 & \text{in } \mathcal{O}, \end{cases} \tag{2.5.4}$$

where we have defined $\mathcal{B}_1 f = \mathcal{C}_{\Lambda_1} f - f \sum_{n=1}^N \eta_n \mathbf{1}_{\Omega_n}$ with

$$\Lambda_1(x) := \alpha\nu_1 + \tau(x) \quad \text{and we remark that} \quad \frac{\tau_0}{2} \leq \Lambda(x) \leq 2\mathcal{E}_{\mathfrak{F}^0} + \tau_1 \quad \text{for all } x \in \Omega.$$

At the level of a priori estimates we proceed as during the proof of Lemma 2.1 and we have that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} F_t^2 \widetilde{\omega}^2 \leq -\frac{\tau_0}{2}\int_{\mathcal{O}} |\nabla_v (F\widetilde{\omega})|^2 + \int_{\mathcal{O}} \varpi^{\mathcal{L}_1}_{\widetilde{\omega},2} F^2 \widetilde{\omega}^2 + \alpha(\nu_1 - \nu_2)\int_{\mathcal{O}} (\Delta_v f_2) F \widetilde{\omega}^2,$$

where

$$\varpi^{\mathcal{L}_1}_{\widetilde{\omega},2} = \varpi^{\mathcal{B}_1}_{\widetilde{\omega},2} + \left(\frac{v \cdot \nabla_x \widetilde{\omega}}{\widetilde{\omega}}\right) + \varpi_{\mathscr{G}}.$$

Again as during the proof of Lemma 2.1 we deduce that $\varpi^{\mathcal{L}_1}_{\widetilde{\omega},2} \le \kappa_1$, for some constant $\kappa_1 > 0$ independent of ν_1, ν_2 and α. Using now integration by parts, we compute

$$\begin{aligned}
\int_{\mathcal{O}} (\Delta_v f_2) F \widetilde{\omega}^2 &= -\int_{\mathcal{O}} \nabla f_2 \cdot \nabla(F\widetilde{\omega})\,\widetilde{\omega} - \int_{\mathcal{O}} \nabla f_2 \cdot \nabla_v \widetilde{\omega}\,(F\,\widetilde{\omega}) \\
&\le \|\nabla_v f_2\|_{L^2_\omega(\mathcal{O})} \|\nabla_v (F\widetilde{\omega})\|_{L^2(\mathcal{O})} + \int_{\mathcal{O}} f_2\, \nabla_v (F\widetilde{\omega}) \nabla_v \widetilde{\omega} + \int_{\mathcal{O}} f_2\, F\, \widetilde{\omega}\, \Delta_v \widetilde{\omega} \\
&\le \|\nabla_v f_2\|_{L^2_\omega(\mathcal{O})} \|\nabla_v (F\widetilde{\omega})\|_{L^2(\mathcal{O})} + \left\| \frac{\nabla_v \widetilde{\omega}}{\widetilde{\omega}} \right\|_{L^\infty(\mathcal{O})} \|\nabla_v (F\widetilde{\omega})\|_{L^2(\mathcal{O})} \|f_2\|_{L^2_{\widetilde{\omega}}(\mathcal{O})} \\
&\quad + \left\| \frac{\Delta_v \widetilde{\omega}}{\widetilde{\omega}} \right\|_{L^\infty(\mathcal{O})} \|F\|_{L^2_{\widetilde{\omega}}(\mathcal{O})} \|f_2\|_{L^2_{\widetilde{\omega}}(\mathcal{O})}
\end{aligned}$$

where we have used the Cauchy Schwartz inequality in the second and third line. We recall from the Step 6 of the proof of Lemma 2.1 that we may write $\widetilde{\omega} = \wp \omega_A$ and $\omega_A = \wp_A \omega$ where $\wp^2 := 1 + (n_x \cdot v)/(2\langle v \rangle^4)$ and $\wp_A^2 := 1 + \chi_A(\mathscr{M}_\Theta^{-1} \omega^{-2} - 1)$. We then have that

$$\left| \frac{\Delta_v \widetilde{\omega}}{\widetilde{\omega}} \right| \le \left| \frac{\Delta_v \wp}{\wp} \right| + \left| \frac{\Delta_v \wp_A}{\wp_A} \right| + \left| \frac{\Delta_v \omega}{\omega} \right| + 2\left| \frac{\nabla_v \wp}{\wp} \cdot \frac{\nabla_v \wp_A}{\wp_4} \right| + 2\left| \frac{\nabla_v \wp}{\wp} \cdot \frac{\nabla_v \omega}{\omega} \right| + 2\left| \frac{\nabla_v \omega}{\omega} \cdot \frac{\nabla_v \wp_A}{\wp_4} \right|,$$

thus using our hypothesis on ω, the compact support of $\wp_A$, and the very definition of $\wp$ together with (2.3.15) we deduce that

$$\|\nabla_v \widetilde{\omega}/\widetilde{\omega}\|_{L^\infty(\mathcal{O})} + \|\Delta_v \widetilde{\omega}/\widetilde{\omega}\|_{L^\infty(\mathcal{O})} \le C_\omega < \infty, \tag{2.5.5}$$

for some constant $C_\omega > 0$. Using the previous informations together with the Young inequality we deduce

$$\begin{aligned}
\int_{\mathcal{O}} (\Delta_v f_2) F \widetilde{\omega}^2 \le\ & \frac{a_1}{2} \|\nabla_v (F\widetilde{\omega})\|^2_{L^2(\mathcal{O})} + \frac{1}{2a_1} \|\nabla_v f_2\|^2_{L^2_\omega(\mathcal{O})} + \frac{a_2 C_\omega}{2} \|\nabla_v (F\widetilde{\omega})\|^2_{L^2(\mathcal{O})} \\
&+ \frac{C_\omega}{2a_2} \|f_2\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \frac{a_3 C_\omega}{2} \|F\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \frac{C_\omega}{2a_3} \|f_2\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})}
\end{aligned}$$

for any constants $a_j > 0$, $j = 1, 2, 3$. Then we choose $a_1 = 1/(\alpha|\nu_1 - \nu_2|)$, $a_2 = 1/(\alpha C_\omega |\nu_1 - \nu_2|)$ and $a_3 = 1/(\alpha|\nu_1 - \nu_2|)$ and putting together the previous informations we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \int_{\mathcal{O}} F_t^2\, \widetilde{\omega}^2 \le -\frac{\tau_0}{2} \int_{\mathbb{R}^d} |\nabla_v (F\widetilde{\omega})|^2 + \int_{\mathcal{O}} \left(\varpi^{\mathcal{L}_1}_{\widetilde{\omega},2} + \frac{C_\omega}{2} \right) F^2 \widetilde{\omega}^2 \\
+ \frac{\alpha^2}{2} |\nu_1 - \nu_2|^2 \Big[\|\nabla_v f_2\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + (C_\omega^2 + C_\omega) \|f_2\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} \Big].
\end{aligned}$$

Using the Grönwall lemma and (2.3.8) we deduce that

$$\begin{aligned}
\|F\|^2_{L^2_\omega(\mathcal{O})} \le 2c_A^2 \frac{\alpha^2}{2} |\nu_1 - \nu_2|^2 \int_0^t e^{(\kappa_1 + C_\omega/2)(t-s)} \\
\times \Big[\|\nabla_v f_{2,s}\|_{L^2_\omega(\mathcal{O})} + (C_\omega^2 + C_\omega) \|f_{2,s}\|_{L^2_\omega(\mathcal{O})} \Big]\, \mathrm{d}s.
\end{aligned} \tag{2.5.6}$$

We obtain (2.5.3) by using (2.3.5) with $p = 2$ and (2.3.6) from Lemma 2.1.

We conclude by remarking that (2.5.6) is still valid for weak solutions by arguing as follows. Using [49, Theorem 2.8] (see also for instance the proof of Proposition 2.4) we deduce that F is also a renormalized solution of Equation (2.5.4). Applying then the renormalized formulation associated to the previous equation with $\beta(s) = s^2$, $\varphi = \widetilde{\omega}^2 \chi_R$ for any $R > 0$ and where $\widetilde{\omega}$ is as defined in (2.3.7), we deduce that arguing as before, passing to the limit as $R \to \infty$, and using the integral version of the Grönwall lemma we obtain (2.5.6). □

Then we have the tools to prove the following continuity result.

Proposition 2.2. *The map $\mathscr{F}:\mathbb{R}\to\mathbb{R}$ is continuous. More precisely, for every $\varepsilon>0$ there is $\delta>0$ such that if $|\nu_1-\nu_2|\leq\delta$ then*

$$|\mathscr{F}(\nu_1)-\mathscr{F}(\nu_2)|\leq\varepsilon.$$

Proof. We fix an admissible weight function ω and we define $C_0=d^{-1}\|\langle v\rangle^2\omega^{-1}\|_{L^2}<\infty$ and $T>0$ to be specified later. There holds

$$\begin{aligned}
|\mathscr{F}(\nu_1)-\mathscr{F}(\nu_2)| &\leq \frac{1}{d}\int_{\mathcal{O}}|v|^2|\mathfrak{F}^\alpha_{\nu_1}-\mathfrak{F}^\alpha_{\nu_2}| \quad\leq C_0\|\mathfrak{F}^\alpha_{\nu_1}-\mathfrak{F}^\alpha_{\nu_2}\|_{L^2_\omega(\mathcal{O})}\\
&\leq C_0\|\mathfrak{F}^\alpha_{\nu_1}-f_{1,T}\|_{L^2_\omega(\mathcal{O})}+C_0\|f_{1,T}-f_{2,T}\|_{L^2_\omega(\mathcal{O})}+C_0\|\mathfrak{F}^\alpha_{\nu_2}-f_{2,T}\|_{L^2_\omega(\mathcal{O})}\\
&\leq C_0C_1e^{-\lambda T}\|\mathfrak{F}^\alpha_{\nu_1}-f_0\|_{L^2\omega(\mathcal{O})}+C_0C_2\alpha^2|\nu_1-\nu_2|^2e^{\kappa T}\|f_0\|_{L^2_\omega(\mathcal{O})}\\
&\quad +C_0C_1e^{-\lambda T}\|\mathfrak{F}^\alpha_{\nu_2}-f_0\|_{L^2_\omega(\mathcal{O})}
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality in the first line, the triangular inequality to obtain the second, and Theorem 2.1.1 and Lemma 2.1 on the last line and we remark that $C_1,\lambda>0$ are given by Theorem 2.1.1 and $\kappa, C_2>0$ are given by Lemma 2.1.
Then for every fixed $\varepsilon>0$ we choose T large enough such that $C_0C_1e^{-\lambda T}\leq\varepsilon/3$ and we choose $\delta=\sqrt{\varepsilon/(3C_0C_2e^{\kappa T})}$ so that there holds

$$|\mathscr{F}(\nu_1)-\mathscr{F}(\nu_2)|\leq\varepsilon.$$

and this completes the proof. ☐

Preservation of lower and upper bounds

We dedicate this sub-subsection to prove the following proposition.

Proposition 2.3. *Let $\nu\in[0,2\mathcal{E}_{\mathfrak{F}^0}]$, there is $\alpha^\star\in(0,1/2)$ such that for every $\alpha\in(0,\alpha^\star)$ there holds $\mathscr{F}(\nu)\in[0,2\mathcal{E}_{\mathfrak{F}^0}]$.*

Proof. We fix during the proof an admissible weight function ω and an arbitrary $0\leq f_0\in L^2_\omega(\mathcal{O})$ with $\langle\!\langle f_0\rangle\!\rangle=1$. On the one hand from the weak maximum principle provided by Proposition 2.6 we deduce that

$$0\leq\mathcal{E}_f\leq C^0_\omega\|f\|_{L^2_\omega(\mathcal{O})},$$

where $C^0_\omega=d^{-1}\|\langle v\rangle^2\omega^{-1}\|_{L^2(\mathbb{R}^d)}<\infty$ and we remark that we have used the Cauchy-Schwartz inequality to obtain the second inequality. We observe that

$$\alpha\mathcal{C}_\nu f+(1-\alpha)\mathcal{C}_\tau f=\alpha(\nu-\tau)\Delta_v f+\mathcal{C}_\tau f,$$

and we will compare a solution f to Equation (2.5.1) and a solution ϕ to the linear equation

$$\begin{cases}
\partial_t\phi = -v\cdot\nabla_x\phi+\mathcal{C}_\tau\phi+\mathscr{G}\phi & \text{in } \mathcal{U},\\
\gamma_-\phi = \mathscr{R}\gamma_+\phi & \text{on } \Gamma_-,\\
\phi_{t=0} = f_0 & \text{in } \mathcal{O}.
\end{cases}\tag{2.5.7}$$

Step 1. We first remark that Proposition 2.2 and Theorem 2.1.1 provide the existence of $\mathfrak{F}^\alpha_\nu$ and $\mathfrak{F}^0$, respective stationary solutions to Equations (2.5.1) and (2.5.7), furthermore we also have the existence of some constants $\lambda, C_1>0$ independent of α and ν such that

$$\|f_t-\mathfrak{F}^\alpha_\nu\|_{L^2_\omega(\mathcal{O})}\leq C_1e^{-\lambda t}\|f_0-\mathfrak{F}^\alpha_\nu\|_{L^2_\omega(\mathcal{O})}\quad\text{and}\quad\|\phi_t-\mathfrak{F}^0\|_{L^1_\omega(\mathcal{O})}\leq C_1e^{-\lambda t}\|f_0-\mathfrak{F}^0\|_{L^1_\omega(\mathcal{O})},\tag{2.5.8}$$

for every $t \geq 0$. We then set $\psi = f - \phi$ and we observe that ψ is a weak solution of the following kinetic equation

$$\begin{cases} \partial_t \psi &= -v \cdot \nabla_x \psi + \mathcal{C}_\tau \psi + \mathscr{G}\psi + \alpha(\nu - \tau(x))\Delta_v f & \text{in } \mathcal{U} \\ \gamma_- \psi &= \mathscr{R}\gamma_+ \psi & \text{on } \Gamma_- \\ \psi_{t=0} &= 0 & \text{in } \mathcal{O}. \end{cases} \tag{2.5.9}$$

At the level of a priori estimates, we introduce the modified weight function $\widetilde{\omega}$ as defined in (2.3.7), and arguing as during the proof of Lemma 2.1 we have that

$$\frac{d}{dt}\int_{\mathcal{O}} \psi_t^2\, \widetilde{\omega}^2 \leq -\frac{\tau_0}{2}\int_{\mathcal{O}} |\nabla_v(\psi\widetilde{\omega})|^2 + \int_{\mathcal{O}} \varpi^{\mathcal{L}}_{\widetilde{\omega},2}\, \psi^2 \widetilde{\omega}^2 + \alpha \int_{\mathcal{O}} (\nu - \tau(x))\,(\Delta_v f)\psi\, \widetilde{\omega}^2, \tag{2.5.10}$$

an there is a constant κ_1, independent of ν_1, ν_2 and α, such that $\varpi^{\mathcal{L}}_{\widetilde{\omega},2} \leq \kappa_1$. On the other hand we compute

$$\begin{aligned} \left|\int_{\mathcal{O}} (\Delta_v f)\psi\, \widetilde{\omega}^2\right| &\leq \left|\int_{\mathcal{O}} \nabla_v f \cdot \nabla_v(\psi\widetilde{\omega})\, \widetilde{\omega}\right| + \left|\int_{\mathcal{O}} \nabla_v f \cdot \nabla_v \widetilde{\omega}\, (\psi\widetilde{\omega})\right| \\ &\leq \frac{1}{2}(1 + C_\star)\, \|\nabla_v f\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \frac{1}{2}\int_{\mathcal{O}} |\nabla_v(\psi\widetilde{\omega})|^2 + \frac{C_\star}{2}\|\psi\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})}, \end{aligned} \tag{2.5.11}$$

where we have used integration by parts and the triangular inequality to obtain the first line and the Cauchy-Scwhartz inequality together with the Young inequality to obtain the second. Moreover we remark that we have set $C_\star = \|(\nabla_v \widetilde{\omega})/\widetilde{\omega}\|_{L^\infty(\mathcal{O})}$, which is finite due to the analysis leading to (2.3.15). Putting together (2.5.10) and (2.5.11) we then obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{O}} \psi_t^2\, \widetilde{\omega}^2 \leq -\left(\frac{\tau_0}{2} - \alpha\,(\tau_1 + 2\mathcal{E}_{\mathfrak{F}^0})\right)\int_{\mathcal{O}} |\nabla_v(\psi\widetilde{\omega})|^2 + \left(\kappa_1 + \frac{C_\star}{2}\right)\int_{\mathcal{O}} \psi^2 \widetilde{\omega}^2 \\ + \frac{\alpha}{2}(1 + C_\star)\,(\tau_1 + 2\mathcal{E}_{\mathfrak{F}^0})\int_0^t \int_{\mathcal{O}} |\nabla_v f|^2\, \widetilde{\omega}^2. \end{aligned}$$

We set then $\alpha_1^\star = \tau_0/(4\,(\tau_1 + 2\mathcal{E}_{\mathfrak{F}^0}))$ and we observe that using (2.3.8) and the Grönwall lemma we deduce that

$$\begin{aligned} \|\psi_t\|^2_{L^2_\omega(\mathcal{O})} &\leq c_A^2 \frac{\alpha}{2}(1 + C_\star)\,(\tau_1 + 2\mathcal{E}_{\mathfrak{F}^0})\int_0^t \int_{\mathcal{O}} e^{\kappa_2(t-s)}|\nabla_v f_s|^2 \omega^2 \\ &\leq c_A^2\, C\,\frac{\alpha}{2}(1 + C_\star)\,(\tau_1 + 2\mathcal{E}_{\mathfrak{F}^0})\; e^{\kappa_2 t}\left(\|f_0\|^2_{L^2_\omega(\mathcal{O})} + \int_0^t \|f_s\|^2_{L^2_\omega(\mathcal{O})}\right) \\ &\leq c_A^2\, C\,\frac{\alpha}{2}(1 + C_\star)\,(\tau_1 + 2\mathcal{E}_{\mathfrak{F}^0})\; e^{\kappa_2 t}\left(1 + C^2 e^{2\kappa t}\right)\|f_0\|^2_{L^2_\omega(\mathcal{O})} \end{aligned}$$

where we have used (2.3.6) to obtain the second inequality, we have used (2.3.5) with $p = 2$ to obtain the third, and we remark that we have set $\kappa_2 = \kappa_1 + C_\star/2$, and the constants $\kappa, C > 0$ are given by Lemma 2.1. Finally we define the constant

$$C_t^2 := c_A^2\, \frac{C}{2}(1 + C_\star)\,(\tau_1 + 2\mathcal{E}_{\mathfrak{F}^0})\; e^{\kappa_2 t}\left(1 + C^2 e^{2\kappa t}\right).$$

We conclude this step by arguing that the previous estimate holds for weak solutions of Equation (2.5.7). Using [49, Theorem 2.8] (see also for instance the proof of Proposition 2.4) we deduce that h is also a renormalized solution of Equation (2.5.9). Applying then the renormalized formulation associated to the previous equation with $\beta(s) = s^2$, $\varphi = \widetilde{\omega}^2 \chi_R$ for any $R > 0$, we have that arguing as during this step, passing to the limit as $R \to \infty$, and using the integral version of the Grönwall lemma we obtain the same estimate.

Step 2. We take $T>0$ to be defined later and we compute by using the Cauchy-Schwartz inequality

$$\begin{aligned}\frac{1}{d}\int_{\mathcal{O}}|v|^2\mathfrak{F}^\alpha_\nu \mathrm{d}x\mathrm{d}v &\le \frac{1}{d}\int_{\mathcal{O}}|v|^2|\mathfrak{F}^\alpha_\nu - f_T|\mathrm{d}x\mathrm{d}v + \frac{1}{d}\int_{\mathcal{O}}|v|^2|f_T-\phi_T|\mathrm{d}x\mathrm{d}v\\ &\quad + \frac{1}{d}\int_{\mathcal{O}}|v|^2|\phi_T - \mathfrak{F}^0|\mathrm{d}x\mathrm{d}v + \frac{1}{d}\int_{\mathcal{O}}|v|^2|\mathfrak{F}^0|\mathrm{d}x\mathrm{d}v\\ &\le C^0_\omega\|\mathfrak{F}^\alpha_\nu - f_T\|_{L^2_\omega(\mathcal{O})} + C^0_\omega\|f_T-\phi_T\|_{L^2_\omega(\mathcal{O})} + C^0_\omega\|\phi_T-\mathfrak{F}^0\|_{L^2_\omega(\mathcal{O})} + \mathcal{E}_{\mathfrak{F}^0}\\ &\le C^0_\omega C_1 e^{-\lambda T}\|\mathfrak{F}^\alpha_\nu - f_0\|_{L^2_\omega(\mathcal{O})} + \alpha^{1/2}C_T\|f_0\|_{L^2_\omega(\mathcal{O})} + C^0_\omega C_1 e^{-\lambda T}\|f_0-\mathfrak{F}^0\|_{L^2_\omega(\mathcal{O})}\\ &\quad + \mathcal{E}_{\mathfrak{F}^0}\end{aligned}$$

where we have used (2.5.8) to obtain the last inequality and we recall that $C^0_\omega>0$ is defined at the beginning of the proof. We set then $T>0$ such that

$$C^0_\omega C_1 e^{-\lambda T}\|f_0-\mathfrak{F}^\alpha_\nu\|_{L^2_\omega(\mathcal{O})} \le \frac{\mathcal{E}_{\mathfrak{F}^0}}{3}, \qquad C^0_\omega C_1 e^{-\lambda T}\|f_0-\mathfrak{F}^0\|_{L^2_\omega(\mathcal{O})} \le \frac{\mathcal{E}_{\mathfrak{F}^0}}{3},$$

we choose $\alpha_2^\star$ small enough such that

$$(\alpha_2^\star)^{1/2}C_T\|f_0\|_{L^2_\omega(\mathcal{O})} \le \frac{1}{3}\mathcal{E}_{\mathfrak{F}^0},$$

and we conclude by setting $\alpha^\star = \min(\alpha_1^\star,\alpha_2^\star)$. □

Proof of Theorem 2.1.2

We take $\alpha\in(0,\alpha^\star)$, where $\alpha^\star>0$ is given by Proposition 2.3 and we remark that Proposition 2.2 implies in particular that the image set of the map $\mathscr{F}$ is a compact set contained in $[0,2\mathcal{E}_{\mathfrak{F}^0}]$. Using this together with Propositions 2.2 and 2.3, we may apply a fixed point theorem (for instance a real version of the Schauder fixed point theorem) and we conclude that there is $\nu^\star\in[0,2\mathcal{E}_{\mathfrak{F}^0}]$ such that $\mathscr{F}(\nu^\star)=\nu^\star$. □

2.6 Perturbation around the equilibrium.

Throughout the sequel we take $\alpha\in(0,\alpha^\star)$ where $\alpha^\star>0$ is given by Theorem 2.1.2, we introduce a function $g:\mathcal{U}\to\mathbb{R}$ such that $\langle\!\langle g_t\rangle\!\rangle_{\mathcal{O}}=0$ for all $t\ge0$, and the initial datum

$$h_0:\mathcal{O}\to\mathbb{R} \qquad \text{such that} \qquad \langle\!\langle h_0\rangle\!\rangle_{\mathcal{O}}=0, \tag{2.6.1}$$

and we will study the following equation

$$\begin{cases}\partial_t h &= -v\cdot\nabla_x h + \mathcal{C}_{\Lambda^\star}h + \mathscr{G}h + \alpha\mathcal{E}_g\Delta_v h + \alpha\mathcal{E}_h\Delta_v\mathfrak{F}^\alpha & \text{in } \mathcal{U}\\ \gamma_- h &= \mathscr{R}\gamma_+ h & \text{on } \Gamma_-\\ h_{t=0} &= h_0 & \text{in } \mathcal{O},\end{cases} \tag{2.6.2}$$

where $\Lambda^\star = \alpha\mathcal{E}_{\mathfrak{F}^\alpha}+(1-\alpha)\tau$, as introduced in (2.1.15) during Subsection 2.1.4. In particular, we observe that, due to Theorem 2.1.2, there holds $\tau_0/2\le\Lambda^\star\le 2\mathcal{E}_{\mathfrak{F}^0}+\tau_1$, where it is worth remarking that the previous upper and lower bounds of $\Lambda^\star$ are independent on α.

Remark 2.1. At a formal level we note that, by arguing as during (2.1.12), Equation (2.6.2) conserves mass, therefore a solution h to Equation (2.6.2) will satisfy that $\langle\!\langle h_t\rangle\!\rangle_{\mathcal{O}}=0$ for all $t\ge0$.

Remark 2.2. Moreover, and still at a formal level, if $g = h$ then a solution h of Equation (2.6.2) satisfies that $f = \mathfrak{F}^\alpha + h$ is a solution to Equation (2.1.1)-(2.1.2)-(2.1.4).

We will dedicate the rest of this section to prove the well-posedness of Equation (2.6.2) under suitable assumptions. During the sequel we will use the notations

$$\mathcal{Q}_g h = \mathcal{P}h + \mathfrak{N}_g h, \quad \text{and} \quad \mathscr{Q}_g h = \mathscr{P}h + \mathfrak{N}_g h, \tag{2.6.3}$$

where

$$\mathscr{P}h = -v \cdot \nabla_x h + \mathcal{P}h, \quad \mathcal{P}h = \mathcal{C}_{\Lambda^\star} h + \mathscr{G}h, \quad \text{and} \quad \mathfrak{N}_g h = \alpha \mathcal{E}_g \Delta_v h + \alpha \mathcal{E}_h \Delta_v \mathfrak{F}^\alpha. \tag{2.6.4}$$

2.6.1 A priori growth estimate

We will prove first that under suitable assumptions for the function g we can control the growth of the solution in time.

Proposition 2.3. *Let ω be an admissible weight function, there are constants $\varepsilon_1, \kappa, C > 0$ such that if $\|g\|_{L^2_\omega(\mathcal{O})} \le \varepsilon$ for any $\varepsilon \in (0, \varepsilon_1)$, then there is a constant $C > 0$ such that for any solution h to Equation* (2.6.2) *there holds*

$$\|h_t\|_{L^2_\omega(\mathcal{O})} \le C e^{\kappa t} \|h_0\|_{L^2_\omega(\mathcal{O})} \qquad \forall t \ge 0, \tag{2.6.5}$$

together with the energy estimate on the gradient

$$\int_0^t \|\nabla_v h_s\|^2_{L^2_\omega(\mathcal{O})} ds \lesssim_C \|h_0\|^2_{L^2_\omega(\mathcal{O})} + \int_0^t \|h_s\|^2_{L^2_\omega(\mathcal{O})} ds \qquad \forall t > 0. \tag{2.6.6}$$

Proof. The proof follows that of Lemma 2.1. We introduce the modified weight function $\widetilde{\omega}$ as defined in (2.3.7) and we write

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} h^2 \,\widetilde{\omega}^2 = \langle \mathcal{P}h, h\rangle_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \frac{1}{2}\int_{\mathcal{O}} h^2 \,(v \cdot \nabla_x \widetilde{\omega}^2) - \frac{1}{2}\int_\Sigma (\gamma h)^p \,\widetilde{\omega}^2 (n_x \cdot v) + \langle \mathfrak{N}_g h, h\rangle_{L^2_{\widetilde{\omega}}(\mathcal{O})}. \tag{2.6.7}$$

We split the proof into 3 Steps.

Step 1. (Control of the terms coming from the operator $\mathscr{P}$) Repeating the arguments from the Steps 1, 2, 3, 4 and 5 of the proof of Lemma 2.1 we know that there is $\kappa > 0$ such that

$$\langle \mathcal{P}h, h\rangle_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \frac{1}{2}\int_{\mathcal{O}} h^2 \,(v \cdot \nabla_x \widetilde{\omega}^2) - \frac{1}{2}\int_\Sigma (\gamma h)^p \,\widetilde{\omega}^2 (n_x \cdot v) \le -\frac{\tau_0}{2}\|\nabla_v (h\,\widetilde{\omega})\|_{L^2(\mathcal{O})} + \kappa \|h\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})}.$$

Step 2. (Control of the term $\mathfrak{N}_g$) We will prove during this step that for every $a^* > 0$ there holds

$$\langle \mathfrak{N}_g h, h\rangle_{L^2_{\widetilde{\omega}}(\mathcal{O})} \le \alpha C_\omega \left(\|g\|_{L^2_\omega(\mathcal{O})} + \frac{a^*}{2}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})}\right) \|\nabla_v (h\widetilde{\omega})\|^2_{L^2(\mathcal{O})} + \alpha C_\omega \left[\left(1 + \frac{1}{2a^*}\right)\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} + \|g\|_{L^2_\omega(\mathcal{O})}\right] \|h\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})},$$

for some constant $C_\omega > 0$. Indeed we have

$$\langle \mathfrak{N}_g h, h\rangle_{L^2_{\widetilde{\omega}}(\mathcal{O})} = \alpha\left(\mathcal{E}_g \langle \Delta_v h, h\rangle_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \mathcal{E}_h \langle \Delta_v \mathfrak{F}^\alpha, h\rangle_{L^2_{\widetilde{\omega}}(\mathcal{O})}\right) =: \alpha(\mathcal{N}_1 + \mathcal{N}_2),$$

and we will control each term separately. On the one hand using integration by parts we compute

$$\begin{aligned}\mathcal{N}_1 &= \mathcal{E}_g \int_{\mathcal{O}} \Delta h \, h \, \widetilde{\omega}^2 = \mathcal{E}_g \left(- \int_{\mathcal{O}} |\nabla_v (h\widetilde{\omega})|^2 + \int_{\mathcal{O}} h^2 \widetilde{\omega}^2 \, \frac{|\nabla_v \widetilde{\omega}|^2}{\widetilde{\omega}^2} \right) \\ &\leq C^1_\omega \, \|g\|_{L^2_\omega(\mathcal{O})} \left(\|\nabla_v (h\widetilde{\omega})\|^2_{L^2(\mathcal{O})} + \left\| \frac{\nabla_v \widetilde{\omega}}{\widetilde{\omega}} \right\|^2_{L^\infty(\mathbb{R}^d)} \|h\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} \right),\end{aligned}$$

where we have set $C^1_\omega = \|v\, \omega^{-1}\|_{L^2}$, and we have used the Cauchy-Schwartz inequality to obtain the second line. On the other hand we have that

$$\begin{aligned}\mathcal{N}_2 &= \mathcal{E}_h \int_{\mathcal{O}} \Delta \mathfrak{F}^\alpha \, h \, \widetilde{\omega}^2 = \mathcal{E}_h \left(- \int_{\mathcal{O}} \nabla_v \mathfrak{F}^\alpha \cdot \nabla_v (h\widetilde{\omega}) \, \widetilde{\omega} - \int_{\mathcal{O}} (\nabla_v \mathfrak{F}^\alpha \cdot \nabla_v \widetilde{\omega}) \, h\widetilde{\omega} \right) \\ &\leq C^1_{\widetilde{\omega}} \|h\|_{L^2_{\widetilde{\omega}}(\mathcal{O})} \left(\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_{\widetilde{\omega}}(\mathcal{O})} \|\nabla_v (h\widetilde{\omega})\|_{L^2(\mathcal{O})} + C^2_{\widetilde{\omega}} \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_{\widetilde{\omega}}(\mathcal{O})} \|h\|_{L^2_{\widetilde{\omega}}(\mathcal{O})} \right) \\ &\leq \frac{a^* C^1_{\widetilde{\omega}} \, c_A}{2} \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \|\nabla_v (h\widetilde{\omega})\|^2_{L^2(\mathcal{O})} + c_A C^1_{\widetilde{\omega}} \left(C^2_{\widetilde{\omega}} + \frac{1}{2a^*} \right) \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \|h\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})},\end{aligned}$$

where $C^2_{\widetilde{\omega}} = \|(\nabla_v \widetilde{\omega})/\widetilde{\omega}\|_{L^\infty}$ and we have used integration by parts on the first line, the Cauchy-Schwartz inequality to obtain the second and the Young inequality together with (2.3.8) to obtain the third line.

We then remark that $C^1_\omega < \infty$ due to the very definition of ω and $C^1_{\widetilde{\omega}} + C^2_{\widetilde{\omega}} < \infty$ by arguing as during (2.5.5). Altogether the previous computations imply the inequality presented at the beginning of this step.

Step 3. (Conclusion) We choose $\varepsilon_1 = \tau_0/(8C_\omega)$ and $a^* = \tau_0/(8C_\omega \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})})$ where we recall that C_ω is given by the Step 2. Then, since $\|g\|_{L^2_\omega(\mathcal{O})} \leq \varepsilon$ with $\varepsilon \leq \varepsilon_1$, we deduce that

$$\frac{1}{2}\frac{d}{dt} \int_{\mathcal{O}} h^2 \, \widetilde{\omega}^2 \leq -\frac{\tau_0}{4} \|\nabla_v (h\widetilde{\omega})\|^2_{L^2(\mathcal{O})} + (\kappa + \kappa^\star) \|h\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})},$$

where

$$\kappa^\star = C_\omega \left(\left(1 + \frac{1}{2a^*} \right) \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} + \|g\|_{L^2_\omega(\mathcal{O})} \right).$$

We remark that $\kappa^\star$ is finite due to (2.1.9), we conclude the proof by remarking that (2.6.5) is a consequence of the Grönwall lemma and (2.6.6) is obtained by arguing as during the Step 6 of the proof of Lemma 2.1. ☐

2.6.2 Well-posedness in a weighted L^2 framework

In this subsection we will extend the well-posedness theory presented in Subsection 2.3.2 to a framework fitting Equation (2.6.2).

We note that the main difficulty comes from the presence of a term including $\Delta_v \mathfrak{F}^\alpha \in L^2_x H^{-1}_v$. We remark, however, that the main technical tools to achieve such a well-posedness result have been recently developed in [49, Subsection 2.3], and we will be using them during the proof of the following proposition.

Proposition 2.4. *Let ω be an admissible weight function. There is $\varepsilon_2 > 0$ such that if $\|g\|_{L^2_\omega(\mathcal{U})} \leq \varepsilon$ for any $\varepsilon \in (0, \varepsilon_2)$ there holds that for any initial datum $h_0 \in L^2_\omega(\mathcal{O})$, there exists a unique global weak solution $h \in C(\mathbb{R}_+, L^2_\omega(\mathcal{O})) \cap \mathscr{H}_\omega(\mathcal{U})$ to Equation* (2.6.2)*, where* *we recall that the Hilbert space* $\mathscr{H}_\omega$ *is defined in* (2.3.17)*. More precisely, for every test function* $\varphi \in \mathcal{D}(\bar{\mathcal{U}})$ *there holds*

$$\int_{\mathcal{U}} \nabla_v \varphi \cdot \nabla_v h \, (\Lambda^\star + \alpha \mathcal{E}_g) + \int_{\mathcal{U}} h \, (-\partial_t \varphi + v \cdot \nabla_v \varphi - \mathscr{G}^* \varphi - v \cdot \nabla_x \varphi) \\ - \int_{\mathcal{U}} \mathcal{E}_h \, \nabla_v \mathfrak{F}^\alpha \cdot \nabla_v \varphi + \int_\Gamma \gamma h \, \varphi (n_x \cdot v) = \int_{\mathcal{O}} h_0 \, \varphi(0, \cdot) \quad (2.6.8)$$

where we remark that the trace γh *is defined by [49, Theorem 2.8] and satisfies* $\gamma h \in L^2_\omega(\Gamma, d\xi^2_\omega)$ *as well as the Maxwell boundary condition* (2.1.2) *pointwisely. Finally there also holds* $h(0, \cdot) = h_0$ *pointwisely.*

Remark 2.5*.* The proof of Proposition 2.4 is based on [45, Theorem 3.3] and [49, Theorem 2.11] using the a priori estimate from Proposition 2.3 and [49, Proposition 2.8].

Remark 2.6*.* In particular, Proposition 2.4 implies the existence of a strongly continuous semigroup, which will be denoted as $S_{\mathscr{Q}_g}$, associated to the solutions of Equation (2.6.2).

Proof of Proposition 2.4. We proceed to the proof into four steps.

Step 1. We consider $\widetilde{\omega}$ as defined in (2.3.7), a finite $T > 0$, $\mathcal{H} \in L^2_{t,x,v} + L^2_{t,x} H^{-1}_v$ such that $\mathcal{H} = \mathcal{H}_0 + \mathrm{div}_v \, \mathcal{H}_1$ with $\mathcal{H}_0, \mathcal{H}_1 \in L^2_{\widetilde{\omega}}(\mathcal{U}_T)$, and the boundary data $\mathcal{H}_b \in L^2(\Gamma_{T,-}; d\xi^1_{\widetilde{\omega}})$. We consider the inflow problem

$$\begin{cases} \partial_t h &= -v \cdot \nabla_x h + \widetilde{\Lambda} \Delta_v h + v \cdot \nabla_v h + \mathcal{H} & \text{in } \mathcal{U}_T \\ \gamma_- h &= \mathcal{H}_b & \text{on } \Gamma_{T,-} \\ h_{t=0} &= h_0 & \text{in } \mathcal{O}, \end{cases} \quad (2.6.9)$$

where $\widetilde{\Lambda} = \alpha \mathcal{E}_{\mathfrak{F}^\alpha} + \alpha \mathcal{E}_g + (1 - \alpha)\tau$. By setting $\varepsilon^1_2 = \tau_0 / (2 \| \langle v \rangle^2 \omega^{-1} \|_{L^2})$ we observe that if $\|g\|_{L^2_\omega(\mathcal{U})} \le \varepsilon$ for $\varepsilon \in (0, \varepsilon^1_2)$, there holds

$$\frac{\tau_0}{4} \le \widetilde{\Lambda} \le 2 \mathcal{E}_{\mathfrak{F}^0} + \frac{\tau_0}{2} + \tau_1.$$

We define, for a constant $\lambda > 0$, the bilinear form $\mathscr{E} : \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T) \times C^1_c([0, T) \times \mathcal{O} \cup \Gamma_{T,-}) \to \mathbb{R}$ by

$$\mathscr{E}(h, \varphi) := \int_{\mathcal{U}_T} \widetilde{\Lambda} \, \nabla_v h \cdot \nabla_v (\varphi \widetilde{\omega}^2) + \int_{\mathcal{U}_T} h \left(\lambda \varphi \, \widetilde{\omega}^2 - (\partial_t \varphi) \, \widetilde{\omega}^2 + \mathrm{div}_v (v \varphi \, \widetilde{\omega}^2) - v \cdot \nabla_x (\varphi \, \widetilde{\omega}^2) \right).$$

We observe that, due to our choice of ε^1_2, we can follow the same computations as during the proof of Lemma 2.1 and we obtain that there is $\lambda_0 \in \mathbb{R}$ such that

$$\mathscr{E}(\varphi, \varphi) \ge (\lambda - \lambda_0) \|\varphi\|^2_{L^2_{\widetilde{\omega}}} + \|\varphi\|^2_{H^{1,\dagger}_{\widetilde{\omega}}},$$

for any $\varphi \in C^1_c([0, T) \times \mathcal{O} \cup \Gamma_-)$. Taking then $\lambda > \lambda_0$, the bilinear form $\mathscr{E}$ is coercive and using Lions' variant of the Lax-Milgram theorem from [126, Chap III, §1], we deduce the existence of a function $h_\lambda \in \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T)$ which satisfies the variational equation

$$\mathscr{E}(h_\lambda, \varphi) = \int_{\Gamma_{T,-}} \mathcal{H}_b \, e^{-\lambda t} \, \varphi \, \widetilde{\omega}^2 \, \mathrm{d}\xi^1_1 + \int_{\mathcal{O}} h_0 \, \varphi(0, \cdot) \, \widetilde{\omega}^2 \, \mathrm{d}v \, \mathrm{d}x + \int_{\mathcal{U}_T} \mathcal{H} \, \varphi \, \widetilde{\omega}^2 \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t,$$

for every $\varphi \in C^1_c([0, T) \times \mathcal{O} \cup \Gamma_-)$ and where we remark that the last term is defined as the product between functions in $L^2_{t,x} H^{-1}_v$ and $L^2_{t,x} H^1_v$ in $\mathcal{U}_T$.

Defining now $h := h_\lambda e^{\lambda t}$ we deduce that $h \in \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T)$ is a weak solution to the inflow problem (2.6.9) and by using [49, Proposition 2.8-(3)] with the choices $\sigma_{ij} = \widetilde{\Lambda}$, $\nu = v$ and $G = \mathcal{H}$, we further have that $h \in C([0,T), L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T)$ and it has a trace $\gamma h \in L^2(\Gamma_T; \mathrm{d}\xi^1_{\widetilde{\omega}}\,\mathrm{d}t)$. Another consequence of [49, Proposition 2.8-(3)] is that h is a renormalized solution to Equation (2.6.9), i.e there holds

$$\int_{\mathcal{U}_t} \Big(\beta(h)\left(-\partial_t\varphi - v\cdot\nabla_x\varphi - \widetilde{\Lambda}\Delta_v\varphi + \mathrm{div}_v(v\varphi)\right) + \beta''(h)\,\widetilde{\Lambda}|\nabla_v h|^2\varphi\Big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \quad (2.6.10)$$
$$+\int_{\Gamma_t}\beta(\gamma\,h)\,\varphi\,(n_x\cdot v)\,\mathrm{d}v\,\mathrm{d}\sigma_x\,\mathrm{d}s + \left[\int_{\mathcal{O}}\beta(h_t)\varphi(t,\cdot)\,\mathrm{d}x\,\mathrm{d}v\right]_0^t = \int_{\mathcal{U}_t}\mathcal{H}\,\beta'(h)\,\varphi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s$$

for any $t \in [0,T]$, any renormalizing function $\beta \in \mathfrak{B}$, and any test function $\varphi \in \mathcal{D}(\bar{\mathcal{U}}_T)$.
Using now the renormalized formulation (2.6.10) with $\beta(s) = s^2$ and $\varphi = \widetilde{\omega}^2\chi_R$ for any $R > 0$, we proceed similarly as during the proof of Proposition 2.3, we take the limit as $\mathbb{R} \to \infty$, and using the integral version of the Grönwall lemma we obtain that

$$\|h\|^2_{L^2_{\widetilde{\omega}}(\mathcal{U}_t)} \le e^{\lambda_0 t}\|h_0\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + C\int_0^t e^{\lambda_0(t-s)}\left(\|\mathcal{H}_{0s}\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \|\mathcal{H}_{1s}\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \|\mathcal{H}_{bs}\|^2_{L^2(\Sigma_-;\mathrm{d}\xi^1_{\widetilde{\omega}})}\right)\mathrm{d}s,$$

for every $t \in [0,T]$. Considering then $h_1, h_2 \in C([0,T), L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T)$ two solutions of Equation (2.6.9), we take $\psi = h_1 - h_2$ and we remark that ψ solves Equation (2.6.9) in the weak sense with $h(0) = \mathcal{H} = \mathcal{H}_b = 0$. We immediately deduce that $\psi \equiv 0$ from the above energy estimate, thus the uniqueness of the solution to Equation (2.6.9).

Step 2. We take $\delta \in (0,1)$, $h^0 = 0$ and we define the sequence h^k recurrently as follows

$$\begin{cases} \partial_t h^{k+1} &= -v\cdot\nabla_x h^{k+1} + \widetilde{\Lambda}\Delta_v h^{k+1} + v\cdot\nabla_v h^{k+1} + \mathcal{H}^k & \text{in } \mathcal{U}_T \\ \gamma_- h^{k+1} &= \delta\,\mathscr{R}\gamma_+ h^k & \text{on } \Gamma_{T,-} \\ h^{k+1}_{t=0} &= h_0 & \text{in } \mathcal{O}, \end{cases} \quad (2.6.11)$$

where $\mathcal{H}^k = dh^k + \mathscr{G}h^k + \mathcal{E}_{h^k}\,\Delta_v\mathfrak{F}^\alpha$. We observe that the sequence h^k is well defined by using an induction argument: given $h^k \in C([0,T], L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T)$ with a trace $\gamma h^k \in L^2(\Gamma_T; \mathrm{d}\xi^1_{\widetilde{\omega}}\,\mathrm{d}t)$ we have that $\mathcal{H}^k \in L^2_{t,x}H^1_v + L^2_{t,x}H^{-1}_v$ and since $\mathscr{R}$ satisfies (2.3.21), as proved during the Step 1 of the proof of Lemma 2.1, we also have that $\mathscr{R}\gamma_+ h^k \in L^2(\Gamma_{T,-}, \mathrm{d}\xi^1_{\widetilde{\omega}}\,\mathrm{d}t)$. Applying then the results from Step 1 we obtain the existence of $h^{k+1} \in C([0,T), L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}$ with an associated trace function $\gamma h^{k+1} \in L^2(\Gamma_T; \mathrm{d}\xi^1_{\widetilde{\omega}}\,\mathrm{d}t)$, renormalized solution of Equation (2.6.11).

We take then $\varepsilon_2 = \min(\varepsilon_1, \varepsilon_2^1)$ where ε_1 is given by Proposition 2.3, and using the renormalized formulation (2.6.10) with the choices of $\beta(s) = s^2$ and $\varphi = \widetilde{\omega}^2\chi_R$ for any $R > 0$ we can follow the computations performed during the proof of Proposition 2.3, we pass to the limit as $R \to \infty$, and we get the energy estimate

$$\|h^{k+1}_t\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \|h^{k+1}\|^2_{H^{1,\dagger}_\omega(\mathcal{O})} + \|\gamma_+ h^{k+1}\|^2_{L^2(\Gamma_{t,+};d\xi^1_{\widetilde{\omega}})} \le \|h^{k+1}_0\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})}$$
$$+\,\delta\|\gamma_+ h^k\|^2_{L^2(\Gamma_{t,+};d\xi^1_{\widetilde{\omega}})} + \lambda_0\|h^{k+1}\|^2_{L^2_{\widetilde{\omega}}(\mathcal{U}_t)} + \lambda_1\|h^k\|^2_{L^2_{\widetilde{\omega}}(\mathcal{U}_t)}, \quad (2.6.12)$$

for every $t \in [0,T]$ and some constants $\lambda_0, \lambda_1 > 0$. Using then the integral version of the Grönwall lemma we obtain

$$\|h^{k+1}_t\|^2_{L^2_{\tilde{\omega}}(\mathcal{O})} + \int_0^t \left(\|\gamma_+ h^{k+1}_s\|^2_{L^2(\Sigma_+;d\xi^1_{\widetilde{\omega}})} + \|h^{k+1}_s\|^2_{H^{1,\dagger}_{\tilde{\omega}}(\mathcal{O})}\right) e^{\lambda_0(t-s)}\,ds$$
$$\le \|h_0\|^2_{L^2_{\tilde{\omega}}(\mathcal{O})}e^{\lambda_0 t} + \int_0^t\left(\delta\|\gamma_+ h^k_s\|^2_{L^2_{\tilde{\omega}}(\Sigma_-;d\xi_1)} + \lambda_1\|h^k_s\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})}\right)e^{\lambda_0(t-s)}\,ds.$$

Choosing then $T > 0$ small enough, the previous estimate implies that h^k and γh^k are Cauchy sequences in the spaces $C([0,T], L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T)$ and $L^2(\Gamma_T; \mathrm{d}\xi^1_{\widetilde{\omega}}\, \mathrm{d}t)$ respectively. Therefore we deduce that there are some functions $h \in C([0,T), L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}$ and $\bar{\gamma} \in L^2(\Gamma_T; \mathrm{d}\xi^1_{\widetilde{\omega}}\, \mathrm{d}t)$ such that $h^k \to h$ strongly in $C([0,T), L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}$ and $\gamma h^k \to \bar{\gamma}$ strongly in $L^2(\Gamma_T; \mathrm{d}\xi^1_{\widetilde{\omega}}\, \mathrm{d}t)$ as $k \to \infty$.

By taking then β one-to-one, any $\varphi \in \mathcal{D}(\bar{\mathcal{U}}_T)$, and using the Lebesgue dominated convergence theorem, we may pass to the limit in the weak formulation of Equation (2.6.11) and we obtain that h is a weak solution of the problem

$$\begin{cases} \partial_t h &= \mathscr{Q}_g h & \text{in } \mathcal{U}_T \\ \gamma_- h &= \delta \mathscr{R} \gamma_+ h & \text{on } \Gamma_{T,-} \\ h_{t=0} &= h_0 & \text{in } \mathcal{O}, \end{cases} \tag{2.6.13}$$

where we recall that $\mathscr{Q}_g$ is defined in (2.6.3). Using again [49, Proposition 2.8-(3)] we have that h is also a renormalized solution of Equation (2.6.13) with a trace function γh and we immediately deduce that $\gamma h = \bar{\gamma}$ a.e in Γ_T.

We now take again $\beta(s) = s^2$ and $\varphi := (n_x \cdot v)\langle v\rangle^{-2}\omega^2(v)$ in the renormalize formulation of Equation (2.6.13) and following the same arguments leading to the energy estimate (2.6.12) we obtain that

$$\int_\Gamma (\gamma h)^2 \,\mathrm{d}\xi^2_\omega \,\mathrm{d}t \lesssim \|h_0\|^2_{L^2_\omega(\mathcal{O})} e^{\lambda_2 T}. \tag{2.6.14}$$

for some constant $\lambda_2 > 0$.

Moreover, using the renormalized formulation with the choices $\beta(s) = s^2$ and $\varphi = \widetilde{\omega}^2 \chi_R$ for any $R > 0$, arguing as during the proof of Proposition 2.3, and taking the limit as $R \to \infty$, we obtain that there is $\kappa > 0$ for which there holds

$$\|h_t\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \int_0^t \left\{ (1-\delta)\|\gamma h_s\|^2_{L^2(\Sigma_+; \mathrm{d}\xi^1_{\widetilde{\omega}})} + \|h_s\|^2_{H^{1,\dagger}_{\widetilde{\omega}}(\mathcal{O})} \right\} e^{\kappa(t-s)}\, \mathrm{d}s \le e^{\kappa t}\|h_0\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} \tag{2.6.15}$$

for every $t \in [0,T]$. We remark then that the linearity of Equation (2.6.13) together with the estimate (2.6.15) give the uniqueness of the solution.

Step 3. For a sequence $\delta_k \in (0,1)$, $\delta_k \nearrow 1$, we consider $h_k \in C([0,T), L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}$ as the renormalized solutions to the modified Maxwell reflection boundary condition problem

$$\begin{cases} \partial_t h_k &= \mathscr{Q}_g h_k & \text{in } \mathcal{U}_T \\ \gamma_- h_k &= \delta_k \mathscr{R} \gamma_+ h_k & \text{on } \Sigma_{T,-} \\ h_{k,t=0} &= h_0 & \text{in } \mathcal{O}, \end{cases} \tag{2.6.16}$$

obtained by using Step 2. By following the arguments leading to the estimate (2.6.15), we have that there is $\kappa > 0$ such that h_k satisfies

$$\|h_{kt}\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \int_0^t \left\{ (1-\delta_k)\|\gamma h_{ks}\|^2_{L^2(\Sigma_+; \mathrm{d}\xi^1_{\widetilde{\omega}})} + \|h_{ks}\|^2_{H^{1,\dagger}_{\widetilde{\omega}}(\mathcal{O})} \right\} e^{\kappa(t-s)}\, \mathrm{d}s \le e^{\kappa t}\|h_0\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} \tag{2.6.17}$$

for any $t \in [0,T]$ and any $k \ge 1$. Moreover, for any constants $T, R > 0$ we may use the renormalized formulation as during the proof of Proposition 2.3 and we obtain that

$$\int_0^t \|\nabla_v h_{ks}\|_{L^2_\omega(\mathcal{O})} \mathrm{d}s \lesssim e^{\kappa t}\|h_0\|_{L^2_\omega(\mathcal{O})},$$

for any $0 < t \leq T$. Additionally, from (2.6.14), we also have the following energy estimate on the boundary

$$\int_{\Gamma_T} (\gamma h_k)^2 \,\mathrm{d}\xi^2_{\tilde{\omega}} \,\mathrm{d}t \lesssim \|h_0\|^2_{L^2_{\tilde{\omega}}(\mathcal{O})} e^{\lambda_2 T}.$$

From the above estimates, we deduce that, up to the extraction of a subsequence, there exist $h \in \mathscr{H}_{\tilde{\omega}}(\mathcal{U}_T) \cap L^2_{\tilde{\omega}}(\mathcal{U}_T)) \cap L^2([0,T] \times \Omega; H^1_\omega(\mathbb{R}^d))$ and $\nu_\pm \in L^2(\Gamma_{T,\pm}; \mathrm{d}\xi^2_{\tilde{\omega}}\mathrm{d}t)$ such that

$$h_k \rightharpoonup h \text{ weakly in } \mathscr{H}_{\tilde{\omega}}(\mathcal{U}_T) \cap L^2_{\tilde{\omega}}(\mathcal{U}_T) \cap L^2([0,T] \times \Omega; H^1_\omega(\mathbb{R}^d)), \tag{2.6.18}$$

and

$$\gamma_\pm h_k \rightharpoonup \nu_\pm \text{ weakly in } L^2(\Gamma_{T,\pm}; \mathrm{d}\xi^2_{\tilde{\omega}}\mathrm{d}t).$$

We now establish more properties regarding the convergence of the previous subsequences to their limit in Steps 3.1 and 3.2 and we will conclude the existence of weak solutions in Step 3.4.

Step 3.1. (Strong convergence in $L^2(\mathcal{U}_T)$) Since $h_k \in L^2([0,T] \times \Omega; H^1_\omega(\mathbb{R}^d))$ and satisfies Equation (2.6.16) in the weak sense, for any truncated (in t and x) version $\left(\bar{h}_k\right)$ of (h_k) we may apply [29, Theorem 1.3] and we deduce that $\left(\bar{h}_k\right)$ is bounded in $H^{1/4}(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$. Using now the version of the Rellich-Kondrachov theorem for fractional Sobolev spaces (see for instance [74, Corollary 7.2] or [148, Lemma 6.11]) we deduce that

$$h_k \to h \text{ strongly in } L^2([0,T] \times \mathcal{O}_R), \tag{2.6.19}$$

for any $R > 0$ and where we have defined $\mathcal{O}_R := \{(x,v) \in \mathcal{O};\, d(x,\Omega^c) > 1/R,\, |v| < R\}$. Let now $\varsigma = \omega\langle v\rangle^{-1/2}$, we have that the sequence h_k is tight in $L^2_\varsigma(\mathcal{U}_T)$. Indeed we observe that

$$\|h_{kt}\|_{L^2_\varsigma(\mathcal{O}^c_R)} \leq \frac{1}{\langle R\rangle^{1/2}} \|h_{kt}\|_{L^2_\omega(\mathcal{O}^c_R)} \leq \frac{1}{\langle R\rangle^{1/2}} \|h_{kt}\|_{L^2_\omega(\mathcal{O})} \lesssim \frac{1}{\langle R\rangle^{1/2}} e^{\kappa t} \|h_0\|_{L^2_\omega(\mathcal{O})}, \tag{2.6.20}$$

for every $t \in [0,T]$, and where we remark that we have used (2.6.17) together with (2.3.8) to obtain the last inequality. Combining (2.6.19) and (2.6.20) we classically obtain that $h_k \to h$ strongly in $L^2_\varsigma(\mathcal{U}_T)$ as $k \to \infty$. Using this convergence, the Cauchy-Schwartz inequality, and the fact that $\langle v\rangle^{2+1/2}\omega^{-1} \in L^2(\mathbb{R}^d)$, we deduce then that

$$\mathcal{E}_{h_k} \to \mathcal{E}_h \qquad \text{as } k \to \infty. \tag{2.6.21}$$

Step 3.2. (Pointwise convergence for the boundary term) Because $\langle v\rangle\omega^{-1} \in L^2(\mathbb{R}^d)$, we have that $L^2(\Gamma_T; \mathrm{d}\xi^2_{\tilde{\omega}} \,\mathrm{d}t) \subset L^1(\Gamma_T; \mathrm{d}\xi^1_1 \,\mathrm{d}t)$ and recalling that, from the very definition of the Maxwell reflection operator given by (2.1.2), there holds

$$\mathscr{R} : L^1(\Sigma_+; d\xi^1_1) \to L^1(\Sigma_-; d\xi^1_1), \quad \|\mathscr{R}\|_{L^1(\Sigma; d\xi^1_1)} \leq 1,$$

we deduce that $\mathscr{R}(\gamma h_{k+}) \rightharpoonup \mathscr{R}(\mathfrak{h}_+)$ weakly in $L^1(\Gamma_{T,-}; \mathrm{d}\xi^1_1 \,\mathrm{d}t)$. On the other hand, from [45, Theorem 3.2], we have $\gamma h_k \rightharpoonup \gamma h$ weakly in $L^2_{\mathrm{loc}}(\Gamma_T; \mathrm{d}\xi^2_1 \,\mathrm{d}t)$. Using both convergences in the boundary condition $\gamma_- h_k = \mathscr{R}(\gamma_+ h_k)$, we obtain that $\gamma_- h = \mathscr{R}(\gamma_+ h)$ a.e.

Step 3.3. (Conclusion of the existence of weak solutions) Using (2.6.18) and (2.6.21) we may take the limit in the weak formulation of Equation (2.6.16) and we obtain that h is a weak solution of Equation (2.6.16) complemented with the Maxwell reflection boundary condition and associated to the initial datum h_0, i.e there holds (2.6.8).

Moreover, [49, Proposition 2.8-(2)] implies that $h \in C([0,T]; L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T)$ and that h is also a renormalized solution of Equation (2.6.16). Repeating the arguments leading to (2.6.17) we deduce the energy estimate

$$\|h_t\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + 2\int_0^t \|h_{ks}\|^2_{H^{1,\dagger}_{\widetilde{\omega}}(\mathcal{O})} e^{\kappa(t-s)}\,\mathrm{d}s \le e^{\kappa t}\|h_0\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} \qquad \forall t \in [0,T]. \tag{2.6.22}$$

Step 4. We consider now two solutions h_1 and $h_2 \in C([0,T]; L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T)$ to Equation (2.6.2) associated to the same initial datum h_0. We now define $\psi := h_2 - h_1$ and we note that $\psi \in C([0,T]; L^2_{\widetilde{\omega}}(\mathcal{O})) \cap \mathscr{H}_{\widetilde{\omega}}(\mathcal{U}_T)$ is a weak solution to Equation (2.6.2) associated to the initial datum $\psi(0) = 0$. We immediately get from (2.6.22) that $\psi \equiv 0$ a.e.

Finally, by repeating this result in every time interval $[jT, (j+1)T]$ for $j \in \mathbb{N}$, we obtain the existence and uniqueness of a global weak solution to Equation (2.6.2) $\square$

2.7 Hypodissipativity

We start this section by taking a glance at the equation

$$\begin{cases} \partial_t h &= \mathscr{P} h & \text{in } \mathcal{U} \\ \gamma_- h &= \mathscr{R}\gamma_+ h & \text{on } \Gamma_- \\ h_{t=0} &= h_0 & \text{in } \mathcal{O}, \end{cases} \tag{2.7.1}$$

where we recall that h_0 is given by (2.6.1), and $\mathscr{P}$ is defined in (2.6.4). We have then the following lemma.

Lemma 2.1. *Consider ω and admissible weight function. The following statements hold.*

(1) Let $h_0 \in L^2_\omega(\mathcal{O})$, there is $h \in L^2_\omega(\mathcal{U}) \cap \mathscr{H}_\omega(\mathcal{U})$ unique global weak solution to Equation (2.7.1)*.*

In particular, there is a strongly continuous continuous semigroup $S_{\mathscr{P}} : L^2_\omega(\mathcal{O}) \to L^2_\omega(\mathcal{O})$ associated to the solutions of Equation (2.7.1)*. Moreover there are constants $\lambda > 0$ and $C_P > 0$ such that there holds*

$$\|S_{\mathscr{P}}(t)h_0\|_{L^2_\omega(\mathcal{O})} \le C_P\, e^{-\lambda t}\|h_0\|_{L^2_\omega(\mathcal{O})}, \tag{2.7.2}$$

and

$$\int_0^t \|\nabla_v S_{\mathscr{P}}(s)h_0\|^2_{L^2_\omega(\mathcal{O})} ds \lesssim_{C_P} \|h_0\|^2_{L^2_\omega(\mathcal{O})} + \int_0^t \|S_{\mathscr{P}}(s)h_0\|^2_{L^2_\omega(\mathcal{O})} ds, \tag{2.7.3}$$

for all $t > 0$.

(2) There is a function $\mathfrak{H} \in L^2(\Omega, H^1(\mathbb{R}^d))$ unique steady solution of Equation (2.7.1)*. Moreover for every admissible weight function ς there is a constant $C_{\mathfrak{H}} > 0$ such that*

$$\|\mathfrak{H}\|_{L^2_\varsigma(\mathcal{O})} \le C_{\mathfrak{H}}. \tag{2.7.4}$$

(3) For any final time $T > 0$ and any final datum $\psi_T \in L^2_m(\mathcal{O})$, with $m = \omega^{-1}$, there is $\psi \in L^2_m(\mathcal{U}_T) \cap \mathscr{H}_m(\mathcal{U}_T)$ unique global weak solution of the dual backwards in time equation

$$\begin{cases} \partial_t \psi &= \mathscr{P}^*\psi := v\cdot\nabla_x\psi + \Lambda^\star\Delta_v\psi - v\cdot\nabla_v\psi + \mathscr{G}^*\psi & \textit{in } \mathcal{U}_T \\ \gamma_+\psi &= \mathscr{R}^*\gamma_-\psi & \textit{on } \Gamma_{T,+} \\ \psi_{t=T} &= \psi_T & \textit{in } \mathcal{O}, \end{cases} \tag{2.7.5}$$

where we recall that $\mathscr{G}^*$ *is defined in* (2.3.45) *and* $\mathscr{R}^*$ *is defined* (2.3.34)*. In particular, there is a strongly continuous semigroup* $S_{\mathscr{P}^*} : L^2_m(\mathcal{O}) \to L^2_m(\mathcal{O})$ *which is dual to the semigroup* $S_{\mathscr{P}}$*, i.e there holds* (2.3.35)*. Moreover, there is a constant* $C^*_P > 0$ *such that*

$$\|S_{\mathscr{P}^*}(0)\,\psi_T - \langle \psi_T, \mathfrak{H}\rangle\|_{L^2_m(\mathcal{O})} \le C^*_P\, e^{-\lambda T} \|\psi_T - \langle \psi_T, \mathfrak{H}\rangle\|_{L^2_m(\mathcal{O})}, \tag{2.7.6}$$

and

$$\int_0^T \|\nabla_v S_{\mathscr{P}^*}(0)\psi_T\|^2_{L^2_m(\mathcal{O})} ds \lesssim_{C^*_P} \|\psi_T\|^2_{L^2_m(\mathcal{O})} + \int_0^T \|S_{\mathscr{P}^*}(0)\psi_T\|^2_{L^2_m(\mathcal{O})} ds. \tag{2.7.7}$$

Proof. We first remark that Equation (2.7.1) fits the framework of Equation (2.1.14) with $\Lambda = \Lambda^\star$. Therefore *(1)* and *(2)* are a consequence of Lemma 2.1, Theorem 2.3.3, and Theorem 2.4.1. Finally, *(3)* is a consequence of Lemma 2.8, Proposition 2.10, Proposition 2.12, and Proposition 2.3. □

We now aim to prove that, under suitable assumptions on the function g and on the parameter α, the decay properties of the semigroup $S_{\mathscr{P}}$ can be extended to $S_{\mathscr{Q}_g}$ by treating the term $\mathfrak{N}_g$ as a perturbation. The core idea of the proof is to define a new norm, in the spirit of [46, Proposition 3.6], [50, Proposition 3.2] and [139, Proposition 4.1], that leverages both the dissipativity of $\mathscr{P}$ and the control on $\mathscr{Q}_g$ established in Proposition 2.3.

Proposition 2.2. *Let* ω *be an admissible weight function. There are constants* $\alpha^{\star\star} \in (0, \alpha^\star)$*,* $\eta > 0$ *and* $\varepsilon_3 > 0$ *such that for every* $\alpha \in (0, \alpha^{\star\star})$*, if* $\|g_t\|_{L^\infty_\omega(\mathcal{O})} \le \varepsilon$ *holds for any* $\varepsilon \in (0, \varepsilon_3)$*, then if* $h_0 \in L^2_\omega(\mathcal{O})$*, every solution* h *of Equation* (2.6.2) *satisfies*

$$\|h_t\|_{L^2_\omega(\mathcal{O})} \le C e^{-\eta t} \|h_0\|_{L^2_\omega(\mathcal{O})} \qquad \forall t \ge 0, \tag{2.7.8}$$

for some constant $C \ge 1$*.*

Proof. We introduce $\beta > 0$ to be fixed later, and we define the norm

$$|\!|\!| h_t |\!|\!|^2 = \beta \|h_t\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \int_0^\infty \|S_{\mathscr{P}}(\tau) h_t\|^2_{L^2_\omega(\mathcal{O})} \mathrm{d}\tau,$$

where we recall that $S_{\mathscr{P}}$ is given by Lemma 2.1-(1) and $\widetilde{\omega}$ is defined in (2.3.7). We first observe that the norm $|\!|\!|\cdot|\!|\!|$ is well defined due to (2.7.2), and we also have that

$$\beta c_A^{-1} \|h_t\|^2_{L^2_\omega(\mathcal{O})} \le \beta \|h_t\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} \le |\!|\!| h_t |\!|\!|^2 \le \left(\beta c_A + \frac{(C_P)^2}{2\lambda} \right) \|h_t\|^2_{L^2_\omega(\mathcal{O})}, \tag{2.7.9}$$

where we have used (2.3.8) to obtain the first inequality, and (2.7.2) together with (2.3.8) again to obtain the last inequality.

We take $\varepsilon_3^1 = \min(\varepsilon_1, \varepsilon_2)$, where $\varepsilon_1 > 0$ is given by Proposition 2.3 and $\varepsilon_2 > 0$ is given by Proposition 2.4, and we assume during the sequel that $\|g_t\|_{L^2_\omega(\mathcal{O})} \le \varepsilon$ for $\varepsilon \in (0, \varepsilon_3^1)$. We recall that $\mathscr{Q}_g$ is defined in (2.6.3), and arguing as during the proof of Proposition 2.3 (see also the arguments leading to Proposition 2.4) we have that there is $\kappa > 0$ such that

$$\begin{aligned}
\frac{d}{dt} |\!|\!| h_t |\!|\!|^2 &= \beta \langle h_t, \mathscr{Q}_g\, h_t \rangle_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \frac{d}{dt} \int_0^\infty \|S_{\mathscr{P}}(\tau) h_t\|^2_{L^2_\omega(\mathcal{O})} \mathrm{d}\tau \\
&\le -\beta \frac{\tau_0}{2} \|\nabla_v (h_t \widetilde{\omega})\|^2_{L^2(\mathcal{O})} + \beta\kappa \|h_t\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \frac{d}{dt} \int_0^\infty \|S_{\mathscr{P}}(\tau) h_t\|^2_{L^2_\omega(\mathcal{O})} \mathrm{d}\tau \\
&\le -\beta \frac{\tau_0}{2} \|\nabla_v h_t\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \beta \left(\kappa + C^0_\omega \frac{\tau_0}{2} \right) \|h_t\|^2_{L^2_{\widetilde{\omega}}(\mathcal{O})} + \frac{d}{dt} \int_0^\infty \|S_{\mathscr{P}}(\tau) h_t\|^2_{L^2_\omega(\mathcal{O})} \mathrm{d}\tau \\
&\le -\beta c_A^{-1} \frac{\tau_0}{2} \|\nabla_v h_t\|^2_{L^2_\omega(\mathcal{O})} + \beta c_A \left(\kappa + C^0_\omega \frac{\tau_0}{2} \right) \|h_t\|^2_{L^2_\omega(\mathcal{O})} + \frac{d}{dt} \int_0^\infty \|S_{\mathscr{P}}(\tau) h_t\|^2_{L^2_\omega(\mathcal{O})} \mathrm{d}\tau
\end{aligned}$$

where we have have used (2.3.8) to obtain the last line and we have defined the positive constant $C^0_\omega = \|(\nabla_v \widetilde{\omega})/\widetilde{\omega}\|_{L^\infty} < \infty$, due to (2.3.15). We then divide the rest of the proof into four steps.

Step 1. We observe that Lemma 2.1-(1) and the Lebesgue dominated convergence theorem imply that

$$\frac{d}{dt}\int_0^\infty \|S_{\mathscr{P}}(\tau)h_t\|^2_{L^2_\omega(\mathcal{O})}\mathrm{d}\tau = \int_0^\infty \frac{d}{dt}\|S_{\mathscr{P}}(\tau)h_t\|^2_{L^2_\omega(\mathcal{O})}\mathrm{d}\tau.$$

Using Remark 2.6 we notice that we may write $h(t) = S_{\mathscr{Q}_g}h_0$, where $S_{\mathscr{Q}_g}$ is the strongly continuous semigroup given by Proposition 2.4 and Remark 2.6. Moreover from Lemma 2.1-(1) we also know that $S_{\mathscr{P}}$ is a strongly continuous semigroup. Thus, using [150, Chapter 1, Corollary 1.4-(d), Corollary 2.3, and Theorem 2.4], we may perform the following computations

$$\begin{aligned}\frac{d}{dt}S_{\mathscr{P}}(\tau)h_t = S_{\mathscr{P}}(\tau)\frac{d}{dt}S_{\mathscr{Q}_g}(t)h_0 &= S_{\mathscr{P}}(\tau)\left((\mathscr{P}+\mathfrak{N}_g)h_t\right)\\ &= S_{\mathscr{P}}(\tau)\mathscr{P}h_t + S_{\mathscr{P}}(\tau)\mathfrak{N}_g h_t = \frac{d}{\mathrm{d}\tau}S_{\mathscr{P}}(\tau)h_t + S_{\mathscr{P}}(\tau)\mathfrak{N}_g h_t, \end{aligned} \tag{2.7.10}$$

where we have used repeatedly the results from [150] to obtain the previous chain of equalities. Using (2.7.10) we deduce then that

$$\begin{aligned}\frac{d}{dt}\|S_{\mathscr{P}}(\tau)h_t\|^2_{L^2_\omega(\mathcal{O})} &= 2\left\langle \frac{d}{dt}S_{\mathscr{P}}(\tau)h_t,\, S_{\mathscr{P}}(\tau)h_t\right\rangle_{L^2_\omega(\mathcal{O})}\\ &= 2\left\langle \frac{d}{\mathrm{d}\tau}S_{\mathscr{P}}(\tau)h_t + S_{\mathscr{P}}(\tau)\mathfrak{N}_g h_t,\, S_{\mathscr{P}}(\tau)h_t\right\rangle_{L^2_\omega(\mathcal{O})}\\ &= \frac{d}{\mathrm{d}\tau}\|S_{\mathscr{P}}(\tau)h_t\|^2_{L^2_\omega(\mathcal{O})} + 2\left\langle S_{\mathscr{P}}(\tau)\mathfrak{N}_g h_t,\, S_{\mathscr{P}}(\tau)h_t\right\rangle_{L^2_\omega(\mathcal{O})},\end{aligned}$$

and we now proceed to compute each term separately. On the one hand, due to the fundamental theorem of calculus, [150, Corollary 1.4-(d)], and Lemma 2.1-(1), we obtain that

$$\int_0^\infty \frac{d}{d\tau}\|S_{\mathscr{P}}(\tau)h_t\|^2_{L^2_\omega(\mathcal{O})}\mathrm{d}\tau = -\|h_t\|^2_{L^2_\omega(\mathcal{O})}.$$

On the other hand, Lemma 2.1-(3) implies the existence of $S_{\mathscr{P}^*}$, dual semigroup to $S_{\mathscr{P}}$, thus we may compute

$$\int_0^\infty \left\langle S_{\mathscr{P}}(\tau)\mathfrak{N}_g h_t,\, S_{\mathscr{P}}(\tau)h_t\right\rangle_{L^2_\omega(\mathcal{O})}\mathrm{d}\tau = \int_0^\infty \left\langle \mathfrak{N}_g h_t,\, S_{\mathscr{P}^*}(0)\left[(S_{\mathscr{P}}(\tau)h_t)\omega^2\right]\right\rangle_{L^2(\mathcal{O})}\mathrm{d}\tau =: P_1 + P_2,$$

and we control now P_1 and P_2 separately.

Step 2. (Control of P_2) We set $m = \omega^{-1}$ and we have

$$\begin{aligned}P_2 &:= \int_0^\infty \left\langle \alpha\mathcal{E}_{h_t}\Delta_v\mathfrak{F}^\alpha,\, S_{\mathscr{P}^*}(0)\left[(S_{\mathscr{P}}(\tau)h_t)\omega^2\right]\right\rangle_{L^2(\mathcal{O})}\mathrm{d}\tau\\ &= -\int_0^\infty \alpha\mathcal{E}_{h_t}\langle\nabla_v\mathfrak{F}^\alpha, \nabla_v\left(S_{\mathscr{P}^*}(0)\left[(S_{\mathscr{P}}(\tau)h_t)\omega^2\right]\right)\rangle_{L^2(\mathcal{O})}\mathrm{d}\tau\\ &\le \alpha C^1_\omega\int_0^\infty \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v\mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})}\left\|\nabla_v\left(S_{\mathscr{P}^*}(0)\left[(S_{\mathscr{P}}(\tau)h_t)\omega^2\right]\right)\right\|_{L^2_m(\mathcal{O})}\mathrm{d}\tau,\end{aligned}$$

where we remark that we have used integration by parts to obtain the second line and the Cauchy-Schwartz inequality to obtain the third one. Moreover we have defined $C^1_\omega = \|\langle v\rangle^2\omega^{-1}\|_{L^2}$ and we deduce that $C^1_\omega < \infty$ due to the very definition of ω.

We now remark that $(S_{\mathscr{P}}(\tau)h_t)\omega^2 \in L^2_m(\mathcal{O})$, indeed there holds

$$\left(\int_{\mathcal{O}} (S_{\mathscr{P}}(\tau)h_t)^2\omega^4 m^2\right)^{1/2} = \|S_{\mathscr{P}}(\tau)h_t\|_{L^2_\omega(\mathcal{O})} \lesssim e^{-\lambda\tau+\kappa t}\|h_0\|_{L^2_\omega(\mathcal{O})},$$

where we have successively used the fact that $m = \omega^{-1}$, (2.7.2) and Proposition 2.3. Coming back then to P_2 we further compute

$$\begin{aligned}
P_2 &\leq \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \lim_{A\to\infty}\int_0^A \left\|\nabla_v\left(S_{\mathscr{P}^*}(0)\left[(S_{\mathscr{P}}(\tau)h_t)\,\omega^2\right]\right)\right\|_{L^2_m(\mathcal{O})} \mathrm{d}\tau \\
&\leq \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \lim_{A\to\infty}\left\|(S_{\mathscr{P}}(A)h_t)\omega^2\right\|_{L^2_m(\mathcal{O})} \\
&\quad + \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \lim_{A\to\infty}\int_0^A \left\|S_{\mathscr{P}^*}(0)\left[(S_{\mathscr{P}}(\tau)h_t)\,\omega^2\right]\right\|_{L^2_m(\mathcal{O})} \mathrm{d}\tau \\
&\leq \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \int_0^\infty \left\|S_{\mathscr{P}^*}(0)\left[(S_{\mathscr{P}}(\tau)h_t)\,\omega^2\right] - \langle S_{\mathscr{P}}(\tau)h_t, \mathfrak{H}\rangle_{L^2_\omega(\mathcal{O})}\right\|_{L^2_m(\mathcal{O})} \mathrm{d}\tau \\
&\quad + \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \int_0^\infty \langle S_{\mathscr{P}}(\tau)h_t, \mathfrak{H}\rangle_{L^2_\omega(\mathcal{O})}\,\|m\|_{L^2(\mathcal{O})}\,\mathrm{d}\tau \\
&=: P_2^1 + P_2^2
\end{aligned}$$

where we have used (2.7.7) to obtain the second inequality and the triangular inequality to obtain the last inequality. On the one hand we have

$$\begin{aligned}
P_2^1 &:= \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \int_0^\infty \left\|S_{\mathscr{P}^*}(0)\left[(S_{\mathscr{P}}(\tau)h_t)\,\omega^2\right] - \langle S_{\mathscr{P}}(\tau)h_t, \mathfrak{H}\rangle_{L^2_\omega(\mathcal{O})}\right\|_{L^2_m(\mathcal{O})} \mathrm{d}\tau \\
&\leq \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} C_P^* \int_0^\infty e^{-\lambda\tau}\left\|(S_{\mathscr{P}}(\tau)h_t)\,\omega^2 - \langle S_{\mathscr{P}}(\tau)h_t, \mathfrak{H}\rangle_{L^2_\omega(\mathcal{O})}\right\|_{L^2_m(\mathcal{O})} \mathrm{d}\tau \\
&\leq \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} C_P^* \int_0^\infty e^{-\lambda\tau}\,\|S_{\mathscr{P}}(\tau)h_t\|_{L^2_\omega(\mathcal{O})}\,\mathrm{d}\tau \\
&\quad + \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} C_P^* \int_0^\infty \langle S_{\mathscr{P}}(\tau)h_t, \mathfrak{H}\rangle_{L^2_\omega(\mathcal{O})}\,\|m\|_{L^2(\mathcal{O})}\,\mathrm{d}\tau
\end{aligned}$$

where we have used (2.7.6) to obtain the first inequality, the triangular inequality to obtain the second inequality. Using then (2.7.2) and the Cauchy-Schwartz inequality we further have that

$$\begin{aligned}
P_2^1 &\leq \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} C_P^* C_P \int_0^\infty e^{-2\lambda\tau}\,\|h_t\|_{L^2_\omega(\mathcal{O})}\,\mathrm{d}\tau \\
&\quad + \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} C_P^* \int_0^\infty \|S_{\mathscr{P}}(\tau)h_t\|_{L^2_\omega(\mathcal{O})}\,\|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})}\,\|m\|_{L^2(\mathcal{O})}\,\mathrm{d}\tau \\
&\leq \alpha \frac{C^1_\omega}{2\lambda}\|h_t\|^2_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} C_P^* C_P \\
&\quad + \alpha C^1_\omega \|h_t\|_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} C_P^*\,\|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})}\,\|m\|_{L^2(\mathcal{O})}\,C_P \int_0^\infty e^{-\lambda\tau}\,\|h_t\|_{L^2_\omega(\mathcal{O})}\,\mathrm{d}\tau \\
&\leq \alpha C^1_\omega \|h_t\|^2_{L^2_\omega(\mathcal{O})}\|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} C_P^* C_P \left(\frac{1}{2\lambda} + \frac{1}{\lambda}\,\|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})}\,\|m\|_{L^2(\mathcal{O})}\right)
\end{aligned}$$

where we have used again (2.7.2) to obtain the last inequality.

On the other hand we compute

$$\begin{aligned}
P_2^2 &:= \alpha C_\omega^1 \|h_t\|_{L^2_\omega(\mathcal{O})} \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \int_0^\infty \langle S_{\mathscr{P}}(\tau) h_t, \mathfrak{H}\rangle_{L^2_\omega(\mathcal{O})} \, \|m\|_{L^2(\mathcal{O})} \, \mathrm{d}\tau \\
&\leq \alpha C_\omega^1 \|h_t\|_{L^2_\omega(\mathcal{O})} \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \int_0^\infty \|S_{\mathscr{P}}(\tau) h_t\|_{L^2_\omega(\mathcal{O})} \, \|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})} \, \|m\|_{L^2(\mathcal{O})} \, \mathrm{d}\tau \\
&\leq \alpha C_\omega^1 \|h_t\|_{L^2_\omega(\mathcal{O})} \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \, \|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})} \, \|m\|_{L^2(\mathcal{O})} \, C_P \int_0^\infty e^{-\lambda\tau} \, \|h_t\|_{L^2_\omega(\mathcal{O})} \, \mathrm{d}\tau \\
&\leq \alpha C_\omega^1 \|h_t\|^2_{L^2_\omega(\mathcal{O})} \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} \frac{1}{\lambda} \, \|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})} \, \|m\|_{L^2(\mathcal{O})} \, C_P
\end{aligned}$$

where we have used successively the Cauchy-Schwartz inequality to obtain the second line and (2.7.2) to obtain the third line.
Putting together the previous computations we observe that we have obtained

$$P_2 \leq \alpha C_\omega^1 \|h_t\|^2_{L^2_\omega(\mathcal{O})} \|\nabla_v \mathfrak{F}^\alpha\|_{L^2_\omega(\mathcal{O})} C_P \left(\frac{C_P^*}{2\lambda} + \frac{C_P^* + 1}{\lambda} \, \|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})} \, \|m\|_{L^2(\mathcal{O})} \right),$$

and this concludes this step.

Step 3. (Control of P_1) We compute now for the term P_1 as follows

$$\begin{aligned}
P_1 &:= \int_0^\infty \alpha \mathcal{E}_{g_t} \langle \Delta_v h_t, S_{\mathscr{P}^*}(0) \left[S_{\mathscr{P}}(\tau) h_t \omega^2 \right] \rangle_{L^2(\mathcal{O})} d\tau \\
&= -\int_0^\infty \alpha \mathcal{E}_{g_t} \left\langle \nabla_v h_t, \nabla_v \left(S_{\mathscr{P}^*}(0) \left[\left(S_{\mathscr{P}}(\tau) h_t \right) \omega^2) \right] \right) \right\rangle_{L^2(\mathcal{O})} \mathrm{d}\tau \\
&= -\int_0^\infty \alpha \mathcal{E}_{g_t} \left\langle \nabla_v h_t, \nabla_v \left(S_{\mathscr{P}^*}(0) \left[\left(S_{\mathscr{P}}(\tau) h_t \right) \omega^2) \right] \right) \right\rangle_{L^2(\mathcal{O})} \mathrm{d}\tau \\
&\leq \alpha C_\omega^1 \int_0^\infty \|g_t\|_{L^2_\omega(\mathcal{O})} \|\nabla_v h_t\|_{L^2_\omega(\mathcal{O})} \|\nabla_v (S_{\mathscr{P}^*}(0) \left[\left(S_{\mathscr{P}}(\tau) h_t \right) \omega^2 \right]) \|_{L^2_m(\mathcal{O})} \mathrm{d}\tau,
\end{aligned}$$

where we have successively used integration by parts and the Cauchy-Schwartz inequality. Arguing exactly as for the term P_2 during the Step 2 we deduce that

$$P_1 \leq \alpha C_\omega^1 \|h_t\|_{L^2_\omega(\mathcal{O})} \|g_t\|_{L^2_\omega(\mathcal{O})} \|\nabla_v h_t\|_{L^2_\omega(\mathcal{O})} C_P \left(\frac{C_P^*}{2\lambda} + \frac{C_P^* + 1}{\lambda} \, \|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})} \, \|m\|_{L^2(\mathcal{O})} \right).$$

Using then the Young inequality we further have that

$$\begin{aligned}
P_1 &\leq \frac{\alpha}{2} C_\omega^1 \|g_t\|_{L^2_\omega(\mathcal{O})} \|\nabla_v h_t\|^2_{L^2_\omega(\mathcal{O})} C_P \left(\frac{C_P^*}{2\lambda} + \frac{C_P^* + 1}{\lambda} \, \|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})} \, \|m\|_{L^2(\mathcal{O})} \right) \\
&\quad + \frac{\alpha}{2} C_\omega^1 \|h_t\|^2_{L^2_\omega(\mathcal{O})} \|g_t\|_{L^2_\omega(\mathcal{O})} C_P \left(\frac{C_P^*}{2\lambda} + \frac{C_P^* + 1}{\lambda} \, \|\mathfrak{H}\|_{L^2_\omega(\mathcal{O})} \, \|m\|_{L^2(\mathcal{O})} \right).
\end{aligned}$$

Step 4. (Choice of parameters and conclusion) Putting together the computations performed during the Steps 1, 2 and 3, and using (2.1.9) and (2.7.4), we have that

$$\begin{aligned}
\frac{d}{dt} \|\!|h_t|\!\|^2 &\leq -\beta c_A^{-1} \frac{\tau_0}{2} \|\nabla_v h_t\|^2_{L^2_\omega(\mathcal{O})} + \beta c_A \left(\kappa + C_\omega^0 \frac{\tau_0}{2} \right) \|h_t\|^2_{L^2_\omega(\mathcal{O})} - \|h_t\|^2_{L^2_\omega(\mathcal{O})} \\
&\quad + \alpha C_\omega^1 \|h_t\|^2_{L^2_\omega(\mathcal{O})} C_{\mathfrak{F}^\alpha} C_P \left(\frac{C_P^*}{2\lambda} + \frac{C_P^* + 1}{\lambda} C_{\mathfrak{H}} \, \|m\|_{L^2(\mathcal{O})} \right) \\
&\quad + C_\omega^1 \|g_t\|_{L^2_\omega(\mathcal{O})} \|\nabla_v h_t\|^2_{L^2_\omega(\mathcal{O})} C_P \left(\frac{C_P^*}{2\lambda} + \frac{C_P^* + 1}{\lambda} C_{\mathfrak{H}} \, \|m\|_{L^2(\mathcal{O})} \right) \\
&\quad + C_\omega^1 \|h_t\|^2_{L^2_\omega(\mathcal{O})} \|g_t\|_{L^2_\omega(\mathcal{O})} C_P \left(\frac{C_P^*}{2\lambda} + \frac{C_P^* + 1}{\lambda} C_{\mathfrak{H}} \, \|m\|_{L^2(\mathcal{O})} \right),
\end{aligned}$$

where the constants $C_{\mathfrak{F}^\alpha} > 0$ and $C_{\mathfrak{H}} > 0$ are given by (2.1.9) and (2.7.4) respectively. We then choose

$$\begin{aligned}
\alpha &= \frac{1}{4}\left(C_\omega^1 C_{\mathfrak{F}^\alpha} C_P \left(\frac{C_P^*}{2\lambda} + \frac{C_P^*+1}{\lambda} C_{\mathfrak{H}} \|m\|_{L^2(\mathcal{O})}\right)\right)^{-1}, \\
\beta &= \frac{1}{4} c_A^{-1}\left(\kappa + C_\omega^0 \frac{\tau_0}{2}\right), \\
\varepsilon_3^2 &= \min\left(\frac{1}{4}, \frac{\beta c_A^{-1}}{4}\right)\left(C_\omega^1 C_P \left(\frac{C_P^*}{2\lambda} + \frac{C_P^*+1}{\lambda} C_{\mathfrak{H}} \|m\|_{L^2(\mathcal{O})}\right)\right)^{-1},
\end{aligned}$$

and we set $\varepsilon_3 = \min(\varepsilon_3^1, \varepsilon_3^2)$. This selection of parameters implies that

$$\frac{d}{dt}|||h_t|||^2 \leq -\frac{1}{4}\|h_t\|^2_{L^2_\omega(\mathcal{O})},$$

and we conclude the proof by using (2.7.9) and the Grönwall lemma. □

2.8 Proof of Theorem 2.1.3

We consider the non-linear equation

$$\left\{\begin{array}{rcll}
\partial_t h &=& \mathscr{Q}_h h & \text{in } \mathcal{U} \\
\gamma_- h &=& \mathscr{R}\gamma_+ h & \text{on } \Gamma_- \\
h_{t=0} &=& h_0 & \text{in } \mathcal{O},
\end{array}\right. \tag{2.8.1}$$

where we recall that $\mathscr{Q}_h$ is defined in (2.6.3), and we will dedicate this section to prove an equivalent version of Theorem 2.1.3 in terms of the solutions of Equation (2.8.1).

Indeed, we recall from Remark 2.2 that if there is a fixed point for the map that associates g to a solution h of Equation (2.6.2), then $f = \mathfrak{F}^\alpha + h$ will be a solution of Equation (2.1.1)-(2.1.2)-(2.1.4). Moreover the decay estimate (2.1.10) will be a direct consequence of Proposition 2.2.

Proposition 2.1. *Let ω be an admissible weight function. There is a constant $\delta > 0$ such that, for any $\alpha \in (0, \alpha^{\star\star})$, where $\alpha^{\star\star} > 0$ is given by Proposition 2.2, and any initial datum $h_0 \in L^2_\omega(\mathcal{O})$ such that*

$$\|h_0\|_{L^2_\omega(\mathcal{O})} \leq \delta,$$

there is $h \in L^2_\omega(\mathcal{U})$ unique global weak solution of Equation (2.8.1) *in the sense of Proposition 2.4. Furthermore, the decay estimate* (2.7.8) *holds for the solutions of Equation* (2.8.1).

Remark 2.2. The proof follows the main ideas from the proof of [49, Theorem 1.1].

Proof. We define the ball

$$\mathcal{Z} := \{g \in L^2_\omega(\mathcal{U}),\ \|g_t\|_{L^2_\omega(\mathcal{O})} \leq \varepsilon_3 \quad \forall t \geq 0\},$$

where $\varepsilon_3 > 0$ is given by Proposition 2.2, and it is worth remarking that $\varepsilon_3 \leq \min(\varepsilon_1, \varepsilon_2)$, where $\varepsilon_1, \varepsilon_2 > 0$ are given by Propositions 2.3 and 2.4 respectively, thus making them valid. We set $\delta = \varepsilon_3/C_0$ where $C_0 \geq 1$ is given by Proposition 2.2, and we define the map

$$\Phi : \mathcal{Z} \to \mathcal{Z}, \quad g \mapsto \Phi(g) = G := S_{\mathscr{Q}_g} h_0.$$

It is worth emphasizing that, due to our choice of δ, Proposition 2.2 will ensure that $S_{\mathscr{Q}_g} h_0 \in \mathcal{Z}$ for every $g \in \mathcal{Z}$. Finally, we endow $\mathcal{Z}$ with the weak topology induced by $L^2_\omega(\mathcal{O})$, which makes $\mathcal{Z}$ a convex and compact set.

Step 1. (Continuity of the map Φ) We consider a sequence (g_n) in $\mathcal{Z}$ such that $g_n \rightharpoonup g$ weakly in $\mathcal{Z}$ as $n \to \infty$ and we define $G_n := S_{\mathscr{Q}_{g_n}} h_0$. Using Proposition 2.2 together with (2.6.6), we have that

$$\|G_n(t,\cdot)\|_{L^2_\omega(\mathcal{O})} + \|\nabla_v G_n\|_{L^2_\omega(\mathcal{U})} \leq C_0 \|h_0\|_{L^2_\omega} \leq \varepsilon_3 \quad \forall\, t \geq 0,$$

so that $G_n \in \mathcal{Z} \cap L^2(\mathbb{R}_+ \times \Omega; H^1_\omega(\mathbb{R}^d))$. Thus there exist a subsequence $(G_{n'})$ and a function $G \in \mathcal{Z} \cap L^2(\mathbb{R}_+ \times \Omega; H^1_\omega(\mathbb{R}^d))$ such that

$$G_{n'} \rightharpoonup G \quad \text{weakly in } \mathcal{Z} \cap L^2(\mathbb{R}_+ \times \Omega; H^1_\omega(\mathbb{R}^d)) \text{ as } n' \to \infty. \tag{2.8.2}$$

We emphasize that $G_{n'}$ solves

$$\partial_t G_{n'} = \mathscr{P} G_{n'} + \mathfrak{N}_{g_{n'}} G_{n'}, \quad \gamma_- G_{n'} = \mathscr{R} \gamma_+ G_{n'}, \quad (G_{n'})_{|t=0} = h_0, \tag{2.8.3}$$

in the sense provided by Proposition 2.4. Therefore, by arguing as during the Step 3.1 of the proof of Proposition 2.4, we have that for any truncated (in t and x) version $(\bar{G}_{n'})$ of $(G_{n'})$, we may apply [29, Theorem 1.3], which gives that $(\bar{G}_{n'})$ is bounded in $H^{1/4}(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{R}^d_v)$. Using then the version of the Rellich-Kondrachov theorem for fractional Sobolev spaces (see for instance [74, Corollary 7.2] or [148, Lemma 6.11]) we deduce that

$$(G_{n'}) \text{ is relatively compact in } L^2((0,T) \times \mathcal{O}_R). \tag{2.8.4}$$

for any $T, R > 0$, and we recall that $\mathcal{O}_R := \{(x,v) \in \mathcal{O};\, d(x, \Omega^c) > 1/R,\, |v| < R\}$.

Moreover, setting $\varsigma = \omega \langle v \rangle^{-1/2}$ we observe that the sequence $(G_{n'})$ is tight in $L^2_\varsigma(\mathcal{U})$. Indeed we have that

$$\|G_{n'}t\|_{L^2_\varsigma(\mathcal{O}^c_R)} \leq \frac{1}{\langle R \rangle^{1/2}} \|G_{n'}t\|_{L^2_\omega(\mathcal{O}^c_R)} \leq \frac{1}{\langle R \rangle^{1/2}} \|G_{n'}t\|_{L^2_\omega(\mathcal{O})} \lesssim \frac{1}{\langle R \rangle^{1/2}} e^{-\eta t} \|h_0\|_{L^2_\omega(\mathcal{O})}, \tag{2.8.5}$$

for every $t \geq 0$, and where we remark that we have used (2.7.8) to obtain the last inequality and $\eta > 0$ is given by Proposition (2.2). Integrating (2.8.5) in the time interval (T, ∞) we further deduce that

$$\|G_{n'}t\|_{L^2_\varsigma((T,\infty) \times \mathcal{O}^c_R)} \lesssim \frac{1}{\langle R \rangle^{1/2}} e^{-2\eta T} \|h_0\|_{L^2_\omega(\mathcal{O})}, \tag{2.8.6}$$

which gives the tightness of the sequence $G_{n'}$ in $L^2_\varsigma(\mathcal{U})$. We then classically deduce from (2.8.4) and (2.8.6) that

$$G_{n'} \to G \;\text{ strongly in }\; L^2_\varsigma((0,\infty) \times \mathcal{O}) \text{ as } n' \to \infty.$$

Using this convergence, the Cauchy-Schwartz inequality, and the fact that $\langle v \rangle^{5/2} \omega^{-1} \in L^2(\mathbb{R}^d)$, we deduce then that

$$\mathcal{E}_{G_{n'}} \to \mathcal{E}_G \qquad \text{as } n' \to \infty. \tag{2.8.7}$$

Using (2.8.2) and (2.8.7) we may then pass to the limit in the weak formulation associated to Equation (2.8.3) and we obtain that G solves

$$\partial_t G = \mathscr{P} G + \mathfrak{N}_g G, \quad \gamma_- G = \mathscr{R} \gamma_+ G, \quad G_{|t=0} = h_0,$$

in the sense of Proposition 2.4. Moreover, from Proposition 2.2 and Proposition 2.3 we also have that

$$\|G\|_{L^2_\omega(\mathcal{U})} \leq \varepsilon_3, \qquad \text{and} \qquad \int_0^\infty \int_{\mathcal{O}} |\nabla_v G|^2 \omega^2 \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \lesssim \|h_0\|^2_{L^2_\omega(\mathcal{O})}.$$

Finally, due to the uniqueness given by Proposition 2.4 we get that G is the only possible function in $L^2_\omega(\mathcal{U})$ such that $G = S_{\mathscr{Q}_g} h_0$. By the uniqueness of the possible limit we deduce that the map Φ is continuous.

Step 2. (Conclusion) We may apply now Schauder's fixed point theorem in the space $\mathcal{Z}$ with the map Φ, and we obtain that there exists $h \in \mathcal{Z}$ such that $h = \Phi(h)$. Moreover, due to the very definition of Φ, we have that h is a global weak solution of Equation 2.8.1 in the sense of Theorem 2.4, and it satisfies the decay estimate (2.7.8). This concludes the proof. $\square$

Part II

The Boltzmann equation in the hydrodynamic limit regime

Chapter 3

The Boltzmann equation on smooth and cylindrical domains.

In this chapter we study the well-posedness of the Boltzmann equation near its hydrodynamic limit on a bounded domain. We consider two types of domains, namely C^2 domains with Maxwell boundary conditions where the accommodation coefficient is a continuous space dependent function $\iota \in [\iota_0, 1]$ for any $\iota_0 \in (0, 1]$, or cylindrical domains with diffusive reflection on the bases of the cylinder and specular reflection on the rest of the boundary. Furthermore, we work with polynomial, stretched exponential and inverse gaussian weights to construct the Cauchy theory near the equilibrium. We remark that all methods are quantitative thus all the constants are constructive and tractable.

3.1 Introduction

This chapter investigates the well-posedness and long-time behavior of the Boltzmann equation in the regime close to the *hydrodynamic limit*. We consider this problem in bounded spatial domains equipped with Maxwell boundary conditions, and where we distinguish between two types of geometries: general C^2 domains and cylindrical domains.

We remark that we take an interest in cylindrical domains motivated by physical applications, the mathematical complexity of dealing with irregular domains, and as a first step towards more complex settings in future works.

Our strategy to achieve our results employs and extends the *hypocoercivity* techniques developed in [21], as well as the *stretching method* introduced in [110]. In particular, we remark that during this chapter we craft a more delicate $L^2 - L^\infty$ theory than the one employed in [110] and [109], which allows us to achieve quantitative decay estimates towards equilibrium.

3.1.1 Framework

We consider a small $\varepsilon > 0$ and we study the following Boltzmann equation

$$\varepsilon \partial_\tau F = -v \cdot \nabla_y F + \varepsilon^{-1} \mathcal{Q}(F, F) \quad \text{in } \mathcal{U} := (0, +\infty) \times \Omega \times \mathbb{R}^3, \tag{3.1.1}$$

where $F = F(\tau, y, v)$ is a density function representing particles which at time $\tau \in (0, \infty)$ are located at position $y \in \Omega \subset \mathbb{R}^3$ and moving with velocity $v \in \mathbb{R}^3$.

The presence of the small parameter $\varepsilon > 0$ in the equation reflects the fact that the system is close to its *hydrodynamic limit*. For a detailed discussion of the physical interpretation and the main mathematical results concerning this type of limit, we refer the reader to [156] and the references therein.

The *Boltzmann collision operator* $\mathcal{Q}$ represents the collisions between particles inside Ω, and is given by the bilinear form

$$\mathcal{Q}(G,H) := \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\,\left[G(v_*')H(v') + H(v_*')G(v') - G(v_*)H(v) - G(v)H(v_*)\right] \mathrm{d}\sigma \mathrm{d}v_*,$$

where we have defined

$$v' := v - ((v-v_*)\cdot\sigma)\sigma, \qquad v_*' := v_* + ((v-v_*)\cdot\sigma)\sigma,$$

with $\sigma \in \mathbb{S}^2$, and the *collision kernel* $\mathcal{B} = \mathcal{B}(|v-v_*|,\sigma)$. We remark that $\mathcal{B}$ describes the *type of interaction* particles exhibit and, during this chapter, we will choose the so-called *hard spheres* model by taking

$$\mathcal{B}(|v-v_*|,\sigma) := |(v-v_*)\cdot\sigma|.$$

We assume Ω to be a bounded domain such that $|\Omega|= 1$, and we assume that there exists $\delta \in W^{1,\infty}(\mathbb{R}^3,\mathbb{R})$ in such a way that $\Omega = \{y \in \mathbb{R}^3,\ \delta(y) > 0\}$, and $|\delta(y)| = \mathrm{dist}(y, \partial\Omega)$ on a neighborhood of the boundary. We then define the normal outward vector

$$n_y = n(y) := -\frac{\nabla\delta(y)}{|\nabla\delta(y)|} \quad \text{for almost every } y \in \bar{\Omega}.$$

We further define the boundary set $\Sigma = \partial\Omega \times \mathbb{R}^d$ and we distinguish between the sets of *outgoing* (Σ_+), *incoming* (Σ_-), and *grazing* (Σ_0) velocities at the boundary by

$$\Sigma_\pm := \{(y,v) \in \Sigma,\ \pm n_y \cdot v > 0\}, \quad \text{and} \quad \Sigma_0 := \{(y,v) \in \Sigma,\ n_y\cdot v = 0\}.$$

Furthermore, we denote $\Gamma := (0,\infty)\times\Sigma$ and accordingly $\Gamma_\pm := (0,\infty)\times\Sigma_\pm$. We define γF as the trace function associated with F over Γ and $\gamma_\pm F := \mathbf{1}_{\Gamma_\pm}\gamma F$.

We then complement the Boltzmann equation (3.1.1) with the *Maxwell boundary condition*

$$\gamma_- F(\tau,y,v) = \mathscr{R}\gamma_+F(\tau,y,v) := (1-\iota(y))\mathscr{S}\gamma_+F(\tau,y,v) + \iota(y)\mathscr{D}\gamma_+F(\tau,y,v) \quad \text{on } \Gamma_-, \tag{3.1.2}$$

where we have defined the *accommodation coefficient* $\iota : \partial\Omega \to [0,1]$, the *specular reflection* operator

$$\mathscr{S}\gamma_+F(\tau,y,v) := \gamma_+F(\tau,y,\mathcal{V}_y v) \quad \text{with} \quad \mathcal{V}_y v = v - 2(n_y\cdot v)n_y,$$

and the *diffusive reflection* operator

$$\mathscr{D}\gamma_+F(\tau,y,v) := \mathscr{M}(v)\widetilde{\gamma_+F} \quad \text{where} \quad \widetilde{\gamma_+F} = \int_{\mathbb{R}^d}\gamma_+F(\tau,y,u)(n(y)\cdot u)_+ du.$$

We have also defined the *Maxwellian* distributions

$$\mathscr{M} := \sqrt{2\pi}\mathcal{M}, \quad \text{with} \quad \mathcal{M} = \mathcal{M}(v) := (2\pi)^{-3/2}e^{-\frac{|v|^2}{2}},$$

and it is worth remarking that $\widetilde{\mathscr{M}} = 1$.

We present now the two types of *geometric assumptions* for our domain Ω, and the respective choice for the accommodation coefficient in each case.

(H1) Assume $\Omega \subset \mathbb{R}^3$ is an open C^2 domain, and $\delta \in C^2(\mathbb{R}^3, \mathbb{R}) \cap W^{3,\infty}(\mathbb{R}^3, \mathbb{R})$. Moreover, take $\iota \in C(\partial\Omega)$ and assume that there is $\iota_0 \in (0, 1]$ such that for every $y \in \partial\Omega$ there holds $\iota(y) \in [\iota_0, 1]$.

(H2) Assume $\Omega = (-L, L) \times \Omega_0$, for some $L > 0$ and where $\Omega_0 \subset \mathbb{R}^2$ is the 2-dimensional ball of radius $\mathfrak{R} > 0$ centered at the origin. In this case we also define

$$\Lambda_1 := \{-L\} \times \Omega_0, \quad \Lambda_2 := \{L\} \times \Omega_0, \quad \Lambda_3 := (-L, L) \times \partial\Omega_0,$$

and $\Lambda := \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$. Furthermore, we impose *mixed* boundary conditions by taking $\iota = \mathbf{1}_{\Lambda_1 \cup \Lambda_2}$, i.e purely diffusive boundary condition on the bases of the cylinder $(\Lambda_1 \cup \Lambda_2)$, and specularity on the lateral surface (Λ_3).

Finally, we complement Equation (3.1.1)-(3.1.2) with the initial condition

$$F(\tau = 0, \cdot) = F_0 \quad \text{in } \mathcal{O} := \Omega \times \mathbb{R}^3, \tag{3.1.3}$$

for some function F_0 satisfying $\langle\langle F_0 \rangle\rangle_{\mathcal{O}} := \int_{\mathcal{O}} F_0 \,\mathrm{d}y\mathrm{d}v = 1$.

3.1.2 Main result and discussion

In order to express our main result we need to introduce the set of the so-called *admissible weight functions* given by:

- *Polynomial weights:* We consider $\omega(v) = (1 + |v|^2)^{q/2}$ with $q > q_\iota^\star$, for some explicit $q_\iota^\star > 0$ defined in Remark 3.2.
- *Stretched exponential weights:* We consider $\omega(v) = e^{\zeta|v|^s}$ with $s \in (0, 2)$ and $\zeta > 0$.
- *Inverse gaussian weights:* We consider $\omega(v) = e^{\zeta|v|^2}$ with $\zeta \in (0, 1/2)$.

Moreover, for a given measure space $(Z, \mathscr{Z}, \mu)$, a weight function $\rho : Z \to (0, \infty)$, and an exponent $p \in [1, \infty]$, we define the weighted Lebesgue spaces $L^p_\rho(Z)$ associated to the norm

$$\|g\|_{L^p_\rho(Z)} = \|\rho g\|_{L^p(Z)}. \tag{3.1.4}$$

In this framework we have the following result for the Boltzmann equation.

Theorem 3.1.1. *Consider either Assumption (H1) or Assumption (H2) to hold, and et ω be an admissible weight function. There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is $\eta(\varepsilon) \in (0, 1)$, satisfying $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for every $F_0 \in L^\infty_\omega(\mathcal{O})$ satisfying*

$$\|F_0 - \mathcal{M}\|_{L^\infty_\omega(\mathcal{O})} \leq (\eta(\varepsilon))^2,$$

there exists a function $F \in L^\infty_\omega(\mathcal{U})$ unique global solution to the Boltzmann equation (3.1.1)–(3.1.2)–(3.1.3) *in the distributional sense. Furthermore, there is a constructive constant $\theta > 0$ such that*

$$\|F_\tau - \mathcal{M}\|_{L^\infty_\omega(\mathcal{O})} \leq e^{-\theta\tau} \eta(\varepsilon) \qquad \forall \tau \geq 0.$$

The precise sense of the solution given by Theorem 3.1.1 is constructed in Theorem 3.1.3, after taking $f = F - \mathcal{M}$, performing the change of variables from Subsection 3.1.3, and studying the resulting equation.

We now briefly discuss the history of the progress on the study of the Boltzmann equation, and we pay particular attention to boundary value problems. This has been an active field of mathematical research since the groundbreaking works of L. Boltzmann in [28] and J.C. Maxwell in [131].

We first mention the framework of renormalized solutions introduced by DiPerna and Lions in [75] for any arbitrary data. Even though, particularly in this setting, renormalized solutions are weaker than standard distributional solutions, this is the only known way of providing well-posedness for the Boltzmann equation for arbitrary data. Later on, there were many works developing this setting for the Boltzmann equation and, as a byproduct enlarging, the notion of renormalizing solutions for general kinetic equations, for references on this subject we refer to [165, Chapter 1, Section 5.6] by C. Villani.

On renormalized solutions for the Boltzmann equation in bounded domains we cite the works by S. Mischler in [137, 138], and we also refer to the results presented in the review paper of C. Cercignani [55], and the references therein.

For other results concerning the Boltzmann equation (and variants), either on the torus or on the whole space, we refer to [100, 102, 105, 107, 143, 163].

Regarding solely the long-time asymptotic behavior of (given) solutions we mention [71] by L. Desvilletes and C. Villani, where they obtain relaxation to a Maxwellian global equilibrium for positivity and regular solutions.

On the existence of weak distributional solutions in bounded domains and their long-time asymptotics, we mention the pioneering paper by Y. Guo [108], where the boundary conditions are either inflow, bounce-back, specular, or diffusive. It is worth mentioning though, that Guo assumes the spatial domain to be convex and analytic, which are quite demanding hypothesis in applications.

In the last decade there have been several studies obtaining the same results as those by [108] with the objective of relaxing their geometrical assumptions: we cite [35] where M. Briant studied a Boltzmann equations with diffusive reflexion along a C^1 domain—we note that he also studies the problem for specular reflections but under the extra assumption of having an analytical and convex boundary. We also refer to the work of C. Kim and D. Lee in [118] for C^3 convex domains with specular reflections.

We also mention [58] where H. Chen and R. Duan consider a problem for a problem with mixed boundary conditions. It is worth remarking that such mixed boundary conditions are similar in objective as those of our Assumption (H2), however instead of a cylinder they consider a C^1 domain.

For results on the Boltzmann equation in irregular bounded domains we cite [9] by L. Arkeryd and A. Heintz for a general conservative boundary condition, and, more particularly in the setting of cylindrical domains, we further cite [119], where C. Kim and D. Lee. construct weak solutions for the Boltzmann equation complemented with specular boundary conditions.

Regarding the problem under Maxwell boundary conditions we have the result from M. Briant and Y. Guo in [36], under the conditions of having an accommodation coefficient bounded below by $\sqrt{2/3}$, and for a C^1 bounded domain.

For Boltzmann equations with long-range interaction (no Grad's cutofff assumption) we cite the more recent paper [67] where D. Deng studies the problem within a C^3 bounded domain and complemented with a Maxwell boundary condition and a fixed accommodation coefficient $\iota \in (0, 1]$.

Moreover, we also have the preprint from Y. Guo and F. Zhou [110], and their recently updated version with J. Jung [109], where they study a Boltzmann equation near the hydrodynamic limit, in a C^2 domain, complemented with Maxwell boundary conditions with a constant accommodation coefficient within $[0, 1]$. They construct a perturbative well-posedness theory for this problem and they also study its hydrodynamic limit.

The framework for our chapter is motivated by, and therefore closely related to, the one developed in [109, 110]. However, within the study of the Boltzmann equation, our result constitutes a genuine generalization of theirs in several key aspects. First, we craft a more delicate $L^2 - L^\infty$ theory than the one presented in [109, 110] for the linearized problem, allowing us to derive explicit and constructive decay rates to the equilibrium, going beyond the boundedness results previously obtained. Second, we also carry out the analysis in cylindrical domains, introducing the presence of geometric irregularities to the obtention of the estimates, thereby generalizing the stretching method from [109, 110] for the cylindrical setting. Third, even though we do not achieve the full range of $[0, 1]$ for the accommodation coefficient in smooth domains, we allow ι to be a spatially dependent continuous function, and within cylindrical domains we further have it to be discontinuous.

This last fact makes Theorem 3.1.1 also a generalization of [36], where we recall that the accommodation coefficient had an imposed lower bound of $\sqrt{2/3}$. It is also worth mentioning that during the obtention of our main result we have extended, in Section 3.2, the results from [21] by constructing hypocoercivity estimates in cylindrical domains satisfying Assumption (RH2). Furthermore, we also provide well-posedness results for initial data with a wide range of decaying tail at infinity, including polynomial.

Notably, we emphasize that, to the best of our knowledge, this is the first well-posedness result for the Boltzmann equation in cylindrical domains with Maxwell boundary conditions and a discontinuous accommodation coefficient ranging over the entire interval $[0, 1]$.

At last, we explain the main motivation for considering cylindrical domains in the study of the Boltzmann equation. From a physics standpoint, this geometry is fundamental both theoretically and practically, due to the fact that many real world systems naturally exhibit cylindrical shapes. Moreover, we are particularly interested by future investigations of the Boltzmann equation in cylindrical domains where each base, Λ_1 and Λ_2, presents diffusive boundary condition associated with different temperatures. This setup falls within the physical framework of *non-equilibrium thermodynamics*, and raises mathematically interesting questions regarding the existence of non-equilibrium steady states and their qualitative properties, such as uniqueness and stability. For a physics-oriented discussion of this topic we refer to [94, 117] and [93, Chapter 9]. We refer to [7, 8, 10, 42, 44, 82, 83] for similar settings in the framework of the Boltzmann equation and to [19, 41, 43, 45, 84–86] and the references therein for related results in other kinetic models.

3.1.3 Transformation of the problem

To study the Boltzmann Equation (3.1.1)-(3.1.2)-(3.1.3), we define the function $\bar{F}$ such that

$$\bar{F}(\tau, y, v) = F(\tau, y, v) - \mathcal{M}(v),$$

and we make the changes of variables $\tau = \varepsilon^2 t$, $y = \varepsilon x$, so we introduce $\Omega^\varepsilon := \{\varepsilon^{-1} y,\, y \in \Omega\}$ and $\mathcal{O}^\varepsilon = \Omega^\varepsilon \times \mathbb{R}^3$. Then we naturally define

$$f(t, x, v) := \bar{F}(\varepsilon^2 t, \varepsilon x, v),$$

so that f satisfies the *linearized rescaled Boltzmann equation*

$$\partial_t f = -v \cdot \nabla_x f + \mathscr{C} f + \mathcal{Q}(f, f) \quad \text{in } \mathcal{U}^\varepsilon := (0, +\infty) \times \mathcal{O}^\varepsilon, \tag{3.1.5}$$

where we have defined the *linearized Boltzmann operator* $\mathscr{C} f := \mathcal{Q}(\mathcal{M}, f) + \mathcal{Q}(f, \mathcal{M})$. We decompose now

$$\mathscr{C} f = K f - \nu f,$$

where, on the one hand, we have the non-local operator

$$\begin{aligned} Kf = Kf(\cdot, v) &:= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathcal{B}\left[\mathcal{M}(v_*')f(v') + \mathcal{M}(v')f(v_*') - \mathcal{M}(v)f(v_*)\right] d\sigma dv_* \\ &= \int_{\mathbb{R}^3} k(v, v_*) f(v_*) dv_*, \end{aligned}$$

with the kernel function

$$\begin{aligned} k = k(v, v_*) := \sqrt{\frac{2}{\pi}} |v - v_*|^{-1} \exp\left(-\frac{1}{8}\frac{(|v_*|^2 - |v|^2)^2}{|v - v_*|^2} - \frac{1}{8}|v - v_*|^2 - \frac{|v|^2}{4} + \frac{|v_*|^2}{4} \right) \\ - \frac{1}{2}|v - v_*| \exp\left(-\frac{|v|^2}{2} \right), \end{aligned}$$

see for instance [56, Theorem 7.2.1] for a derivation of k, up to a conjugate change of scale. On the other hand, we have

$$\nu = \nu(v) := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\,\mathcal{M}(v_*) d\sigma dv_*,$$

and there are constants $\nu_0, \nu_1 > 0$ such that

$$\nu_0 \le \nu_0 \langle v \rangle \le \nu(v) \le \nu_1 \langle v \rangle, \tag{3.1.6}$$

where we define $\langle v \rangle := \sqrt{1 + |v|^2}$, and we refer to [102, Section 4] for a derivation of this.

Remark 3.2. Furthermore, also from [102, Section 4], we note that possible choices for these constants are $\nu_0 = 4\pi\sqrt{2/(e\pi)}$ and $\nu_1 = 16\pi$. This motivates the definition of $\nu_* := \nu_1/\nu_0 = 2\sqrt{2e\pi}$ and subsequently

$$q_\iota^\star = \begin{cases} \dfrac{5\iota_0 + 8\nu_* + \sqrt{128\pi\mathfrak{C}_0\iota_0\nu_* + (8\nu_* - 3\iota_0)^2}}{2\iota_0} & \text{if (H1) holds} \\[2ex] \dfrac{5 + 16\nu_* + \sqrt{9 + 160\nu_* + 256\nu_*\left(1 + \pi\mathfrak{C}_0\right) + 256\nu_*^2}}{2} & \text{if (H2) holds,} \end{cases} \tag{3.1.7}$$

where $\nu_* = 2\sqrt{2e\pi}$, and $\mathfrak{C}_0 > 0$ is such that $\omega\mathscr{M} \le \mathfrak{C}_0$.

We translate now our framework towards this new setting. We recall that for any $x \in \partial\Omega^\varepsilon$ there holds $y = \varepsilon x \in \partial\Omega$, thus we define $\delta^\varepsilon : \Omega^\varepsilon \to \mathbb{R}$, $\delta^\varepsilon(x) := \delta(y)$, and we observe that there holds almost everywhere

$$n(y) = -\frac{\nabla_y \delta(y)}{|\nabla_y \delta(y)|} = -\frac{\nabla_y \delta(\varepsilon x)}{|\nabla_y \delta(\varepsilon x)|} = -\frac{\nabla_x[\delta(\varepsilon x)]}{|\nabla_x[\delta(\varepsilon x)]|} = -\frac{\nabla_x \delta^\varepsilon(x)}{|\nabla \delta^\varepsilon(x)|} =: n(x) = n_x, \tag{3.1.8}$$

which is nothing but saying that the normal vector on a rescaled point of the boundary set $\partial\Omega^\varepsilon$ coincides with the one of the corresponding point on the original boundary set $\partial\Omega$.

We define then $\Sigma^\varepsilon := \partial\Omega^\varepsilon \times \mathbb{R}^3$ and we define accordingly the sets

$$\Sigma_\pm^\varepsilon := \{(x, v) \in \Sigma^\varepsilon,\ \pm n_x \cdot v > 0\}, \quad \Sigma_0^\varepsilon := \{(x, v) \in \Sigma^\varepsilon,\ n_x \cdot v = 0\},$$

and $\Gamma_\pm^\varepsilon := (0, \infty) \times \Sigma_\pm^\varepsilon$. We introduce the rescaled accommodation coefficient $\iota^\varepsilon : \partial\Omega^\varepsilon \to [0, 1]$ defined as $\iota^\varepsilon(x) := \iota(\varepsilon x)$. We have then that the associated boundary conditions for the rescaled Boltzmann Equation (3.1.5) read

$$\gamma_- f = \mathscr{R}\gamma_+ f = (1 - \iota^\varepsilon)\mathscr{S}\gamma_+ f + \iota^\varepsilon \mathscr{D}\gamma_+ f \quad \text{on } \Gamma_-^\varepsilon, \tag{3.1.9}$$

by abusing notation on the fact that we will maintain unchanged the symbols of the boundary reflection operators.

We translate now the geometrical assumptions (H1) and (H2) into the rescaled setting.

(RH1) $\Omega^\varepsilon \subset \mathbb{R}^3$ is an open C^2 domain, and $\delta^\varepsilon \in C^2(\mathbb{R}^3, \mathbb{R}) \cup W^{3,\infty}$. Moreover, $\iota^\varepsilon \in C(\partial\Omega)$ and such that for every $x \in \partial\Omega^\varepsilon$, $\iota^\varepsilon(x) \in [\iota_0, 1]$ with $\iota_0 \in (0,1]$.

(RH2) $\Omega^\varepsilon = (-L^\varepsilon, L^\varepsilon) \times \Omega_0^\varepsilon$, with $L^\varepsilon := \varepsilon^{-1}L$ and $\Omega_0^\varepsilon := \varepsilon^{-1}\Omega_0$, i.e is the 2-dimensional ball of radius $\varepsilon^{-1}\mathfrak{R}$ centered at the origin. We also define

$$\Lambda_1^\varepsilon := \{-L^\varepsilon\} \times \Omega_0^\varepsilon, \quad \Lambda_2^\varepsilon := \{L^\varepsilon\} \times \Omega_0^\varepsilon, \quad \Lambda_3^\varepsilon := (-L^\varepsilon, L^\varepsilon) \times \partial\Omega_0^\varepsilon,$$

and $\Lambda^\varepsilon := \Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon \cup \Lambda_3^\varepsilon$. Moreover, we take $\iota^\varepsilon = \mathbf{1}_{\Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon}$.

Finally we complement Equation (3.1.5)-(3.1.9) with the initial condition

$$f(t=0,x,v) = f_0(x,v) := F_0(\varepsilon x, v) - \mathcal{M}(v) \quad \text{in } \mathcal{O}^\varepsilon, \tag{3.1.10}$$

and we remark that due to the hypothesis on F_0 we have that $\langle\!\langle f_0 \rangle\!\rangle_{\mathcal{O}^\varepsilon} = 0$.

We state now an equivalent version of Theorem 3.1.1 in the rescaled setting, and this will be the result we will prove during this chapter.

Theorem 3.1.3. *Consider either Assumption (RH1) or Assumption (RH2) to hold, and let ω be an admissible weight function. There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there is $\eta(\varepsilon) \in (0,1)$, satisfying $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for every $f_0 \in L^\infty_\omega(\mathcal{O}^\varepsilon)$ satisfying*

$$\|f_0\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)} \leq (\eta(\varepsilon))^2,$$

there exists $f \in L^\infty_\omega(\mathcal{U}^\varepsilon)$, unique global solution to the linearized rescaled Boltzmann equation (3.1.5)–(3.1.9)–(3.1.10). *Furthermore, there is a constructive constant $\theta > 0$ such that*

$$\|f_t\|_{L^\infty_\omega(\mathcal{O})} \leq e^{-\theta\varepsilon^2 t}\,\eta(\varepsilon) \qquad \forall t \geq 0.$$

In order to explain the precise sense of the solutions given by Theorem 3.1.3 we introduce the subset of admissible weight functions, that we will call *strongly confining*, as follows

$$\mathfrak{W}_1 = \{\omega : \mathbb{R}^3 \to \mathbb{R},\, \omega(v) = e^{\zeta|v|^2}, \text{ with } \zeta \in (1/4, 1/2)\}. \tag{3.1.11}$$

In consequence, we also define the space of *weakly confining* admissible weights by

$$\mathfrak{W}_0 = \{\omega : \mathbb{R}^3 \to \mathbb{R},\ \omega \text{ is an admissible weight function, with } \omega \notin \mathfrak{W}_1\}. \tag{3.1.12}$$

We then remark that the proof of Theorem 3.1.3 is given by Theorem 3.7.1 for strongly confining admissible weight functions and Theorem 3.8.1 for weakly confining ones.

3.1.4 Strategy for the proof of the main result

We explain in this subsection the main ideas used for the proof of Theorem 3.1.3.

In general, the main strategy to construct weak solutions for the Boltzmann equation near Maxwellian stationary solutions, which is outlined in [56, Section 7.1] and applied for instance in [36, 83, 102, 108, 109], can be summarize in three main steps as follows.

(1) Obtaining some decay for the linearized operator $-v \cdot \nabla_x + \mathscr{C}$ in an appropriate Hilbert space, usually done by employing *hypocoercivity* techniques.

(2) Extending the decay towards a (appropriately) weighted L^∞ space, nice enough to provide an algebraic structure to control the Boltzmann collision operator $\mathcal{Q}$.

(3) Constructing weak solutions under the assumption of smallness of the initial data.

We explain now in precise detail the methods and more technical ideas we employ in this chapter to accomplish the previous objectives.

① *Hypocoercivity decay:* We consider the problem

$$\begin{cases} \partial_t f &= \mathscr{L} f := -v \cdot \nabla_x f + \mathscr{C} f \quad \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f &= \mathscr{R} \gamma_+ f \quad \text{on } \Gamma_-^\varepsilon \\ f_{t=0} &= f_0 \quad \text{in } \mathcal{O}^\varepsilon. \end{cases} \tag{3.1.13}$$

On the one hand we observe that, for any nice enough functions $G, H, \varphi : \mathbb{R}^3 \to \mathbb{R}$, the Boltzmann collision operator classically satisfies

$$\int_{\mathbb{R}^3} \mathcal{Q}(G,H)\,\varphi = \frac{1}{8} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B}\left(G'_* H' + H'_* G' - G_* H - G H_*\right)\left(\varphi + \varphi_* - \varphi' - \varphi'_*\right), \tag{3.1.14}$$

where we have used the shorthands

$$\phi = \phi(v), \quad \phi_* = \phi(v_*), \quad \phi' = \phi(v'), \quad \phi'_* = \phi(v'_*),$$

and we recall that v' and v'_* are given in Subsection 3.1.1. The interested reader can consult the derivation of (3.1.14) in [56, Section 3.1].

In particular, if we set $\mathbb{R}^3 \ni v = (v_1, v_2, v_3)$, then (3.1.14) implies that choosing $\varphi = \varphi(v)$ to be either $1, v_1, v_2, v_3$ or $|v|^2$ there holds

$$\int_{\mathbb{R}^3} \mathcal{Q}(G,H)(v)\,\varphi(v)\,\mathrm{d}v = 0. \tag{3.1.15}$$

Therefore, (3.1.15) implies that for the previous choices of φ there holds

$$\int_{\mathbb{R}^3} (\mathscr{C} f)(v)\,\varphi(v)\,\mathrm{d}v = 0. \tag{3.1.16}$$

We now take as a momentary framework the Hilbert space $L^2_{\mathcal{M}^{-1/2}}(\mathbb{R}^3)$ endowed with the scalar product

$$\langle g, h \rangle_{L^2_{\mathcal{M}^{-1/2}}(\mathbb{R}^3)} := \int_{\mathbb{R}^3} g(v)\,h(v)\,\mathcal{M}^{-1}(v)\,\mathrm{d}v,$$

and its associated norm as defined in (3.1.4). We observe that [56, Theorem 7.2.4] implies that we can set $\mathrm{Dom}(\mathscr{C}) := L^2_{\mathcal{M}^{-1/2}}(\mathbb{R}^3)$, that $\mathscr{C}$ is a closed operator on its domain, and (3.1.16) further gives that

$$\ker(\mathscr{C}) = \mathrm{span}\,\{\mathcal{M},\, v_1\mathcal{M},\, v_2\mathcal{M},\, v_3\mathcal{M},\, |v|^2\mathcal{M}\}.$$

This motivates the definition of Πf as the projection of $f \in \mathrm{Dom}(\mathscr{C})$ onto $\ker(\mathscr{C})$ given by

$$\Pi f = \left(\int_{\mathbb{R}^d} f(w)\,\mathrm{d}w\right)\mathcal{M} + \left(\int_{\mathbb{R}^d} w f(w)\,\mathrm{d}w\right)\cdot v\mathcal{M} + \left(\int_{\mathbb{R}^d} \frac{|w|^2 - 3}{\sqrt{6}}\, f(w)\,\mathrm{d}w\right)\frac{|v|^2 - 3}{\sqrt{6}}\,\mathcal{M}.$$

We also note that $\mathscr{C}$ is self-adjoint on its domain and negative, i.e

$$\langle \mathscr{C} f, f\rangle_{L^2_{\mathcal{M}^{-1/2}}(\mathbb{R}^3)} \leq 0, \tag{3.1.17}$$

so that its spectrum is included in $\mathbb{R}_-$, and (3.1.16) holds true for any $f \in \mathrm{Dom}(\mathscr{C})$. Furthermore, $\mathscr{C}$ satisfies a *microscopic coercivity* estimate, more precisely [15, Theorem 1.1] gives that there is $\kappa_0 > 0$ such that for any $f \in \mathrm{Dom}(\mathscr{C})$ one has

$$\langle -\mathscr{C} f, f\ \rangle_{L^2_{\mathcal{M}^{-1/2}}(\mathbb{R}^3)} \geq \kappa_0 \|f^\perp\|^2_{L^2_{\mathcal{M}^{-1/2}}(\mathbb{R}^3)}, \tag{3.1.18}$$

where $f^\perp := f - \Pi f$. Finally, we observe that for any polynomial function $\phi = \phi(v) : \mathbb{R}^3 \to \mathbb{R}$ of degree less or equal to 4, there holds $\mathcal{M}\phi \in \mathrm{Dom}(\mathscr{C})$, and using again [56, Theorem 7.2.4] we deduce that

$$\|\mathscr{C}(\phi\mathcal{M})\|_{L^2_{\mathcal{M}^{-1/2}}(\mathbb{R}^3)} \leq \|\phi\mathcal{M}\|_{L^2_{\mathcal{M}^{-1/2}}(\mathbb{R}^3)}.$$

Therefore, there exists a constant $C_\phi \in (0,\infty)$ such that

$$\|\mathscr{C}(\phi\mathcal{M})\|_{L^2_{\mathcal{M}^{-1/2}}(\mathbb{R}^3)} \leq C_\phi.$$

Finally, we remark that the conservations laws (3.1.16) together with the fact that

$$\|\mathscr{R}\|_{L^1(\Gamma_+^\varepsilon)\to L^1(\Gamma_-^\varepsilon)} = 1,$$

imply that, at least formally, Equation (3.1.13) conserves mass, i.e for any solution f of Equation (3.1.13) there holds $\langle\!\langle f_t\rangle\!\rangle_{\mathcal{O}^\varepsilon} = \langle\!\langle f_0\rangle\!\rangle_{\mathcal{O}^\varepsilon} = 0$ for every $t \geq 0$.

Altogether, the previous analysis implies that $\mathscr{C}$ satisfies the structural assumptions made on [21]. We thus define the weighted Hilbert space $\mathcal{H} := L^2_{\mathcal{M}^{-1/2}}(\mathcal{O}^\varepsilon)$ equipped with the scalar product

$$\langle g, h\rangle_{\mathcal{H}} := \int_{\mathcal{O}} g(x,v)\, h(x,v)\mathcal{M}^{-1}(v)\mathrm{d}v\mathrm{d}x,$$

and we observe that [21, Theorem 1.1] and [21, Theorem 5.1] imply that under Assumption (RH1), and at least at the level of a priori estimates, we immediately have that there is $\kappa > 0$ such that any solution f of Equation (3.1.13) satisfies

$$\|f_t\|_{\mathcal{H}} \leq C e^{\kappa\varepsilon^{-2}t}\|f_0\|_{\mathcal{H}} \qquad \forall t \geq 0, \tag{3.1.19}$$

for a constant $C > 0$, independent of ε. We remark that the constant ε^{-2} on the rate of the exponential decay can be deduce from [21, Theorem 5.1] and the change of variables made in Subsection 3.1.3, see for instance [47, Proposition A.1] for a justification of this fact.

We then dedicate Section 3.2 to prove (3.1.19) under the geometrical Assumption (RH2). The main goal is to obtain H^2 regularity estimates for the solutions of certain elliptic equations introduced in [21, Section 2], for cylindrically shaped domains. It is worth remarking that this is not straightforward because such gain of regularity, even though classical in smooth domains, doesn't necessarily hold for irregular domains, see for instance [101].

Once obtained these regularity estimates, the rest of the computations leading to (3.1.19) follow as an exact repetition of the main ideas and computations from [21].

② *$L^2 - L^\infty$ method for strongly confining weights:* We remark that $\mathscr{L} = \mathscr{T} + K$, where we have defined the *free transport operator*

$$\mathscr{T} h(t,x,v) := -v \cdot \nabla_x h(t,x,v) - \nu(v) h(t,x,v), \tag{3.1.20}$$

for any nice enough function $h : \mathcal{U}^\varepsilon \to \mathbb{R}$. We consider then a function $G : \mathcal{U}^\varepsilon \to \mathbb{R}$, and we study the following perturbed evolution equation

$$\begin{cases} \partial_t f &=& \mathscr{T} f + K f + G & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f &=& \mathscr{R} \gamma_+ f & \text{on } \Gamma_-^\varepsilon \\ f_{t=0} &=& f_0 & \text{in } \mathcal{O}^\varepsilon. \end{cases} \tag{3.1.21}$$

Our main interest is to provide weighted $L^2 - L^\infty$ estimates for the solutions of Equation (3.1.21) by employing the *stretching method* developed in [110], see also [109]. To do this we define the *free transport semigroup*, classically generated by the transport operator $\mathscr{T}$ (see for instance [157, Section 8]), as

$$S_{\mathscr{T}}(t) h(s,x,v) := e^{-\nu(v)t} h(s, x - vt, v) \qquad \text{for any } s, t \in \mathbb{R}_+, \text{ and any } (x,v) \in \mathcal{O}^\varepsilon. \tag{3.1.22}$$

Furthermore, for any sufficiently nice operator $\mathcal{A}$, and any $\sigma \in [0,t]$, we define the convolution operation

$$(S_{\mathscr{T}} *_\sigma \mathcal{A})\, h(t,x,v) := \int_\sigma^t S_{\mathscr{T}}(t-s)\, \mathcal{A}\, h(s,x,v) ds = \int_\sigma^t e^{-\nu(v)(t-s)} \mathcal{A} f(s, x-(t-s)v, v) \mathrm{d}s, \tag{3.1.23}$$

and in particular we will classically denote $S_{\mathscr{T}} * \mathcal{A} := S_{\mathscr{T}} *_0 \mathcal{A}$. We now remark that $S_{\mathscr{T}}$ will satisfy two important properties.

- On the one hand, $S_{\mathscr{T}} * K$ generates an integrability gain of the form $L^2 \to L^\infty$ in the sense provided by Proposition 3.10 and Proposition 3.9, for the cases for smooth and cylindrical domains respectively.
- On the other hand, and as described in (K2), weighted L^∞ estimates on $S_{\mathscr{T}} * G$ generate a gain of weight of ν^{-1}, which is crucial to control the Boltzmann collision operator $\mathcal{Q}$ (see Lemma 3.2).

Exploiting this, the strategy for the $L^2 - L^\infty$ decay transfer will focus on the repeated use of the Duhamel formula on the semigroup $S_{\mathscr{T}}$ for the solutions of Equation (3.1.21), and using the stretching method we will use the effect of the parameter ε on the domain to make small the *measure* of the "set of singular trajectories".

To explain how this works we first observe that the characteristics of Equation (3.1.21) are given by

$$X(s;t,x,v) := x - v(t-s) \quad \text{and} \quad V(s;t,x,v) := v. \tag{3.1.24}$$

Therefore, for any fixed particle with coordinates in $\mathcal{U}^\varepsilon$, we can characterize the coordinates of the last collision against the boundary of Ω^ε. Indeed, let $(t_0, x_0, v_0) \in \mathcal{U}^\varepsilon$ be the coordinates of a particle, we define the time of collision along this trajectory (t_b), the time of life of the particle prior to such collision (t_1), and the position (x_1) and velocity (v_1) at the boundary during this collision, as follows

$$\begin{aligned} t_b(x_0,v_0) &= \inf\{s > 0;\ X(-s,0,x,v) \notin \Omega^\varepsilon\}, \\ t_1(t_0,x_0,v_0) &= t_0 - t_b(x_0,v_0), \\ x_1(t_0,x_0,v_0) &= X(t_1;t,x,v) = x - v(t-t_1), \\ v_1(t_0,x_0,v_0) &= \begin{cases} \mathcal{V}_{x_1}(v_0) & \text{during specular reflection,} \\ v_0^* & \text{during diffuse reflection,} \end{cases} \end{aligned} \tag{3.1.25}$$

where v_0^* stands for an independent variable.

We then note that the smaller ε is chosen, the *flatter* the boundary $\partial\Omega^\varepsilon$ becomes. In particular, any particle following the *backwards trajectories* given by (3.1.25), doesn't collide more than once against the boundary.

Therefore, the objective of the stretching method is to choose ε small enough such that the set of paths that collide more than once against the boundary is *small* in an appropriate sense.

This will be done for smooth domains in Section 3.3, which is mainly following the arguments presented in [110]. Afterwards during Section 3.4, and following a similar methodology, we will generalize this technique for cylindrical domains.

It is worth remarking that such a generalization is not straightforward and requires a more delicate control of the backwards trajectories due to the presence of irregularities at the boundary. In particular, we will need to account for trajectories where the particle changes between the boundary subsets $\Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon$ and Λ_3^ε.

Finally, we emphasize that by combining in a very delicate way the L^2 hypocoercivity decay and the L^∞ control given by the stretching method, we craft a $L^2 - L^\infty$ control that provides a control on the long-time behavior for the solutions of Equation (3.1.21) in a weighted L^∞ space.

③ *Proof of Theorem 3.1.3 for strongly confining weight functions:* The a priori estimates obtained by using the above $L^2 - L^\infty$ method are enough then to construct a well-posedness theory for Equation (3.1.21).

Moreover, using the same a priori estimates and setting $G = \mathcal{Q}(g, g)$, we will prove the well-posedness of Equation (3.1.5)-(3.1.9)-(3.1.10) by using a Banach fixed point argument.

④ *Mathematical setting for weakly confining weights:* In order to obtain the well-posedness for the linearized Boltzmann equation for weakly confining weights, we need to split the problem into a system of equations.

Before doing this we need to define some operators. Consider a parameter $\delta > 0$, and define the sets

$$\begin{aligned} E_1^\delta &= \{(v, v_*, \theta) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2;\ |v| \le \delta^{-1},\ 2\delta \le |v - v_*| \le \delta^{-1},\ |\cos(\theta)| \le 1 - 2\delta\}, \\ E_2^\delta &= \{(v, v_*, \theta) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2;\ |v| \le 2\delta^{-1},\ \delta \le |v - v_*| \le 2\delta^{-1},\ |\cos(\theta)| \le 1 - \delta\}, \end{aligned}$$

and the cut-off function $\chi_\delta \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \mathbb{R})$, such that $\mathbf{1}_{E_1^\delta} \le \chi_\delta \le \mathbf{1}_{E_2^\delta}$. We define then the operators

$$\mathcal{A}_\delta f = K((1 - \chi_\delta) f) \quad \text{and} \quad \mathcal{K}_\delta f = K(\chi_\delta f), \tag{3.1.26}$$

and we remark that $K = \mathcal{A}_\delta + \mathcal{K}_\delta$.

This choice for the splitting of the operator K is taken from [102] and motivated by the ideas developed in [36, Section 6]. Its main interest is to be able to exploit [102, Lemma 4.12], presented in Lemma 3.3, providing dissipative estimates on $\mathcal{A}_\delta$.

Moreover, we remark that the nature of the operator $\mathcal{K}_\delta$ will make it enjoy nice boundedness properties (see for instance Lemma 3.4).

We then observe that if f_1, f_2 are solutions of the system of equations

$$\left\{ \begin{aligned} \partial_t f_1 &= \mathscr{T} f_1 + \mathcal{A}_\delta f_1 + \mathcal{Q}(f_1 + f_2, f_1 + f_2) && \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_1 &= \mathscr{R} \gamma_+ f_1 && \text{on } \Gamma_-^\varepsilon \\ f_{1,t=0} &= f_0 && \text{in } \mathcal{O}^\varepsilon, \end{aligned} \right. \tag{3.1.27}$$

and

$$\begin{cases} \partial_t f_2 &= \mathscr{L} f_2 + \mathcal{K}_\delta f_1 & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_2 &= \mathscr{R}\gamma_+ f_2 & \text{on } \Gamma_-^\varepsilon \\ f_{2,t=0} &= 0 & \text{in } \mathcal{O}^\varepsilon, \end{cases} \tag{3.1.28}$$

respectively then, at least formally, $f = f_1 + f_2$ is a solution of Equation (3.1.5)-(3.1.9)-(3.1.10).

⑤ *A priori estimates for weakly confining weights:* We first remark that Equation (3.1.28) fits the linear framework of Equation 3.1.21 by setting $G = \mathcal{K}_\delta f_1$.

This combined with the boundedness properties of the operator $\mathcal{K}_\delta$ imply that we only need to provide a priori estimates with weakly confining weights for the solutions of the evolution equation

$$\begin{cases} \partial_t f_1 &= \mathscr{T} f_1 + \mathcal{A}_\delta f_1 + G & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_1 &= \mathscr{R}\gamma_+ f_1 & \text{on } \Gamma_-^\varepsilon \\ f_{1,t=0} &= f_0 & \text{in } \mathcal{O}^\varepsilon, \end{cases} \tag{3.1.29}$$

for a function $G : \mathcal{U}^\varepsilon \to \mathbb{R}$.

This will be done by using the stretching method as during Sections 3.3 and 3.4, Lemma 3.3 to control polynomial and stretched exponential admissible weight functions and some extra dissipativity properties of $\mathcal{A}_\delta$ for inverse gaussian weakly confining admissible weight functions (see (K1)).

⑥ *Proof of Theorem 3.1.3 for weakly confining weight functions:* Using the a priori estimates established for weakly confining weights, we construct the solutions of Equation (3.1.27) and Equation (3.1.28) by using two fixed point arguments in the spirit of [36, Theorems 6.1 and 6.9].

3.1.5 Organisation of the chapter

This chapter is structured as follows.

In Section 3.2 we generalize the hypocoercivity theory developed in [21] to cylindrical domains, and as a consequence we deduce a decay estimate for the solutions of Equation (3.1.13) in the Hilbert space $\mathcal{H}$.

During Section 3.3 we consider Assumption (RH1) to hold, and we use the stretching method developed in [110] to obtain a $L^2 - L^\infty$ type of estimate with strongly confining weights, for the solutions of Equation (3.1.21).

In Section 3.4 we consider Assumption (RH2) to hold, and still working with strongly confining weights, we generalize the ideas and methods from Section 3.3. We emphasize that this means that we prove a $L^2 - L^\infty$ type of estimate for the solutions of Equation (3.1.21) within cylindrical domains.

We devote Section 3.5 to study the dissipative Equation (3.1.27) with weakly confining weight functions. We use the stretching method as during Sections 3.3 and 3.4, and the dissipative properties of the operator $\mathcal{A}_\delta$ to obtain weighted L^∞ decay estimates for the solutions of this equation.

Section 3.6 then addresses the well-posedness of transport equations with non-local terms. The focus is particularly in obtaining the well-posedness of the linear equations (3.1.21), (3.1.29) and (3.1.28), and on justifying the validity of the a priori estimates obtained during Sections 3.3, 3.4 and 3.5.

In Section 3.7 we use the results from Sections 3.3, 3.4 and 3.6 to construct the solutions for the Boltzmann equation (3.1.5)-(3.1.9)-(3.1.10) for strongly confining weight functions, proving a first part of Theorem 3.1.3.

Finally, during Section 3.8 we complete the proof of Theorem 3.1.3 by constructing solutions for the Boltzmann equation (3.1.5)-(3.1.9)-(3.1.10) for weakly confining weight functions. In particular, this will be done by using the results from Sections 3.5 and 3.6 to construct the solutions of the system formed by Equation (3.1.27) and Equation (3.1.28).

3.1.6 Notation

During this chapter we use the following notations.

- μ_L is the Lebesgue measure, and when it is clear we will denote it as $|\cdot| = \mu_L$.
- $d\sigma_x$ is the Lebesgue measure on the boundary set $\partial\Omega^\varepsilon$.
- B_r will denote the 3−dimensional ball of radius $r > 0$, and $D := |B_1|$.
- $H^k(Z)$ is the Sobolev space for $L^2(Z)$ functions with $k > 0$ weak derivatives in $L^2(Z)$.
- For a point $z \in \mathbb{R}^3$ we denote its component as z_i or z^i, with $i \in \{1, 2, 3\}$, and we will use the latter only when there is no confusion with the exponentiation operator, otherwise we will clarify it.
- We define the tensor product between two vector $a, b \in \mathbb{R}^3$ as $a \otimes b := (a_i b_j)_{1 \le i,j \le 3} \in \mathbb{M}_3$, the space of square matrices of size 3.
- For $a, b > 0$ we say that $a \lesssim b$ when there is a constant $c > 0$ such that $a \le cb$.
- δ_{ij} is the Kronecker delta function defined by
$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$
- $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ is the canonical basis of $\mathbb{R}^3$, i.e for every $i \in \{1, 2, 3\}$, $\mathbf{e}_i$ is the vector that has for i−th component the Kronecker symbol δ_{ij}.

Throughout this chapter we will denote as C, often defined using subscripts and superscripts (when there is no possible confusion with the exponentiation operation), a constant that might change from one lemma, proposition, or theorem to the other. Furthermore, and unless said otherwise, this constant may change from one line to the other.

3.2 Hypocoercivity

We dedicate this section to prove the following hypocoercivity result.

Theorem 3.2.1. *Consider either Assumption (RH1) or (RH2) to hold. There are constructive constants $\kappa > 0$ and $C \ge 1$ such that for any solution f of Equation* (3.1.13) *there holds*
$$\|f_t\|_{\mathcal{H}} \le Ce^{-\kappa\varepsilon^2 t}\|f_0\|_{\mathcal{H}} \qquad \forall t \ge 0. \tag{3.2.1}$$
Furthermore, there is a norm $|\!|\!|\cdot|\!|\!|$ equivalent to the usual norm of $\mathcal{H}$ uniformly in ε, i.e there is a constant $c > 0$ independent of ε such that
$$c^{-1}\|f\|_{\mathcal{H}} \le |\!|\!|f|\!|\!| \le c\|f\|_{\mathcal{H}}, \tag{3.2.2}$$
and a constant $\kappa^\star > 0$ for which there holds
$$|\!|\!|f_t|\!|\!| \le e^{-\kappa^\star\varepsilon^2 t}|\!|\!|f_0|\!|\!| \qquad \forall t \ge 0. \tag{3.2.3}$$

Remark 3.2. Theorem 3.2.1 under the assumption (RH1) is nothing but [21, Theorem 1.1] and [21, Theorem 5.1] up to the change of variables performed in Subsection 3.1.4. Therefore our proof will focus on proving the hypocoercivity of Equation (3.1.13) under Assumption (RH2).

The L^2 hypocoercivity method we will use during the proof of Theorem 3.2.1 is taken from [21]. The objective behind general hypocoercivity theories is to extend the microscopic coercivity of $\mathscr{C}$, exhibited by (3.1.18), into a coercivity condition for the full operator $\mathscr{L} = -v \cdot \nabla_x + \mathscr{C}$. We refer towards [79, 108, 110, 166] for different approaches to achieve this.

More specifically, the obtention of (3.2.1) during this chapter is done by constructing the norm $|\!|\!|\cdot|\!|\!|$ in such a way that (3.2.3) holds, i.e Equation (3.1.13) is coercive over the Hilbert space L^2 equipped with this new norm. To do this, A. Bernou, K. Carrapatoso, S. Mischler and I. Tristani, introduce in [21], certain classes of elliptic problems whose solutions are used to build $|\!|\!|\cdot|\!|\!|$. In particular, they exploit the classical gain of H^2 regularity for solutions of elliptic equations in smooth domains (see for instance [89], [34], [95], [101, Chapter 2]), to prove the coercivity estimate.

However, such type of regularity gains might not hold true if the domain is not sufficiently regular. For instance, P. Grisvard proved in [101, Chapter 4] that in the case of two and three dimensional polygons there is a maximum regularity we can achieve depending on the amount of edges and vertices, and the values of the interior angles.

In order then to generalize [21, Theorem 1.1] for cylindrical domains, we need to prove that these elliptic equations provide enough regularizing properties for their solutions. This will be done by exploiting the particular geometry of cylindrical domain, together with classical results on the gain of regularity in smooth domains, and introducing a *reflection method* inspired by [34, Lemma 9.2 and Remark 9.3].

3.2.1 Poisson equation in the cylinder

Consider Ω^ε satisfying the geometrical Assumption (RH2) and let $\xi \in L^2(\Omega^\varepsilon)$. During this subsection we study the following Poisson equation

$$\left\{ \begin{array}{rcll} -\Delta u &=& \xi & \text{in } \Omega^\varepsilon \\ (2-\alpha(x))\partial_n u + \alpha(x) u &=& 0 & \text{on } \partial\Omega^\varepsilon, \end{array} \right. \tag{3.2.4}$$

where α is chosen satisfying one of the following conditions:

(P1) either $\alpha = \mathbf{1}_{\Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon}$,

(P2) or we assume $\langle\!\langle \xi \rangle\!\rangle_{\Omega^\varepsilon} = 0$, and we take $\alpha \equiv 0$.

We consider then the functional spaces

$$V_1 := H^1(\Omega^\varepsilon) \qquad \text{and} \qquad V_2 := \left\{ u \in H^1(\Omega^\varepsilon),\ \langle\!\langle u \rangle\!\rangle_{\Omega^\varepsilon} = 0 \right\},$$

from where we define

$$V_k := \left\{ \begin{array}{ll} V_1 & \text{if (P1) holds,} \\ V_2 & \text{if (P2) holds.} \end{array} \right.$$

and we will dedicate the rest of this subsection to prove the following well-posedness and regularity result.

Theorem 3.2.3. *Let Ω^ε be a cylindrical domain as defined in (RH2) and let either (P1) or (P2) boundary conditions hold. For any $\xi \in L^2(\Omega^\varepsilon)$ there exists a unique variational solution $w \in H^2(\Omega^\varepsilon) \cap V_k$ to the Poisson Equation (3.2.4), i.e there holds*

$$\int_{\Omega^\varepsilon} \nabla w(x) \cdot \nabla v(x)\, \mathrm{d}x + \int_{\partial\Omega^\varepsilon} \frac{\alpha(x)}{2-\alpha(x)} w(x) v(x)\, \mathrm{d}\sigma_x = \int_{\Omega^\varepsilon} \xi(x)\, v(x) \mathrm{d}x \qquad \forall v \in V_k. \tag{3.2.5}$$

Furthermore, there holds

$$\|w\|_{H^2(\Omega^\varepsilon)} \leq C\varepsilon^{-2}\|\xi\|_{L^2(\Omega^\varepsilon)},$$

for some constructive constant $C > 0$, independent of ε.

Proof. We split the proof into three steps.

Step 1. (Existence and uniqueness of variational solutions) The analysis for this step is classical and similar to the arguments from the Step 1 of the proof of [21, Theorem 2.2], so we will only sketch it. We introduce the bilinear form

$$a(u, v) := \int_{\Omega^\varepsilon} \nabla u(x) \nabla v(x) \mathrm{d}x + \int_{\partial\Omega^\varepsilon} \frac{\alpha(x)}{2-\alpha(x)} u(x) v(x) \mathrm{d}\sigma_x,$$

and we observe that a is continuous in $V_k \times V_k$. On the other hand, by using the Poincaré-Wirtinger's inequality (see for instance [21, Proposition 2.1]) together with classical scaling arguments, we deduce that

$$a(u, u) \geq C_P\, \varepsilon^2 \|u\|^2_{L^2(\Omega^\varepsilon)},$$

for some constructive constant $C_P > 0$, independent of ε. This classically implies that a is coercive, i.e

$$a(u, u) \gtrsim \varepsilon^2 \|u\|^2_{H^1(\Omega^\varepsilon)}, \tag{3.2.6}$$

thus this gives, by the application of the Lax-Milgram theorem, the existence and uniqueness of a variational solution $w \in V_k$ for the Poisson Equation (3.2.4) in the sense of (3.2.5). Furthermore, from the variational formulation (3.2.5) and the previous coercivity estimate (3.2.6), we further deduce that

$$\|w\|_{H^1(\Omega^\varepsilon)} \lesssim \varepsilon^{-2}\|\xi\|_{L^2(\Omega^\varepsilon)}.$$

In particular, we classically deduce that there is a trace function $w|_{\partial\Omega^\varepsilon} \in L^2(\partial\Omega^\varepsilon)$ and there holds

$$\|w|_{\partial\Omega^\varepsilon}\|_{L^2(\partial\Omega^\varepsilon)} \lesssim \varepsilon^{-2}\|\xi\|_{L^2(\Omega^\varepsilon)}.$$

This concludes the well-posedness of Equation (3.2.4).

Step 2. (Domain reflection) We consider now the extended domain $\widehat{\Omega}^\varepsilon := (-2L^\varepsilon, 2L^\varepsilon) \times \Omega_0^\varepsilon$ and we define

$$\widehat{\Lambda_1^\varepsilon} := \{-2L^\varepsilon\} \times \Omega_0^\varepsilon, \qquad \widehat{\Lambda_2^\varepsilon} := \{2L^\varepsilon\} \times \Omega_0^\varepsilon, \qquad \text{and} \qquad \widehat{\Lambda_3^\varepsilon} := (-2L^\varepsilon, 2L^\varepsilon) \times \partial\Omega_0^\varepsilon.$$

Now for any $x \in \mathbb{R}^3$, we write it as $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^2$ and for any function ϕ with domain in Ω^ε we define a new function $\widehat{\phi}$ with domain in $\widehat{\Omega^\varepsilon}$ as follows

$$\widehat{\phi}(x) = \begin{cases} \phi(-x_1 + 2L^\varepsilon, \bar{x}) & \text{if } x_1 \in (-2L^\varepsilon, -L^\varepsilon) \\ \phi(x_1, \bar{x}) & \text{if } x_1 \in (-L^\varepsilon, L^\varepsilon) \\ \phi(-x_1 - 2L^\varepsilon, \bar{x}) & \text{if } x_1 \in (L^\varepsilon, 2L^\varepsilon). \end{cases} \tag{3.2.7}$$

It is easy to check that, defined this way, $\widehat{w}$ is a variational solution of the extended Poisson problem

$$\begin{cases} -\Delta \widehat{u} &= \widehat{\xi} \quad \text{in } \widehat{\Omega^\varepsilon} \\ (2-\widehat{\alpha}(x))\partial_n \widehat{u} + \widehat{\alpha}(x)\widehat{u} &= 0 \quad \text{on } \partial\widehat{\Omega^\varepsilon}. \end{cases} \tag{3.2.8}$$

As a consequence, repeating the arguments of Step 1, we deduce that $\widehat{w}$ satisfies the following estimates

$$\|\widehat{w}\|_{H^1(\widehat{\Omega^\varepsilon})} \lesssim \varepsilon^{-2}\|\widehat{\xi}\|_{L^2(\widehat{\Omega^\varepsilon})} \quad \text{and} \quad \|\widehat{w}|_{\partial\Omega^\varepsilon}\|_{L^2(\partial\widehat{\Omega}^\varepsilon)} \lesssim \varepsilon^{-2}\|\xi\|_{L^2(\widehat{\Omega}^\varepsilon)}. \tag{3.2.9}$$

Step 3. (Regularity) Using the partition of unity (see for instance [34, Lemma 9.3]) we obtain the existence of a collection of open balls of radius $r > 0$, $(B_r^j)_{j=1}^N \subset \mathbb{R}^3$ covering $\partial\widehat{\Omega^\varepsilon}$, and a family of functions $\varrho_j \in C^\infty(\mathbb{R}^3, \mathbb{R})$ with $j = 0, \ldots, N$, such that

(i) $0 \le \varrho_j \le 1$ for every $j = 0, \ldots, N$, and $\sum_{j=0}^N \varrho_j = 1$ on $\mathbb{R}^3$,

(ii) moreover, $\operatorname{supp} \varrho_0 \subset \mathbb{R}^3 \setminus \partial\widehat{\Omega^\varepsilon}$ and for every $j = 1, \ldots, N$, there holds that $\varrho_j \in C_c^\infty(B_r^j)$.

In particular, we choose $r > 0$ small enough such that the balls B_r^j covering Λ_3^ε do not intersect $\widehat{\Lambda_1^\varepsilon}$ or $\widehat{\Lambda_2^\varepsilon}$, and we call $\mathcal{J}$ the sets of such indexes plus 0, i.e

$$\mathcal{J} = \Big\{ j \in \{1, \ldots, N\} \,\Big|\, \bigcup B_r^j \supset \Lambda_3^\varepsilon \,\Big|\, B_r^j \cap \widehat{\Lambda_1^\varepsilon} = \emptyset \,\Big|\, B_r^j \cap \widehat{\Lambda_2^\varepsilon} = \emptyset \Big\} \cup \{0\}.$$

Then we classically have, see for instance [34, Theorem 9.26 Steps C_1 and C_2], that for every $j \in \mathcal{J}$ there holds $\widehat{w}\varrho_j \in H^2(\widehat{\Omega}^\varepsilon)$ and

$$\|\Delta(\widehat{w}\varrho_j)\|_{H^2(\widehat{\Omega}^\varepsilon)} \le \widehat{C}\varepsilon^{-2}\|\widehat{\xi}\|_{L^2(\widehat{\Omega}^\varepsilon)}, \tag{3.2.10}$$

where $\widehat{C} > 0$ is a constant independent of ε. We deduce then that for every $\varphi \in C_c^\infty(\mathbb{R}^3)$ we have

$$\begin{aligned} \int_{\cup_{j\in\mathcal{J}} B_r^j \cap \widehat{\Omega^\varepsilon}} \nabla\widehat{w} \cdot \nabla\varphi &= \int_{\cup_{j\in\mathcal{J}} B_r^j \cap \widehat{\Omega^\varepsilon}} \nabla\widehat{w} \cdot \nabla\varphi\, \varrho_j \\ &= \sum_{j\in\mathcal{J}} \left(\int_{B_r^j \cap \widehat{\Omega^\varepsilon}} \nabla(\widehat{w}\varrho_j) \cdot \nabla\varphi - \int_{B_r^j \cap \widehat{\Omega^\varepsilon}} \widehat{w}\nabla\varrho_j \cdot \nabla\varphi \right) \\ &= \sum_{j\in\mathcal{J}} \left(\int_{B_r^j \cap \partial\widehat{\Omega^\varepsilon}} \varphi\nabla(\widehat{w}\varrho_j) \cdot n_x - \int_{B_r^j \cap \widehat{\Omega^\varepsilon}} \Delta(\widehat{w}\varrho_j)\varphi \right. \\ &\qquad \left. - \int_{B_r^j \cap \partial\widehat{\Omega^\varepsilon}} \widehat{w}\varphi\nabla\varrho_j \cdot n_x + \int_{B_r^j \cap \widehat{\Omega^\varepsilon}} \operatorname{div}(\widehat{w}\nabla\varrho_j)\varphi \right) \end{aligned}$$

where we have used that $\widehat{w}\varrho_j \in H^2(\widehat{\Omega}^\varepsilon)$, $\widehat{w}\nabla\varrho_j \in H^1(\widehat{\Omega}^\varepsilon)$ and we have performed an integration by parts on the third line. We then compute

$$\begin{aligned} \left| \int_{\partial\widehat{\Omega^\varepsilon}} \varphi\nabla(\widehat{w}\varrho_j) \cdot n_x \right| &= \left| \int_{\partial\widehat{\Omega^\varepsilon}} \varphi\nabla(\widehat{w}) \cdot n_x\varrho_j \right| + \left| \int_{\partial\widehat{\Omega^\varepsilon}} \varphi\nabla(\varrho_j) \cdot n_x\widehat{w} \right| \\ &\le \int_{\partial\widehat{\Omega^\varepsilon}} |\varphi\widehat{w}| \left| \frac{\alpha}{2-\alpha}\varrho_j + \nabla\varrho_j \cdot n_x \right| \end{aligned}$$

where we have used the boundary conditions on the second line. Then by using the Cauchy-Schwartz inequality we have

$$\begin{aligned}\int_{\cup_{j\in\mathcal{J}} B_r^j\cap\widehat{\Omega^\varepsilon}} \nabla\widehat{w}\cdot\nabla\varphi &\lesssim \|\varphi\|_{L^2(\mathbb{R}^3)}\sum_{j\in\mathcal{J}}\Big(\|\Delta(\widehat{w}\varrho_j)\|_{L^2(B_r^j\cap\widehat{\Omega^\varepsilon})}+\|\nabla\widehat{w}\|_{L^2(B_r^j\cap\widehat{\Omega^\varepsilon})}\\ &\quad+\|\widehat{w}\|_{L^2(B_r^j\cap\partial\widehat{\Omega^\varepsilon})}\Big)\\ &\lesssim \|\varphi\|_{L^2(\mathbb{R}^3)}\varepsilon^{-2}\|\widehat{\xi}\|_{L^2(\widehat{\Omega^\varepsilon})}\end{aligned}$$

where we have used (3.2.9) and (3.2.10) to obtain the second inequality. From the previous result we deduce that $\widehat{w}\in H^2(\cup_{j\in\mathcal{J}}B_r^j\cap\widehat{\Omega^\varepsilon})$ and

$$\|\Delta\widehat{w}\|_{L^2(\cup_{j\in\mathcal{J}}B_r^j\cap\widehat{\Omega^\varepsilon})}\lesssim\varepsilon^{-2}\|\widehat{\xi}\|_{L^2(\widehat{\Omega^\varepsilon})},$$

which implies in particular that $w\in H^2(\Omega^\varepsilon)$. Moreover, we remark that the above estimate together with (3.2.9) implies that

$$\|w\|_{H^2(\Omega^\varepsilon)}\le\|\widehat{w}\|_{H^2(\cup_{j\in\mathcal{J}}B_r^j\cap\widehat{\Omega^\varepsilon})}\lesssim\varepsilon^{-2}\|\widehat{\xi}\|_{L^2(\widehat{\Omega^\varepsilon})}\lesssim\varepsilon^{-2}\|\xi\|_{L^2(\Omega^\varepsilon)},$$

and this concludes the proof. □

3.2.2 Lamé system in the cylinder

Consider Ω^ε satisfying Assumption (RH2), and consider $\Xi\in L^2(\Omega^\varepsilon)$. We dedicate this subsection to study the following Lamé system of elliptic equations

$$\begin{cases}-\operatorname{div}(\nabla^s U) = \Xi & \text{in } \Omega^\varepsilon\\ U\cdot n(x) = 0 & \text{on } \partial\Omega^\varepsilon\\ (2-\iota^\varepsilon(x))[\nabla^s U\cdot n(x)-(\nabla^s U : n(x)\otimes n(x))n(x)]+\iota^\varepsilon(x)U = 0 & \text{on } \partial\Omega^\varepsilon.\end{cases}\tag{3.2.11}$$

We consider the functional space

$$\mathscr{U}(\Omega^\varepsilon) := \{U\in H^1(\Omega^\varepsilon),\, U(x)\cdot n(x)=0 \text{ on } \partial\Omega^\varepsilon\},\tag{3.2.12}$$

and we will prove the following well-posedness and regularity result.

Theorem 3.2.4. *Let Ω^ε be a cylindrical domain as described in (RH2). For any $\Xi\in L^2(\Omega^\varepsilon)$ there is a unique variational solution $W\in\mathscr{U}(\Omega^\varepsilon)\cap H^2(\Omega^\varepsilon)$ to the Lamé system* (3.2.11), *i.e for any $V\in\mathscr{U}(\Omega^\varepsilon)$ there holds*

$$\int_{\Omega^\varepsilon}\nabla^s W(x):\nabla^s V(x)\mathrm{d}x+\int_{\partial\Omega^\varepsilon}\frac{\iota^\varepsilon(x)}{2-\iota^\varepsilon(x)}W(x)\cdot V(x)\mathrm{d}\sigma_x=\int_{\Omega^\varepsilon}\Xi(x)\cdot V(x)\mathrm{d}x.\tag{3.2.13}$$

Furthermore, there holds

$$\|W\|_{H^2(\Omega^\varepsilon)}\le C\varepsilon^{-2}\|\Xi\|_{L^2(\Omega^\varepsilon)},\tag{3.2.14}$$

for some constructive constant $C>0$, independent of ε.

Preliminary inequalities

Before proving Theorem 3.2.4 we will prove some preliminary results in order to provide the coercivity for Equation (3.2.11).

One of the main inequalities to study this problem will be Korn's inequality, and we refer to [80, Theorem 3.2, Chapter 3], [61] and [72] for more on the history and applications of this type of inequality.

Lemma 3.5. *Consider a vector-field $U \in H^1(\Omega^\varepsilon)$, the following statements hold.*

(L1) (Poincaré's inequality) *If $U(x) \cdot n(x) = 0$ for* almost *every $x \in \partial\Omega^\varepsilon$ we have that*

$$\varepsilon^2 \|U\|^2_{L^2(\Omega^\varepsilon)} \lesssim \|\nabla U\|^2_{L^2(\Omega^\varepsilon)}.$$

(L2) (Korn's inequality) *There is a constant $C > 0$, independent of ε, for which there holds*

$$\|\nabla U\|^2_{L^2(\Omega^\varepsilon)} \lesssim_C \|\nabla^s U\|^2_{L^2(\Omega^\varepsilon)} + \left\| \sqrt{\frac{\iota^\varepsilon}{2-\iota^\varepsilon}} U \right\|^2_{L^2(\partial\Omega^\varepsilon)}.$$

Proof. We proof each of the statements separately.

(L1). Poincaré's inequality is nothing but a repetition of the compactness arguments used during the proof of [21, Lemma 2.7], and we further deduce the dependence on ε by a standard scaling argument.

(L2). The proof of the Korn's inequality follows by using the exact same arguments as during its proof in [21], except for some details accounted from the fact that the domain is now irregular, thus we make it explicit. We divide the proof into three steps.

Step 1. We observe first that from [72, Section 5] we have

$$\inf_{R \in \mathcal{R}} \|\nabla(U - R)\|^2 \lesssim \|\nabla^s U\|^2,$$

for some constant depending on Ω^ε and where we recall that $\mathcal{R}$ is the space of all infinitesimal rigid displacement fields. Equivalently, we have

$$\|\nabla U\|^2 \lesssim \|\nabla^s U\|^2 + |\langle \nabla^a U \rangle|^2, \tag{3.2.15}$$

where we remark that a scaling argument implies that the constant is independent of ε. It is worth remarking that this result is the same as the one stated in [21, Lemma 2.3].

Step 2. Moreover, there holds

$$|\langle \nabla^a U \rangle|^2 \lesssim \|\nabla^s U\|^2 + \left\| \sqrt{\frac{\iota^\varepsilon}{2-\iota^\varepsilon}} U \right\|^2_{L^2(\partial\Omega)}. \tag{3.2.16}$$

We follow the proof of [21, Lemma 2.4]. We argue by contradiction, and following the same compactness argument as during the proof of [21, Lemma 2.4] we deduce that there exists a function $U \in H^1(\Omega^\varepsilon)$ satisfying $|\langle \nabla^a U \rangle|^2 = 1$, $\|\sqrt{\iota^\varepsilon/(2-\iota^\varepsilon)} U\|^2_{L^2(\partial\Omega)} = 0$, and $\|\nabla^s U\| = 0$. In particular, this implies that $U(x) = Ax + b$ on Ω^ε, and

$$Ax + b = 0 \quad \text{on } \Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon.$$

We fix then $x_0 \in \Lambda_2^\varepsilon$ and we observe that $n(x_0) = \mathbf{e_1}$. Repeating again the arguments from the proof of [21, Lemma 2.4] we deduce that there is a constant $u_0 \in \mathbb{R}$ such that $U(x) = u_0 n(x_0)$ for every $x \in \Omega^\varepsilon$. This, together with the boundary condition $U(x) \cdot n(x) = 0$ on $\Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon$ implies that $U \equiv 0$ which is in contradiction with the fact that $|\langle \nabla^a U \rangle|^2 = 1$. Finally, a scaling argument implies that the previous estimate is uniform in ε.

Step 3. We conclude by putting together the estimates (3.2.15) and (3.2.16) from Steps 1 and 2 respectively. ☐

Proof of Theorem 3.2.4

We divide the proof into two steps.

Step 1. (Existence and uniqueness of variational solutions) We consider the bilinear operator $A : \mathscr{U}(\Omega^\varepsilon) \times \mathscr{U}(\Omega^\varepsilon) \to \mathbb{R}$ defined as

$$A(U,V) = \int_{\Omega^\varepsilon} \nabla^s U(x) : \nabla^s V(x) \mathrm{d}x + \int_{\partial\Omega^\varepsilon} \frac{\iota^\varepsilon(x)}{2-\iota^\varepsilon(x)} U(x) \cdot V(x) \mathrm{d}\sigma_x.$$

We observe that A is continuous on $\mathscr{U}(\Omega^\varepsilon) \times \mathscr{U}(\Omega^\varepsilon)$ and it is also coercive, indeed

$$\begin{aligned} A(U,U) &= \int_{\Omega^\varepsilon} \nabla^s U(x) : \nabla^s U(x) \mathrm{d}x + \int_{\partial\Omega^\varepsilon} \frac{\iota^\varepsilon(x)}{2-\iota^\varepsilon(x)} |U(x)|^2 \mathrm{d}\sigma_x \\ &= \|\nabla^s U\|^2_{L^2(\Omega^\varepsilon)} + \left\| \sqrt{\frac{\iota^\varepsilon}{2-\iota^\varepsilon}} U \right\|^2_{L^2(\partial\Omega^\varepsilon)} \\ &\gtrsim \varepsilon^2 \|U\|^2_{H^1(\Omega^\varepsilon)}. \end{aligned} \tag{3.2.17}$$

where we have applied simultaneously (L1) and (L2) from Lemma 3.5 to obtain the last line. We then may apply the Lax-Milgram theorem and we get that for every $\Xi \in L^2(\Omega^\varepsilon)$ there is a unique variational solution $W \in \mathscr{U}(\Omega^\varepsilon)$ of the Lamé system (3.2.11). Furthermore, using (3.2.17) and the weak formulation (3.2.13) we deduce that

$$\|W\|_{H^1(\Omega^\varepsilon)} \le C_L \, \varepsilon^{-2} \|\Xi\|_{L^2(\Omega^\varepsilon)}. \tag{3.2.18}$$

for some constant $C_L > 0$ independent of ε.

Step 2. (Domain reflection and regularity) Following the same reflection method as for the Poisson equation, we extend the functions component-wise as described in (3.2.7). We observe that $\widehat{\Xi} \in L^2(\widehat{\Omega^\varepsilon})$ and that $\widehat{W}$ is a variational solution of the problem

$$\left\{ \begin{aligned} -\operatorname{div}(\nabla^s \widehat{W}) &= \widehat{\Xi} && \text{in } \widehat{\Omega^\varepsilon} \\ \widehat{W} \cdot n(x) &= 0 && \text{on } \partial\widehat{\Omega^\varepsilon} \\ \left(2 - \widehat{\iota^\varepsilon}\right) [\nabla^s \widehat{W} \cdot n - (\nabla^s \widehat{W} : n \otimes n) n] + \widehat{\iota^\varepsilon} \, \widehat{W} &= 0 && \text{on } \partial\widehat{\Omega^\varepsilon}. \end{aligned} \right. \tag{3.2.19}$$

Moreover, by arguing as in the Step 1 we have that

$$\|\widehat{W}\|_{H^1(\widehat{\Omega^\varepsilon})} \lesssim \varepsilon^{-2} \|\Xi\|_{L^2(\widehat{\Omega^\varepsilon})}. \tag{3.2.20}$$

Repeating the arguments from [21, Theorem 2.11, Steps 3 and 4] together with (3.2.20), we deduce that for every small enough open set $\mathcal{G} \subset (-3L^\varepsilon/2, 3L^\varepsilon/2) \times \Omega_0^\varepsilon$ there holds

$$\|\widehat{W}\|_{H^2(\widehat{\Omega}^\varepsilon \cap \mathcal{G})} \lesssim \varepsilon^{-2} \|\widehat{\xi}\|_{L^2(\widehat{\Omega}^\varepsilon)}.$$

We then choose a finite family of open sets $(\mathcal{G}_j)_{j=1}^N$ with $\mathcal{G}_j \subset (-3L^\varepsilon/2, 3L^\varepsilon/2) \times \Omega_0^\varepsilon$ and such that $\bigcup_{j=1}^N \mathcal{G}_j \supset (-L^\varepsilon, L^\varepsilon) \times \Omega_0$. Using then the above estimate we deduce that $W \in H^2(\Omega^\varepsilon)$ and there holds

$$\|W\|_{H^2(\Omega^\varepsilon)} \le \|\widehat{W}\|_{H^2(\bigcup_{j=1}^N \mathcal{G}_j)} \lesssim \varepsilon^{-2} \|\widehat{\Xi}\|_{L^2(\widehat{\Omega^\varepsilon})} \le 3\varepsilon^{-2} \|\Xi\|_{L^2(\Omega^\varepsilon)}.$$

This completes the proof. □

3.2.3 Proof of Theorem 3.2.1

As discussed during Remark 3.2, Theorem 3.2.1 under hypothesis (RH1) is nothing but [21, Theorem 1.1] and [21, Theorem 5.1]. Moreover, the existence of the norm $|||\cdot|||$, and the fact that it is equivalent to $\|\cdot\|_{\mathcal{H}}$ uniformly in ε, is obtained by arguing exactly as during the proof of [21, Theorem 1.1], but we also refer to [47, Theorem A.1] together with the comments and references on [47, Appendix A].

Finally, the proof of Theorem 3.2.1 under the assumptions (RH2) is mainly a repetition of the proof of [21, Theorem 1.1] using instead the regularity results from Theorem 3.2.3 and Theorem 3.2.4. □

3.3 Stretching method for strongly confining weights in smooth domains

During this section we assume there to hold assumptions (RH1), and we study Equation (3.1.21). We dedicate this section to prove the following result.

Proposition 3.1. *Consider Assumption (RH1) to hold, $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function, $G : \mathcal{U}^\varepsilon \to \mathbb{R}$ satisfying $\langle\langle G_t \rangle\rangle_{\mathcal{O}^\varepsilon} = 0$ for every $t \geq 0$, and let f be a solution of Equation* (3.1.21)*. There are constructive constants $\varepsilon_1, \theta > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ there holds*

$$\|f_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq Ce^{-\theta\varepsilon^2 t}\left(\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \sup_{s\in[0,t]}\left[e^{\theta\varepsilon^2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right]\right) \quad \forall t \geq 0, \tag{3.3.1}$$

for some constant $C = C(\varepsilon) > 0$ such that $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

Remark 3.2. The computations leading to the proof of Proposition 3.1 use the stretching method developed in [110].

Remark 3.3. The exponential decay obtained in Proposition 3.1 is an improvement of [109, Proposition 4.1]. This is a consequence of a more delicate way of combining the hypocoercivity exponential decay from Theorem 3.2.1 and the weighted L^∞ estimate obtained by using the stretching method.

3.3.1 Auxiliary problem in finite time and backwards trajectories

Here and below, during this section, we consider an arbitrary $T > 0$, we take G as defined in Subsection 3.1.4, and we study the following evolution equation

$$\begin{cases} \partial_t f &= \mathscr{L} f + G & \text{in } \mathcal{U}^\varepsilon_T := (0,T)\times\mathcal{O}^\varepsilon \\ \gamma_- f &= \mathscr{R}\gamma_+ f & \text{on } \Gamma^\varepsilon_{-,T} := (0,T)\times\Sigma^\varepsilon_- \\ f_{t=0} &= f_0 & \text{in } \mathcal{O}^\varepsilon. \end{cases} \tag{3.3.2}$$

For any fixed $(t,x,v) \in \mathcal{U}^\varepsilon_T$, we recall the backwards trajectories defined in (3.1.25), and we define the singular set along the backwards trajectory

$$S_x := \{v \in \mathbb{R}^3;\ \exists\, t \in [0,T],\, n(X(t_b(x,v),t,x,v))\cdot v = 0\}. \tag{3.3.3}$$

To understand the geometrical meaning of this set we take $(t_0, x_0, v_0) \in \mathcal{U}^\varepsilon$ as before and we define the point of collision against the boundary as (t_1, x_1, v_1) as given by (3.1.25). Then we observe that $v_0 \in S_{x_0}$ if and only if $(x_1, v_1) \in \Sigma^\varepsilon_0$, or in words the point of collision against the boundary through the backwards trajectory belongs to the singular set Σ^ε_0.

3.3.2 Stretching lemma

We have now the following lemma, which is mainly an application of [109, Lemma 4.3].

Lemma 3.4. *Assume (RH1) to hold. Let* $(t,x,v) \in [0,T] \times \partial\Omega^\varepsilon \times \{|v| \le M, |n_x \cdot v| > \eta\}$ *for some* $M, \eta > 0$ *and such that* $(x,v) \in \Sigma_+^\varepsilon$. *There exists* $\varepsilon_R = \varepsilon_R(\eta, M, T) \lesssim \eta M^{-2} T^{-1}$ *such that for every* $\varepsilon \in (0, \varepsilon_R(\eta, M, T))$ *there is at most one bounce against the boundary* $\partial\Omega_\varepsilon$ *along the backwards trajectory.*

Remark 3.5. Even though the proof of Lemma 3.4 is mainly using the computations from [109, Lemma 4.3] in a very close way to that of [109, Lemmas 4.4 and 4.5], we provide a proof in order to emphasize that the arguments are, in a certain sense, local along trajectories which will allow us to prove, in Section 3.4, a weaker version for certain trajectories within cylindrical domains.

Proof. We proceed in two steps. In Step 1 we will repeat the proof of [109, Lemma 4.3], and we will apply this result in Step 2 to conclude the proof.

Step 1. We will prove in this step that there are universal constants $C_1, C_2 > 0$, only depending on δ, such that

$$t_b(x,v) \;\ge\; \frac{C_1\,|v\cdot n(x)|}{\varepsilon\, C_2\,|v|^2}. \tag{3.3.4}$$

We define (t_1, x_1, v_1) as the point of the first collision through the backwards trajectory starting at (t,x,v) and given by (3.1.25). We recall that $\varepsilon x, \varepsilon x_1 \in \partial\Omega$ and that, under the Assumption (H1), $\delta \in C^2(\bar\Omega)$. Thus the Taylor expansion at εx_1 implies that there is $\widetilde\theta \in (0,1)$ such that

$$\delta(\varepsilon x_1) = \delta(\varepsilon x) + \varepsilon\,\nabla_y\delta(\varepsilon x)\cdot(x_1 - x) + \frac{\varepsilon^2}{2}\,(x_1 - x)\cdot\big[\nabla_y^2\delta(\tilde\theta\,\varepsilon x + (1-\tilde\theta)\,\varepsilon x_1)\big](x_1 - x).$$

Since $\varepsilon x, \varepsilon x_1 \in \partial\Omega$, we have that $\delta(\varepsilon x) = \delta(\varepsilon x_1) = 0$, and since $\nabla_y\delta \neq 0$ on $\partial\Omega$, we deduce that

$$\frac{\nabla_y\delta(\varepsilon x)}{|\nabla_y\delta(\varepsilon x)|}\cdot(x_1 - x) = -\frac{\varepsilon}{2}\,(x_1 - x)\cdot\left(\frac{\nabla_y^2\delta(\tilde\theta\,\varepsilon x + (1-\tilde\theta)\,\varepsilon x_1)}{|\nabla_y\delta(\varepsilon x)|}\right)(x_1 - x).$$

Using then the analysis from (3.1.8), and again the fact that $\delta \in C^2(\bar\Omega)$ and $\nabla_y\delta \neq 0$ on $\partial\Omega$, we further deduce that there are constants $C_1, C_2 > 0$ such that

$$\big|n(x)\cdot(x_1 - x)\big| \;\le\; \varepsilon\,\frac{C_2}{C_1}\,|x_1 - x|^2$$

We recall now that due to its very definition $x_1 = x + t_b(x,v)v$, thus

$$|x_1 - x| = |t_b(x,v)|\,|v| \quad\text{ and }\quad |n(x)\cdot(x_1 - x)| = |t_b(x,v)||n(x)\cdot v|.$$

We conclude (3.3.4) by putting together these previous informations.

Step 2. The conclusion follows by using (3.3.4) and choosing $\varepsilon_R(\eta, M, T) := C_1\eta/(C_2 M T)$ so that for any $\varepsilon \in (0, \varepsilon_R)$ there holds $t_b(x,v) > T$.

In words, we have proven that any particle $(t,x,v) \in [0,T]\times\partial\Omega^\varepsilon\times\{|v| \le M, |n_x\cdot v| > \eta\}$, needs a time larger than the total time T of evolution of the system to collide with the boundary through the backwards trajectory. □

3.3.3 Preliminary lemmas

We present now several results which will be useful during the rest of this chapter.

Lemma 3.6. *Let $h : \mathcal{U}^\varepsilon \to \mathbb{R}$, we consider ω to be an admissible weight function, and we define $\varsigma(v) = e^{\zeta|v|^2}$ for $\zeta \in (0, 1/2)$. The following statements hold.*

(K1) For every $N > 0$ there is $m = m(N) \geq 1$ such that

$$\varsigma(v)|Kh(v)| \leq \frac{1}{N} \|h\|_{L^\infty_\varsigma(\mathbb{R}^3)} + K_m\left(|\varsigma(v)h(v)|\right)$$

where $K_m h := \int_{\mathbb{R}^3} k_m(v, v_*)h(v_*)dv_*$ and

$$k_m(v, v_*) := c_k \mathbf{1}_{|v_*|\leq m}\mathbf{1}_{|v|\leq m}\mathbf{1}_{|v_*-v|\geq 1/m}\left(|v - v_*| + |v - v_*|^{-1}\right), \tag{3.3.5}$$

for some constructive constant $c_k > 0$. It is also worth remarking that $k_m \leq C_k\, m$, where $C_k = 3c_k$.

(K2) For every $\nu_2 \in (0, \nu_0)$, every $(t, x, v) \in \mathcal{U}^\varepsilon$, and every $\sigma \in [0, t]$ there holds

$$\omega(v)\,|S_{\mathscr{T}} *_\sigma h(t, x, v)| \leq \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|h_s\|_{L^\infty_{\omega\nu^{-1}}(\mathcal{O}^\varepsilon)}\right].$$

where we remark that $S_{\mathscr{T}} *_\sigma h$ is in the sense of (3.1.23) taking $\mathcal{A} = Id$,

(K3) Let $0 \leq r \leq s \leq t$, $x \in \Omega^\varepsilon$, $u, v_* \in \mathbb{R}^3$ and consider a function $\mathcal{X} = \mathcal{X}(t, s, r, x, u, v_*)$ such that

$$|\det \nabla_{v_*}\mathcal{X}(v_*)| \geq C_0|s - r|^3,$$

for some constant $C_0 > 0$. Then for any $m_1, m_2 > 0$, and every $\alpha > 0$ there holds

$$\begin{aligned}\int_0^t \int_{|v_*|\leq m_1} \int_0^s e^{-\nu_0(t-r)} \int_{|u|\leq m_2} \omega(u)|h(r, \mathcal{X}, u)|\, \mathrm{d}u\mathrm{d}r\mathrm{d}v_*\mathrm{d}s \\ \leq D^2\, m_1^3\, m_2^3\, \alpha\, t\, e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|h_s\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\right] \\ + D m_1^{3/2} m_2^{3/2}\, \omega(m_2)\, t^2 \alpha^{-3/2} C_0^{-1/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|h_s\|_{L^2(\mathcal{O}^\varepsilon)}\right].\end{aligned}$$

where we recall that $D > 0$ is the volume of the $3-$dimensional ball of radius 1.

(K4) For every $\alpha > 0$ there are $M, \eta > 0$ such that

$$\int_{\mathbb{R}^3} |h(t, x, u)|(n_x \cdot u)_+ \mathrm{d}u \quad \leq \quad \alpha\|h_t\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)} + \int_{\{|u|\leq M,\, |n_x\cdot u|>\eta\}} |h(t, x, u)|(n_x \cdot u)\mathrm{d}u$$

Remark 3.7. It is worth emphasizing that the results from Lemma 3.6 are independent of the geometry of the domain, so they hold under either Assumption (RH1) or Assumption (RH2).

Proof of Lemma 3.6. We proof each of the statements separately.

Proof of (K1). We divide the proof in three steps

Step 1. We observe that

$$k(v,v_*) = \widetilde{k}(v,v_*)\exp\left(-\frac{|v|^2}{4}+\frac{|v_*|^2}{4}\right),$$

where

$$\widetilde{k}(v,v_*) = 2(2\pi)^{-1/2}|v-v_*|^{-1}\exp\left(-\frac{1}{8}\frac{(|v_*|^2-|v|^2)^2}{|v-v_*|^2}-\frac{1}{8}|v-v_*|^2\right) - \frac{1}{2}|v-v_*|\exp\left(-\frac{|v|^2+|v_*|^2}{4}\right),$$

and from [108, Lemma 3], [99] or [56, Equation (2.17)] we have that there is a constant $c_k>0$ such that

$$\widetilde{k}(v,v_*) \le c_k\bar{k}(v,v_*), \tag{3.3.6}$$

with

$$\bar{k}(v,v_*) = (|v-v_*|+|v-v_*|^{-1})\exp\left(-\frac{1}{8}\frac{(|v_*|^2-|v|^2)^2}{|v-v_*|^2}-\frac{1}{8}|v-v_*|^2\right).$$

Furthermore, repeating the exact arguments used during the proof of [108, Lemma 3] we have that for any $\zeta\in(0,1/2)$, the quadratic form

$$\mathscr{Q} := -\frac{1}{8}\frac{(|v_*|^2-|v|^2)^2}{|v-v_*|^2}-\frac{1}{8}|v-v_*|^2-\frac{|v|^2}{4}+\frac{|v_*|^2}{4}+\zeta|v|^2-\zeta|v_*|^2,$$

is definite negative. Thus there is $b>0$ such that

$$\int_{\mathbb{R}^3}\bar{k}(v,v_*)e^{-\frac{|v|^2}{4}+\frac{|v_*|^2}{4}}\exp\left(\frac{b}{8}\frac{(|v_*|^2-|v|^2)^2}{|v-v_*|^2}+\frac{b}{8}|v-v_*|^2\right)\frac{\varsigma(v)}{\varsigma(v_*)}dv_* \le \frac{C_1}{1+|v|}, \tag{3.3.7}$$

for some positive constant $C_1=C_1(b)$.

Step 2. The results from Step 1 in particular imply that

$$\bar{k}(v,v_*)e^{-\frac{|v|^2}{4}+\frac{|v_*|^2}{4}}\frac{\varsigma(v)}{\varsigma(v_*)} = (|v-v_*|+|v-v_*|^{-1})e^{\mathscr{Q}} \le (|v-v_*|+|v-v_*|^{-1}), \tag{3.3.8}$$

thus setting $A_m=\{|v_*|\le m,\,|v|\le m,\,|v_*-v|\ge 1/m\}$ for some $m>0$ to be chosen later, we compute

$$\begin{aligned}
\varsigma(v)|Kh(v)| &\le c_k\int_{\mathbb{R}^3}\bar{k}(v,v_*)e^{-\frac{|v|^2}{4}+\frac{|v_*|^2}{4}}\frac{\varsigma(v)}{\varsigma(v_*)}|h(v_*)|\varsigma(v_*)dv_* \\
&\le c_k\|h\|_{L^\infty_\varsigma(\mathcal{O}^\varepsilon)}\int_{A_m^c}\bar{k}(v,v_*)e^{-\frac{|v|^2}{4}+\frac{|v_*|^2}{4}}\frac{\varsigma(v)}{\varsigma(v_*)}dv_* \\
&\quad +c_k\int_{A_m}(|v-u|+|v-u|^{-1})\varsigma(v_*)|h(v_*)|dv_* \\
&\le \frac{c_kC_1}{1+m}\|h\|_{L^\infty_\varsigma(\mathcal{O}^\varepsilon)}+\int_{\mathbb{R}^3}k_m(v,v_*)\varsigma(v_*)|h(v_*)|dv_* \\
&\le \frac{1}{N}\|h\|_{L^\infty_\varsigma(\mathcal{O}^\varepsilon)}+K_m(|\varsigma(v)h(v)|)
\end{aligned}$$

where we have used (3.3.6) on the first inequality, (3.3.8) to obtain the third, we have used (3.3.7) and the definition of k_m on the fourth inequality, and we have chosen $m=\max(Nc_kC_1,1)$ on the last line. This concludes the proof of (K1)

Proof of (K2): We compute

$$\begin{aligned}
\left|\int_0^t S_{\mathscr{T}}(t-s)h(s,x,v)\omega(v)\mathrm{d}s\right| &\le \int_0^t e^{-\nu(v)(t-s)}\,|h(s,x-v(t-s),v)|\,\omega(v)\,\mathrm{d}s\\
&\le \sup_{s\in[0,t]}\left[e^{\nu_2 s}\|h_s\|_{L^\infty_{\omega\nu^{-1}}(\mathcal{O}^\varepsilon)}\right]\int_0^t \nu(v)e^{-\nu(v)(t-s)-\nu_2 s}\mathrm{d}s\\
&\le \frac{\nu(v)}{\nu(v)-\nu_2}e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|h_s\|_{L^\infty_{\omega\nu^{-1}}(\mathcal{O}^\varepsilon)}\right].
\end{aligned}$$

We then remark that (3.1.6) and the fact that $1\le\langle v\rangle$ imply that

$$\frac{\nu(v)}{\nu(v)-\nu_2}\le\frac{\nu_1}{\nu_0-\nu_2}.$$

This concludes the proof of (K2).

Proof of (K3): We split the integral on the LHS into I_1+I_2 where

$$\begin{aligned}
I_1 &:= \int_0^t\int_{|v_*|\le m_1}\int_{s-\alpha}^s e^{-\nu_0(t-r)}\int_{|u|\le m_2}\omega(u)|h(r,\mathcal{X},u)|\mathrm{d}u\mathrm{d}r\mathrm{d}v_*\mathrm{d}s\\
&\le D^2 m_1^3\,m_2^3\,\alpha\,t\,e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|g_s\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\right],
\end{aligned}$$

and

$$I_2 := \int_0^t\int_{|v_*|\le m_1}\int_0^{s-\alpha} e^{-\nu_0(t-r)}\int_{|u|\le m_2}\omega(u)|h(r,\mathcal{X},u)|\,\mathrm{d}u\mathrm{d}r\mathrm{d}v_*\mathrm{d}s.$$

To control I_2 we perform the change of variables $\mathcal{X}\to x$, and we remark that

$$|\det\nabla_{v_*}\mathcal{X}(v_*)|\ge C_0|s-r|^3>C_0\alpha^3>0,$$

thus we compute

$$\begin{aligned}
I_2 &\le t^2\omega(m_2)e^{-\nu_0 t}\sup_{0\le r\le s-\alpha<s\le t}\left[e^{\nu_0 r}\left(\int_{|v_*|\le m_1}\int_{|u|\le m_2}|h(r,\mathcal{X},u)|\mathrm{d}u\mathrm{d}v_*\right)\right]\\
&\le D m_1^{3/2}m_2^{3/2}\omega(m_2)\,t^2e^{-\nu_0 t}\sup_{0\le r\le s-\alpha<s\le t}\left[e^{\nu_0 r}\left(\int_{|v_*|\le m_1}\int_{|u|\le m_2}|h(r,\mathcal{X},u)|^2\,\mathrm{d}u\mathrm{d}v_*\right)^{1/2}\right]\\
&\le D m_1^{3/2}m_2^{3/2}\omega(m_2)\,t^2e^{-\nu_0 t}\\
&\quad\times\sup_{0\le r\le s-\alpha<s\le t}\left[e^{\nu_0 r}\left(\int_{x\in\Omega^\varepsilon}\int_{|u|\le m_2}|h(r,x,u)|^2\frac{1}{|\det\nabla_{v_*}X(v_*)|}\mathrm{d}u\mathrm{d}x\right)^{1/2}\right]\\
&\le D m_1^{3/2}m_2^{3/2}\omega(m_2)\,t^2C_0^{-1/2}\alpha^{-3/2}e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|h_s\|_{L^2(\mathcal{O}^\varepsilon)}\right].
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality to obtain the second line. This concludes the proof of (K3).

Proof of (K4): We fix $M,\eta>0$ to be defined later and we split

$$\{n_x\cdot u>0\}=\{|u|>M,\ n_x\cdot u>0\}\cup\{|u|\le M,\ |n_x\cdot u|<\eta\}\cup\{|u|\le M,\ |n_x\cdot u|\ge\eta\}.$$

On the one hand, we compute

$$\begin{aligned}\int_{\{|u|>M,\ n_x\cdot u>0\}}|h(t,x,u)||n_x\cdot u|du &\leq \|g_t\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\int_{\{|u|>M,\ n_x\cdot u>0\}}\omega^{-1}(u)|n_x\cdot u|du\\ &\leq \|h_t\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\int_{|u|>M}\omega^{-1}(u)|u|du\\ &\leq o(M)\,\|h_t\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)},\end{aligned}$$

where $o(M)\to 0$ as $M\to\infty$. Similarly, on the set $\{|u|\leq M,\ |n(x_1)\cdot u|<\eta\}$ we compute

$$\begin{aligned}\int_{\{|u|\leq M,\ |n_x\cdot u|<\eta\}}|h(t,x,u)||n_x\cdot u|du &\leq \eta\,\|h_t\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\int_{|u|\leq M}\omega^{-1}(u)du\\ &\leq \eta\,\|h_t\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\int_{|u|\leq M}du\\ &\leq \eta\,M^3D\|h_t\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}.\end{aligned}$$

We take then $M>0$ large enough such that $o(M)\leq\alpha/2$ and we choose $\eta=\alpha/(2M^3D)$. This concludes the proof of (K4) and the proof of Lemma 3.6. $\square$

Moreover, we also have the following lemma in the spirit of (K3).

Lemma 3.8. *Let $0\leq r\leq s\leq t$, $x\in\Omega^\varepsilon$, $u,v_*\in\mathbb{R}^3$ and consider a function $\mathcal{X}=\mathcal{X}(t,s,r,x,u,v_*)$ such that*

$$\det\nabla_{v_*}\mathcal{X}(v_*)=C_0|s-r|^3+C_1\varepsilon,$$

for some constants $C_0,C_1>0$. Then for any $m_1,m_2>0$, and every $\alpha>0$ there is a constructive $\varepsilon_U=\varepsilon_U(\alpha,C_0,C_1):=|C_0|\alpha^3/(2|C_1|)>0$ such that for every $\varepsilon\in(0,\varepsilon_U)$ there holds

$$\begin{aligned}&\int_0^t\int_{|v_*|\leq m_1}\int_0^s e^{-\nu_0(t-r)}\int_{|u|\leq m_2}\omega(u)|h(r,\mathcal{X},u)|\,\mathrm{d}u\mathrm{d}r\mathrm{d}v_*\mathrm{d}s\\ &\qquad\leq D^2\,m_1^3\,m_2^3\,\alpha\,t\,e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|h_s\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\right]\\ &\qquad+2^{1/2}Dm_1^{3/2}m_2^{3/2}\,\omega(m_2)\,t^2\alpha^{-3/2}|C_0|^{-1/2}t^2e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|h_s\|_{L^2(\mathcal{O}^\varepsilon)}\right].\end{aligned}$$

where we recall that $D>0$ is the volume of the $3-$dimensional ball of radius 1.

Remark 3.9. We also notice that Lemma 3.8 is independent of either geometrical Assumption (RH1) or (RH2).

Proof. The proof follows the main ideas of that of (K3). We split the integral on the LHS into I_1+I_2 where

$$\begin{aligned}I_1 &:= \int_0^t\int_{|v_*|\leq m_1}\int_{s-\alpha}^s e^{-\nu_0(t-r)}\int_{|u|\leq m_2}\omega(u)|h(r,\mathcal{X},u)|\mathrm{d}u\mathrm{d}r\mathrm{d}v_*\mathrm{d}s\\ &\leq D^2m_1^3\,m_2^3\,\alpha\,t\,e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|g_s\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\right],\end{aligned}$$

and

$$I_2:=\int_0^t\int_{|v_*|\leq m_1}\int_0^{s-\alpha}e^{-\nu_0(t-r)}\int_{|u|\leq m_2}\omega(u)|h(r,\mathcal{X},u)|\,\mathrm{d}u\mathrm{d}r\mathrm{d}v_*\mathrm{d}s.$$

We now remark that if $s-r\geq\alpha$ and $\varepsilon\in(0,\varepsilon_U)$, with ε_U as defined above, there holds

$$|\det\nabla_{v_*}\mathcal{X}(v_*)|\geq|C_0|\alpha^3-|C_1|\varepsilon\geq\frac{|C_0|}{2}\alpha^3+\left(\frac{|C_0|}{2}\alpha^3-|C_1|\varepsilon\right)>\frac{|C_0|}{2}\alpha^3.$$

To control then I_2 we perform the change of variables $\mathcal{X}\to x$, and using the above computations we have that

$$\begin{aligned}
I_2 &\leq t^2\omega(m_2)e^{-\nu_0 t}\sup_{0\leq r\leq s-\alpha<s\leq t}\left[e^{\nu_0 r}\left(\int_{|v_*|\leq m_1}\int_{|u|\leq m_2}|h(r,\mathcal{X},u)|\mathrm{d}u\mathrm{d}v_*\right)\right]\\
&\leq Dm_1^{3/2}m_2^{3/2}\omega(m_2)\,t^2e^{-\nu_0 t}\sup_{0\leq r\leq s-\alpha<s\leq t}\left[e^{\nu_0 r}\left(\int_{|v_*|\leq m_1}\int_{|u|\leq m_2}|h(r,\mathcal{X},u)|^2\,\mathrm{d}u\mathrm{d}v_*\right)^{1/2}\right]\\
&\leq Dm_1^{3/2}m_2^{3/2}\omega(m_2)\,t^2e^{-\nu_0 t}\\
&\quad\times\sup_{0\leq r\leq s-\alpha<s\leq t}\left[e^{\nu_0 r}\left(\int_{x\in\Omega^\varepsilon}\int_{|u|\leq m_2}|h(r,x,u)|^2\frac{1}{|\det\nabla_{v_*}X(v_*)|}\mathrm{d}u\mathrm{d}x\right)^{1/2}\right]\\
&\leq 2^{1/2}Dm_1^{3/2}m_2^{3/2}\omega(m_2)\,t^2|C_0|^{-1/2}\alpha^{-3/2}e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|h_s\|_{L^2(\mathcal{O}^\varepsilon)}\right].
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality to obtain the second line. □

3.3.4 Regularizing effect of K

We will now use the previous lemmas to prove the regularizing character of the interplay between the free transport semigroup $S_{\mathscr{T}}$ and the non-local operator K.

Proposition 3.10. *Consider Assumption (RH1) to hold, $\omega_1\in\mathfrak{W}_1$ a strongly confining admissible weight function, and let f be a solution of Equation* (3.3.2). *For every $\lambda>0$ there is $\varepsilon_2=\varepsilon_2(\lambda,T)$ such that for every $\varepsilon\in(0,\varepsilon_3)$ and every $\nu_2\in(0,\nu_0)$ we have that for every $(t,x,v)\in\mathcal{U}_T^\varepsilon$ with $v\notin S_x$ there holds*

$$\begin{aligned}
\omega_1(v)|S_{\mathscr{T}}*_\sigma Kf(t,x,v)|\leq{}&\lambda(t+t^2)e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right]+Cte^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\\
&+Ct^2e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right]+C(1+t)e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right],
\end{aligned}$$

for a constant $C=C(\lambda)>0$, and any $\sigma\in[0,t]$ such that $x-v(t-\sigma)\in\bar{\Omega}^\varepsilon$. Furthermore, there holds $C(\lambda)\lesssim\lambda^{-p}$ for some constant $p>0$.

Remark 3.11. The techniques are mainly taken from the computations performed during the proof of [110, Theorem 2.4] and we repeat it in order to later generalize these ideas to cylindrical domains.

Proof of Proposition 3.10. We fix $(t,x,v)\in\mathcal{U}_T^\varepsilon$ with $v\notin S_x$, we set

$$\mathfrak{I}:=\omega_1(v)|S_{\mathscr{T}}*_\sigma Kf(t,x,v)|=\omega_1(v)\left|\int_\sigma^t S_{\mathscr{T}}(t-s)Kf(s,x,v)\mathrm{d}s\right|,$$

and we split the proof into four steps.

Step 1. (A first control on $\mathfrak{I}$) By using (K1) we deduce that for every $N_1 > 0$ there is $m_1(N_1) > 0$ such that

$$\mathfrak{I} \leq \frac{1}{N_1} t e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + S_{\mathscr{T}} *_\sigma K_{m_1} \left(\omega_1(v)|f(t,x,v)| \right), \tag{3.3.9}$$

where we recall that K_{m_1} has been defined in (K1). Moreover, for each $s \in [\sigma, t]$ we denote $x_s := y - v(t-s)$, and for any constant $\alpha_0 > 0$ we define $k_{0,m_1} = \mathbf{1}_{\{|n(x_s)\cdot v_*| \geq \alpha_0\}} k_{m_1}$ and the modified non-local operator

$$K_{0,m_1} h(v) = \int_{\mathbb{R}^3} k_{0,m_1}(v, v_*) h(v_*) \mathrm{d}v_*. \tag{3.3.10}$$

We then have that $S_{\mathscr{T}} *_\sigma K_{m_1} \left(\omega_1(v)|f(t,x,v)| \right) = \mathfrak{I}_0 + \widetilde{\mathfrak{I}}$ where

$$\mathfrak{I}_0 := \int_\sigma^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*) \omega_1(v_*) |f(s,x_s,v_*)| \mathrm{d}v_* \mathrm{d}s,$$

and

$$\begin{aligned} \widetilde{\mathfrak{I}} &:= \int_0^t e^{-\nu(v)(t-s)} \int_{|v_*|\leq m_1, |n_{x_s}\cdot v_*| \leq \alpha_0} k_{m_1}(v,v_*) \omega_1(v) |f(s,x_s,v_*)| \mathrm{d}v_* \mathrm{d}s \\ &\leq C_k m_1 \int_0^t e^{-\nu_0(t-s)} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \mathrm{d}s \int_{-\alpha_0}^{\alpha_0} \mathrm{d}v_*^{par} \int_{|v_*^\perp| \leq m_1} \mathrm{d}v_*^\perp \\ &\leq 2 C_k D m_1^4 \alpha_0 t e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \end{aligned}$$

where we have used the change of variables $v_*^{par} = [n(x_s)\cdot v_*] n(x_s)$ and $v_*^\perp = v_* - v_*^{par}$ to obtain the second line together with (3.3.5). The above computations together with (3.3.9) give that

$$\mathfrak{I} \leq \left(\frac{1}{N_1} + 2 C_k D m_1^4 \alpha_0 \right) t e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + \mathfrak{I}_0.$$

We remark now that [108, Lemma 17] gives that Σ_0^ε and S_{x_s} have Lebesgue measure zero. Due to this fact, we may rewrite $\mathfrak{I}_0$ being integrated over the set $v_* \in \mathbb{R}^3 \setminus S_{x_s}$. This implies that, since $v \notin S_x$ and $v_* \notin S_{x_s}$, we may define (s_1, x_1, v_1), with $(x_1, v) \notin \Sigma_0^\varepsilon$, as the point of the first bounce against the boundary through the backwards trajectory starting at (s, x_s, v_*), and which is given by (3.1.25). In particular, we remark that $x_1 = x_s - v_*(s - s_1)$.

Moreover, it is worth remarking that when $s_1 \leq 0$ we reach the initial time $t = 0$ along the backwards trajectory, on the contrary if $s_1 > 0$ then along the backwards trajectory we reach the boundary at the time $s - s_1$. Using the Duhamel formula we deduce then that

$$\begin{aligned} f(s,x_s,v_*) &= \mathbf{1}_{\{s_1\leq 0\}} S_{\mathscr{T}}(s) f_0(x_s,v_*) + \int_{\max\{0,s_1\}}^s S_{\mathscr{T}}(s-r) K f(r,x_s,v_*) dr \\ &\quad + \int_{\max\{0,s_1\}}^s S_{\mathscr{T}}(s-r) G(r,x_s,v_*) dr + \mathbf{1}_{\{s_1>0\}} S_{\mathscr{T}}(s-s_1) f(s_1,x_1,v_*), \end{aligned} \tag{3.3.11}$$

where we remark that $S_{\mathscr{T}}(s-s_1) f(s_1,x_s,v_*) = e^{-\nu(v_*)(s-s_1)} f(s_1,x_1,v_*)$. Putting (3.3.11) into the expression of $\mathfrak{I}_0$ we have that

$$\mathfrak{I}_0 \leq \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3 + \mathfrak{I}_4, \tag{3.3.12}$$

where we have defined

$$\begin{aligned}
\mathfrak{I}_1 &:= \int_\sigma^t e^{-\nu(t-s)} \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)\,\omega_1(v_*)\,|S_{\mathscr{T}}(s)f_0(x_s,v_*)|\,\mathrm{d}v_*\mathrm{d}s,\\
\mathfrak{I}_2 &:= \int_\sigma^t e^{-\nu(t-s)} \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)\,\omega_1(v_*)\,\left|S_{\mathscr{T}} *_{\max(0,s_1)} Kf(s,x_s,v_*)\right|\mathrm{d}v_*\mathrm{d}s,\\
\mathfrak{I}_3 &:= \int_\sigma^t e^{-\nu(t-s)} \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)\,\omega_1(v_*)\,\left|S_{\mathscr{T}} *_{\max(0,s_1)} G(s,x_s,v_*)\right|\mathrm{d}v_*\mathrm{d}s,\\
\mathfrak{I}_4 &:= \int_\sigma^t e^{-\nu(t-s)} \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)\,e^{-\nu(v_*)(s-s_1)}\,\omega_1(v_*)\,|f(s_1,x_1,v_*)|\,\mathrm{d}v_*\mathrm{d}s,
\end{aligned}$$

and we will control now each term separately. Using the formula of k_{m_1} defined in (3.3.5) we first compute

$$\begin{aligned}
\mathfrak{I}_1 &= e^{-\nu_0 t}\int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*)\omega_1(v_*)|f_0(x-(t-s)v-sv_*,v_*)|\mathrm{d}v_*\mathrm{d}s\\
&\leq C_k D m_1^4 t e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},
\end{aligned}$$

and using (K2) we deduce that for every $\nu_2\in(0,\nu_0)$ we have

$$\begin{aligned}
\mathfrak{I}_3 &\leq \frac{\nu_1}{\nu_0-\nu_2}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right]\int_\sigma^t e^{-\nu_0(t-s)}e^{-\nu_2 s}\int_{\mathbb{R}^3} k_{m_1}(v,v_*)\mathrm{d}v_*\mathrm{d}s\\
&\leq C_k m_1^4 D\frac{\nu_1 t}{\nu_0-\nu_2}e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right].
\end{aligned}$$

Using now (K1) we deduce that for any $N_2>0$ there is $m_2(N_2)>0$ such that

$$\begin{aligned}
\mathfrak{I}_2 &\leq \int_\sigma^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)\left(\frac{1}{N_2}\,s\,e^{-\nu_0 s}\sup_{r\in[0,s]}\left[e^{\nu_0 r}\|f_r\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right]\right)\mathrm{d}v_*\mathrm{d}s\\
&\quad+\int_\sigma^t e^{-\nu(v)(t-s)}\int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)\left(S_{\mathscr{T}} * K_{m_2}(\omega_1(v_*)|f(s,x_s,v_*)|\right)\mathrm{d}v_*\mathrm{d}s.\\
&=: \mathfrak{I}_1^K+\mathfrak{I}_2^K,
\end{aligned}$$

and we will deal with each term separately. On the one hand we compute

$$\begin{aligned}
\mathfrak{I}_1^K &\leq \frac{1}{N_2}e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right]\int_0^t\int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)\,s\,\mathrm{d}v_*\mathrm{d}s\\
&\leq C_k m_1^4 D\frac{t}{N_2}e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right],
\end{aligned}$$

where we have used again the definition of k_{m_1} given in (3.3.5). On the other hand, to control $\mathfrak{I}_2^K$ we first observe that

$$\mathfrak{I}_2^K \leq (C_k)^2 m_1 m_2\int_0^t\int_{|v_*|\leq m_1}\int_0^s e^{-\nu_0(t-r)}\int_{|u|\leq m_2}\omega_1(u)\,|f(r,x_s-(s-r)u,u)|\,\mathrm{d}u\mathrm{d}r\mathrm{d}v_*\mathrm{d}s,$$

and since

$$|\det\nabla_{v_*}(x_s-(s-r)u)| = |\det\nabla_{v_*}(x-(t-s)v_*-(s-r)u)| = |t-s|^3$$

we can apply (K3) and it yields that for every $\alpha_1 > 0$ there holds

$$\mathfrak{I}_2^K \le C_k^2 D^2 m_1^4 m_2^4 \alpha_1 t e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + C_k^2 D m_1^{1+3/2} m_2^{1+3/2}\,\omega_1(m_2)\alpha_1^{-3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right].$$

Altogether we have obtained that

$$\begin{aligned}\mathfrak{I} \;\le\; & \left(\frac{1}{N_1} + 2C_k D m_1^4 \alpha_0 + \frac{C_k D m_1^4}{N_2} + C_k^2 D^2 m_1^4 m_2^4 \alpha_1\right) t e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] \\ & + C_k D m_1^4 t e^{-\nu t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_k D m_1^4 \frac{\nu_1 t}{\nu_0-\nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right] \\ & + C_k^2 D m_1^{1+3/2} m_2^{1+3/2}\omega_1(m_2)\alpha_1^{-3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right] + \mathfrak{I}_4,\end{aligned}$$

where there is only left to control the boundary term $\mathfrak{I}_4$. We remark that

$$\begin{aligned}\mathfrak{I}_4 &= \int_\sigma^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*) e^{-\nu(v_*)(s-s_1)}\omega_1(v_*)|f(s_1,x_1,v_*)|\,\mathrm{d}v_*\mathrm{d}s \\ &\le \int_\sigma^t \int_{|v_*|\le m_1,\,|n(x_s)\cdot v_*|\ge\alpha_0} e^{-\nu_0(t-s_1)} k_{0,m_1}(v,v_*)\omega_1(v_*)|f(s_1,x_1,v_*)|\,\mathrm{d}v_*\mathrm{d}s,\end{aligned}$$

where we recall that the expression on the second line is due to the very definition of k_{0,m_1}. We recall that $x_s - (s-s_1)v_* = x_1$, and since by definition $(x_1,v_*) \in \Sigma_-^\varepsilon$, the Maxwell boundary condition applies and we have that

$$\begin{aligned}f(s_1,x_1,v_*) &= (1-\iota^\varepsilon(x_1))\mathscr{S} f(s_1,x_1,v_*) + \iota^\varepsilon(x_1)\mathscr{D} f(s_1,x_1,v_*) \\ &= (1-\iota^\varepsilon(x_1)) f(s_1,x_1,\mathcal{V}_{x_1}v_*) + \iota^\varepsilon(x_1)\mathscr{M}(v_*)\widetilde{f}(s_1,x_1).\end{aligned}$$

We then have that $\mathfrak{I}_4 \le \mathfrak{I}^S + \mathfrak{I}^D$, where

$$\mathfrak{I}^S := \int_\sigma^t \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)}\omega_1(v_*)(1-\iota^\varepsilon(x_1))\,|f(s_1,x_1,\mathcal{V}_{x_1}v_*)|\,\mathrm{d}v_*\mathrm{d}s, \tag{3.3.13}$$

$$\mathfrak{I}^D := \int_\sigma^t \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)}\omega_1(v_*)\iota^\varepsilon(x_1)\mathscr{M}(v_*)\left|\widetilde{f}(s_1,x_1)\right|\mathrm{d}v_*\mathrm{d}s. \tag{3.3.14}$$

Step 2. (Control of $\mathfrak{I}^S$) For any small parameter $\alpha_{0,1} > 0$ we define the non-local operator

$$K_{1,m_1}h(v) = \int_{\mathbb{R}^3} k_{1,m_1}(v,v_*)h(v_*)\mathrm{d}v_*,$$

where $k_{1,m_1}(v,v_*) = \mathbf{1}_{\{|n(x_1)\cdot v_*|\ge\alpha_{0,1}\}} k_{0,m_1}(v,v_*)$, and we also define

$$\mathfrak{I}_0^S = \int_\sigma^t \int_{\mathbb{R}^3} k_{1,m_1}(v,v_*) e^{-\nu_0(t-s_1)}\omega_1(v_*)\,|f(s_1,x_1,\mathcal{V}_{x_1}v_*)|\,\mathrm{d}v_*\mathrm{d}s.$$

Proceeding then as during the Step 1 of this proof we have that

$$\begin{aligned}\mathfrak{I}^S &\le \mathfrak{I}_0^S + \int_\sigma^t \int_{|n(x_1)\cdot v_*|<\alpha_{0,1}} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)}\omega_1(v_*)\,|f(s_1,x_1,\mathcal{V}_{x_1}v_*)|\,\mathrm{d}v_*\mathrm{d}s \\ &\le \mathfrak{I}_0^S + C_k m_1 \int_\sigma^t e^{-\nu_0(t-s_1)}\|f_{s_1}\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\mathrm{d}s \int_{-\alpha_{0,1}}^{\alpha_{0,1}} \mathrm{d}v_*^{par} \int_{|v_\perp|\le m_1} dv_*^\perp \\ &\le \mathfrak{I}_0^S + 2C_k D m_1^4 \alpha_{0,1} t e^{-\nu_0 t} \sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right],\end{aligned}$$

where we have used the change of variables $v_*^{par} = [n(x_1)\cdot v_*]n(x_1)$ and $v_*^{\perp} = v_* - v_*^{par}$ together with the definition of k_{m_1} given by (3.3.5) to obtain the second line. We remark now that

$$|\mathcal{V}_{x_1} v_*| = |v_*|, \quad \text{and} \quad |n(x_1)\cdot \mathcal{V}_{x_1} v_*| = |n(x_1)\cdot v_*|,$$

and we recall that [108, Lemma 17] gives that the singular set S_{x_1} has Lebesgue measure zero. Therefore we deduce that we can integrate the expression of $\mathfrak{I}_0^S$ on the set

$$\{|\mathcal{V}_{x_1} v_*| \le m_1, |n(x_1)\cdot \mathcal{V}_{x_1} v_*| \ge \alpha_{0,1}\} \setminus S_{x_1}.$$

We can then apply Lemma 3.4 and that there is $\varepsilon_2^1 = \varepsilon_R(\alpha_0, M, T)$ such that for every $\varepsilon \in (0, \varepsilon_2^1)$ there is only one bounce against the backwards trajectory, hence applying the Duhamel formula as before we obtain that

$$f(s_1, x_1, \mathcal{V}_{x_1} v_*) = S_{\mathscr{T}}(s_1) f_0(x_1, \mathcal{V}_{x_1} v_*) + S_{\mathscr{T}} * Kf(s_1, x_1, \mathcal{V}_{x_1} v_*) + S_{\mathscr{T}} * G(s_1, x_1, \mathcal{V}_{x_1} v_*). \tag{3.3.15}$$

Putting together the expression for $\mathfrak{I}_0^S$ and (3.3.15) we have that

$$\mathfrak{I}_0^S \le \mathfrak{I}_1^S + \mathfrak{I}_2^S + \mathfrak{I}_3^S,$$

where

$$\begin{aligned}
\mathfrak{I}_1^S &:= \int_\sigma^t \int_{\mathbb{R}^3} k_{1,m_1}(v, v_*) e^{-\nu_0(t-s_1)} \omega_1(v_*) |S_{\mathscr{T}}(s_1) f_0(x_1, \mathcal{V}_{x_1} v_*)| \mathrm{d}v_* \mathrm{d}s, \\
\mathfrak{I}_2^S &:= \int_\sigma^t \int_{\mathbb{R}^3} k_{1,m_1}(v, v_*) e^{-\nu_0(t-s_1)} \omega_1(v_*) |S_{\mathscr{T}} * Kf(s_1, x_1, \mathcal{V}_{x_1} v_*)| \mathrm{d}v_* \mathrm{d}s, \\
\mathfrak{I}_3^S &:= \int_\sigma^t \int_{\mathbb{R}^3} k_{1,m_1}(v, v_*) e^{-\nu_0(t-s_1)} \omega_1(v_*) |S_{\mathscr{T}} * G(s_1, x_1, \mathcal{V}_{x_1} v_*)| \mathrm{d}v_* \mathrm{d}s,
\end{aligned}$$

and we proceed to control each of these terms separately. First we remark that $\omega_1(\mathcal{V}_{x_1} v) = \omega_1(v)$, thus we deduce that

$$\mathfrak{I}_1^S \le C_k D m_1^4 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and using (K1)-(K2) we also deduce that for every $\nu_2 \in (0, \nu_0)$ there holds

$$\begin{aligned}
\mathfrak{I}_3^S &\le \frac{\nu_1}{\nu_0 - \nu_2} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \int_\sigma^t \int_{\mathbb{R}^3} e^{-\nu_0(t-s_1)} e^{\nu_2 s_1} k_{1,m_1}(v, v_*) \mathrm{d}v_* \mathrm{d}s \\
&\le C_k D m_1^4 \frac{\nu_1 t}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].
\end{aligned}$$

Furthermore, to control $\mathfrak{I}_2^S$ we use (K1) so that for every $N_3 > 0$ there is $m_3(N_3) > 0$ satisfying

$$\mathfrak{I}_2^S \le C_k D m_1^4 \frac{t}{N_3} e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + \mathfrak{I}_{2,0}^S,$$

where we have defined

$$\mathfrak{I}_{2,0}^S = \int_\sigma^t \int_{\mathbb{R}^3} k_{0,m_1}(v, v_*) e^{-\nu_0(t-s_1)}\, S_{\mathscr{T}} * K_{m_3} \left(\omega_1(v_*) |f(s_1, x_1, \mathcal{V}_{x_1} v_*)| \right) \mathrm{d}v_* \mathrm{d}s.$$

To control then $\mathfrak{I}_{2,0}^S$ we first use the expression of k_{m_1} given by (3.3.5) and we have that

$$\mathfrak{I}_{2,0}^S \le C_k^2 m_1 m_3 \int_0^t \int\limits_{|v_*|\le m_1} \int_0^{s_1} e^{-\nu_0(t-r)} \int\limits_{|u_*|\le m_3} \omega_1(u_*) |f(x_1 - (\mathcal{V}_{x_1} v_*)(s_1 - r)), u_*)| \mathrm{d}u_* \mathrm{d}r \mathrm{d}v_* \mathrm{d}s.$$

We then have the following lemma, whose proof will be provided in Appendix 3.9.

Lemma 3.12. *Assume $|v_*|\leq m_1$ for some $m_1>0$, and we define $\mathcal{X}(v_*) := x_1 - \mathcal{V}_{x_1} v_*(s_1 - r)$. There holds*

$$\det \nabla_{v_*}(\mathcal{X}(v_*)) = (s-r)^3 + \mathscr{O}(\varepsilon).$$

We then deduce that Lemma 3.12 together with Lemma 3.8, we deduce that for every $\alpha_2>0$ there is $\varepsilon_2^2 = \varepsilon_U(\alpha_2)>0$ such that for every $\varepsilon\in(0,\ \min(\varepsilon_2^1,\varepsilon_2^2))$ there holds

$$\begin{aligned}\mathfrak{I}^S_{2,0} \leq C_k^2\, D^2\, m_1^4\, m_3^4\, \alpha_2\, t\, e^{-\nu_0 t} \sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right]\\ + 2^{1/2}\omega_1(m_3)\, C_k^2\, D m_1^{1+3/2}\, m_3^{1+3/2}\, \alpha_2^{-3/2}\, t^2\, e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right].\end{aligned}$$

We conclude this step by putting together the estimates to control the specular reflection and we obtain that

$$\begin{aligned}\mathfrak{I}^S \ \leq\ & \left(2C_kDm_1^4\alpha_{0,1} + C_kDm_1^4\frac{t}{N_3} + C_k^2D^2m_1^4m_3^4\alpha_2 t\right) e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right]\\ &+C_kDm_1^4te^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_kDm_1^4\frac{\nu_1 t}{\nu_0-\nu_2}e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right]\\ &+2^{1/2}\omega_1(m_3)C_k^2m_1^{1+3/2}m_3^{1+3/2}\alpha_2^{-3/2}t^2e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right],\end{aligned}$$

for every $\nu_2\in(0,\nu_0)$.

Step 3. (Control of $\mathfrak{I}^D$) From the very definition of $\mathfrak{I}^D$ we first observe the following elementary inequality

$$\mathfrak{I}^D \leq \int_\sigma^t\int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)e^{-\nu_0(t-s_1)}\int_{\mathbb{R}^3}|f(s_1,x_1,u_*)|\,(n(x_1)\cdot u_*)_+\,\mathrm{d}u_*\mathrm{d}v_*\mathrm{d}s,$$

where we have used the fact that $\mathscr{M}\omega_1\leq 1$. We then apply (K4) so that for every $\lambda_1>0$ there are $M_1,\eta>0$ such that

$$\begin{aligned}\mathfrak{I}^D \ \leq\ & \int_\sigma^t\int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)e^{-\nu_0(t-s_1)}\\ &\times\left[\lambda_1\|f_{s_1}\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \int_{|u_*|\leq M_1,\,|n_{x_1}\cdot u|>\eta}|f(s_1,x_1,u_*)|\,(n_{x_1}\cdot u_*)_+\right]\mathrm{d}u_*\mathrm{d}v_*\mathrm{d}s\\ \leq\ & \lambda_1 C_kDm_1^4te^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + \mathfrak{I}^D_0\end{aligned}$$

where we have defined

$$\mathfrak{I}^D_0 := \int_\sigma^t\int_{\mathbb{R}^3} k_{m_1}(v,v_*)e^{-\nu_0(t-s_1)}\int_{|u_*|\leq M_1,\,|n_{x_1}\cdot u|>\eta}|f(s_1,x_1,u_*)|\,(n_{x_1}\cdot u_*)_+\,\mathrm{d}u_*\mathrm{d}v_*\mathrm{d}s.$$

Arguing as during the first Duhamel decomposition of this proof we deduce that we may rewrite the previous integral as being integrated over the set

$$\{|u_*|\leq M_1, |n_{x_1}\cdot u_*|\geq\eta\}\setminus S_{x_1},$$

thus we may apply Lemma 3.4 and we deduce that there is $\varepsilon_2^3 = \varepsilon_R(\eta, M_1, T)$ such that for every $\varepsilon\in(0,\min(\varepsilon_2^1,\varepsilon_2^2,\varepsilon_2^3))$ there is only one bounce against the backwards trajectory. Hence applying Duhamel's principle as in the Step 2 we get that

$$f(s_1,x_1,u_*) = S_{\mathscr{T}}(s_1)f_0(x_1,u_*) + S_{\mathscr{T}} * Kf(s_1,x_1,u_*) + S_{\mathscr{T}} * G(s_1,x_1,u_*). \tag{3.3.16}$$

We then have that

$$\mathfrak{I}_0^D \leq \mathfrak{I}_1^D + \mathfrak{I}_2^D + \mathfrak{I}_3^D,$$

with

$$\begin{aligned}
\mathfrak{I}_1^D &:= \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{|u_*|\leq M_1,\, |n_{x_1}\cdot u|>\eta} |S_{\mathscr{T}}(s_1) f_0(x_1,u_*)| \, (n_{x_1}\cdot u_*)_+ \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s,\\
\mathfrak{I}_2^D &:= \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{|u_*|\leq M_1,\, |n_{x_1}\cdot u|>\eta} |S_{\mathscr{T}} * Kf(s_1,x_1,u_*)| \, (n_{x_1}\cdot u_*)_+ \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s,\\
\mathfrak{I}_3^D &:= \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{|u_*|\leq M_1,\, |n_{x_1}\cdot u|>\eta} |S_{\mathscr{T}} * G(s_1,x_1,u_*)| \, (n_{x_1}\cdot u_*)_+ \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s,
\end{aligned}$$

and we will control each of these terms separately. On the one hand, we observe that

$$\mathfrak{I}_1^D \leq C_k D^2 m_1^4 M_1^4 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and on the other, we have that for every $\nu_2 \in (0,\nu_0)$ there holds

$$\begin{aligned}
\mathfrak{I}_3^D &\leq \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{|u_*|\leq M_1,\, |n_{x_1}\cdot u|>\eta} \omega_1(u_*) \, |S_{\mathscr{T}} * G(s_1,x_1,u_*)| \, |u_*| \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s\\
&\leq \frac{\nu_1}{\nu_0-\nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) \int_{|u_*|\leq M_1} |u_*| \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s\\
&\leq \frac{\nu_1 C_k D^2 m_1^4 M_1^4 t}{\nu_0-\nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right]
\end{aligned}$$

where we have used that $\omega_1 \geq 1$ on the first line, (K2) to obtain the second and the definition of k_{m_1} given by (3.3.5) to obtain the third inequality. To analyze $\mathfrak{I}_2^D$ we use (K1) so that for every $N_4 > 0$ there is $m_4(N_4) > 0$ for which there holds

$$\begin{aligned}
\mathfrak{I}_2^D &\leq \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{|u_*|\leq M_1,\, |n_{x_1}\cdot u|>\eta} \omega_1(u_*) \, |S_{\mathscr{T}} * Kf(s_1,x_1,u_*)| \, |u_*| \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s\\
&\leq \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{|u_*|\leq M_1} |u_*| \int_0^{s_1} e^{-\nu_0(s_1-r)}\\
&\quad \times \left(\frac{1}{N_4} \|f_{s_1}\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \int_{\mathbb{R}^3} k_{m_4}(u_*,u') \omega_1(u') |f(r, x_1-(s_1-r))u_*, u')| \mathrm{d}u' \right) \mathrm{d}r \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s\\
&\leq \frac{t^2 D^2 m_1^4 M_1^4}{N_4} e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) \int_{\{|u_*|\leq M_1\}} |u_*|\\
&\quad \times \int_0^{s_1} e^{-\nu_0(t-r)} \int_{\mathbb{R}^3} k_{m_4}(u_*,u') \omega_1(u') |f(r, x_1-(s_1-r))u_*, u')| \mathrm{d}u' \mathrm{d}r \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s\\
&\leq \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_4} e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + C_k^2 m_1 m_4 M_1 \, \mathfrak{I}_{2,0}^D
\end{aligned}$$

where we have used the fact that $\omega_1 \geq 1$ in the first inequality, (K3) to obtain the second inequality, (3.3.5) to obtain the third and fourth inequalities and we have defined

$$\mathfrak{I}_{2,0}^D = \int_\sigma^t \int_{|v_*|\leq m_1} \int_{|u_*|\leq M_1} \int_0^{s_1} e^{-\nu_0(t-r)} \int_{|u'|\leq m_4} \omega(u') |f(r, x_1-u_*(s_1-r), u')| \mathrm{d}u' \mathrm{d}r \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s.$$

Furthermore, we remark that

$$|\det \nabla_{u_*}(x_1 - u_*(s_1 - r))| = |s_1 - r|^3,$$

thus using (K3) we have that for every $\alpha_4 > 0$ there holds

$$\begin{aligned}\mathfrak{I}^D_{2,0} \leq D^3\, m_1^3\, M_1^3\, m_4^3\, \alpha_4\, t\, e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] \\ + D^2 m_1^3 M_1^{3/2} m_4^{3/2}\, \omega_1(m_4)\, \alpha_4^{-3/2}\, t^2\, e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right].\end{aligned}$$

Putting together the above estimates controlling the diffusion term $\mathfrak{I}^D$ we have that

$$\begin{aligned}\mathfrak{I}^D \quad \leq \quad & \left(\lambda_1 C_k D m_1^4 t + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_4} + C_k^2 D^3 m_1^4 M_1^4 m_4^4 \alpha_4 t\right) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] \\ & + C_k D^2 m_1^4 M_1^4 t e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \frac{\nu_1 C_k D^2 m_1^4 M_1^4 t}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right] \\ & + C_k^2 D^2 m_1^4 m_4^{1+3/2} M_1^{1+3/2} \omega(m_4) \alpha_4^{-3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right],\end{aligned}$$

for every $\nu_2 \in (0, \nu_0)$.

Step 4. (Choice of the parameters) Putting together the estimates from Steps 1, 2 and 3 we get that for every $\nu_2 \in (0, \nu_0)$ there holds

$$\begin{aligned}\mathfrak{I} \leq C_1 \left(t + t^2\right) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + C_2 t e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ + C_3 t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right] + C_4 t e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right],\end{aligned}$$

where

$$\begin{aligned}C_1 \quad &= \quad \frac{1}{N_1} + 2C_k D m_1^4(\alpha_0 + \alpha_{0,1}) + \frac{C_k D m_1^4}{N_2} + C_k^2 D^2 m_1^4 m_2^4 \alpha_1 + C_k D m_1^4 \frac{1}{N_3} \\ & \quad + C_k^2 D^2 m_1^4 m_3^4 \alpha_2 + \lambda_1 C_k D m_1^4 + \frac{C_k^2 D^2 m_1^4 M_1^4}{N_4} + C_k^2 D^3 m_1^4 M_1^4 m_4^4 \alpha_4, \\ C_2 \quad &= \quad C_k D m_1^4 + C_k D m_1^4 + C_k D^2 m_1^4 M_1^4, \\ C_3 \quad &= \quad 9 C_k^2 D m_1^{1+3/2} m_2^{1+3/2} \omega_1(m_2) \alpha_1^{-3/2} + 2^{3/2} \omega_1(m_3) C_k^2 m_1^{1+3/2} m_3^{1+3/2} \alpha_2^{-3/2} \\ & \quad + C_k^2 D^2 m_1^4 m_4^{1+3/2} M_1^{1+3/2} \omega_1(m_4) \alpha_4^{-3/2}, \\ C_4 \quad &= \quad \frac{\nu_1}{\nu_0 - \nu_2} \left(C_k D m_1^4 + C_k D m_1^4 + C_k D^2 m_1^4 M_1^4\right).\end{aligned}$$

We then set

- $N_1 = 9\lambda^{-1}$ and this fixes m_1,
- $\alpha_0 + \alpha_{0,1} = \lambda(18 C_k D m_1^4)^{-1}$,
- $N_2 = 9 C_k D m_1^4 \lambda^{-1}$ fixing m_2,
- $\alpha_1 = \lambda(9 C_k^2 D^2 m_1^4 m_2^4)^{-1}$,
- $N_3 = 9 C_k D m_1^4 \lambda^{-1}$ and this fixes m_3
- $\alpha_2 = \lambda(9 C_k^2 D^2 m_1^4 m_3^4)^{-1}$,
- $\lambda_1 = \lambda(9 C_k D m_1^4)^{-1}$ fixing M_1,
- $N_4 = 9 C_k^2 D^2 m_1^4 M_1^4 \lambda^{-1}$ and this fixes m_4,
- $\alpha_4 = \lambda(9 C_k^2 D^3 m_1^4 m_4^4 M_1^4)^{-1}$,

This implies that $C_1 \le \lambda$, we then define $C = \max(C_2, C_3, C_4)$ and we observe that, since the constants that define C come from (K1), (K3) and (K4), there is a constant $p > 0$ such that $C \lesssim \lambda^{-p}$ and we conclude by setting $\varepsilon_2 = \min(\varepsilon_2^1, \varepsilon_2^2, \varepsilon_2^3)$. □

3.3.5 Pointwise estimate on the trajectories

We now use the regularization estimate given by Proposition 3.10 to obtain a point-wise control on the solutions of Equation (3.3.2).

Proposition 3.13. *Consider Assumption (RH1) to hold, $\omega_1 \in \mathfrak{W}_1$ a strongly confining weight function, and let f be a solution of Equation* (3.3.2)*. For every $\lambda > 0$ there is a constructive $\varepsilon_3 = \varepsilon_3(\lambda, T) \ge 0$ such that for every $\varepsilon \in (0, \varepsilon_3)$, every $t \in [0, T]$, and every $\nu_2 \in (0, \nu_0)$, we have that for every $(x, v) \in \mathcal{O}^\varepsilon$, $v \notin S_x(v)$ there holds*

$$\omega_1(v)|f(t,x,v)| \le \left[1 - \iota_0 + \lambda(1 + t + t^2)\right] e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + C(1+t)e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}$$
$$+ Ct^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s}\|f_s\|_{\mathcal{H}}] + C(1+t)e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right],$$

for some constant $C = C(\lambda) > 0$. Moreover, there is $p > 0$ such that $C \lesssim \lambda^{-p}$.

Proof. We split the proof into four steps.

Step 1. (Duhamel decomposition) We denote the point (t_1, x_1, v_1) as the first collision through the backwards trajectory as defined in (3.1.25), and we remark that we have $x_1 = x - v(t - t_1)$. We observe that if $t_1 \le 0$, then along the backwards trajectory we reach the initial time $t = 0$, on the contrary if $t_1 > 0$ then along the backward trajectory we reach the boundary at the time $t - t_1$. Using Duhamel's formula we then have that

$$f(t,x,v) = \mathbf{1}_{\{t_1\le 0\}} S_{\mathscr{T}}(t) f_0(x,v) + \int_{\max\{0,t_1\}}^t S_{\mathscr{T}}(t-s) K f(s,x,v) \mathrm{d}s$$
$$+ \int_{\max\{0,t_1\}}^t S_{\mathscr{T}}(t-s) G(s,x,v) \mathrm{d}s + \mathbf{1}_{\{t_1>0\}} S_{\mathscr{T}}(t-t_1) f(t_1,x,v), \quad (3.3.17)$$

where we remark that $S_{\mathscr{T}}(t - t_1)f(t_1, x, v) = e^{-\nu(v)(t-t_1)} f(t_1, x_1, v)$. We multiply both sides of (3.3.17) by ω_1 and we get that

$$\begin{aligned} |\omega_1(v) f(t,x,v)| &\le e^{-\nu(v)t}\omega_1(v)|f_0(x,v)| + \omega_1(v)|S_{\mathscr{T}} *_\sigma Kf(t,x,v)| + \omega_1(v)|S_{\mathscr{T}} *_\sigma G(t,x,v)| \\ &\quad + e^{-\nu(v)(t-t_1)}\omega_1(v)|f(t_1,x_1,v)| \\ &=: \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4, \end{aligned}$$

where we have defined $\sigma = \max(0, t_1)$ and we will control each of these terms separately. We first bound

$$\mathscr{I}_1 = e^{-\nu(v)t}\omega_1(v)|f_0(x,v)| \le e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and by using (K2) we have that for every $\nu_2 \in (0, \nu_0)$ there holds

$$\mathscr{I}_3 = \int_\sigma^t S_{\mathscr{T}}(t-s) G(s,x,v) ds \le \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right].$$

Furthermore, using Proposition 3.10 we have that for every $\lambda_1 > 0$ there is $\varepsilon_3^1 := \varepsilon_2(\lambda_1, T) > 0$ such that for every $\varepsilon \in (0, \varepsilon_3^1)$ and every $\nu_2 \in (0, \nu_0)$ there is a constant $C_1 = C_1(\lambda_1) > 0$

such that

$$\begin{aligned}\mathscr{I}_3 &= \omega_1(v)|S_{\mathscr{T}} *_\sigma Kf(t,x,v)| \\ &\leq \lambda_1(t+t^2)e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + C_1 t e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ &\quad + C_1 t^2 e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right] + C_1(1+t)e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right],\end{aligned}$$

furthermore there are $c_1, p_1 > 0$ such that $C_1 \leq c_1\lambda_1^{-p_1}$. To control now the boundary term $\mathscr{I}_4$ we have that $(x_1, v) \in \Sigma_-^\varepsilon$, thus the Maxwell boundary condition gives that

$$\begin{aligned}f(t_1,x_1,v) &= (1-\iota^\varepsilon(x_1))\mathscr{S}f(t_1,x_1,v) + \iota^\varepsilon(x_1)\mathscr{D}f(t_1,x_1,v) \\ &= (1-\iota^\varepsilon(x_1))f(t_1,x_1,\mathcal{V}_{x_1}v) + \iota^\varepsilon(x_1)\mathscr{M}(v)\widetilde{f}(t_1,x_1),\end{aligned}$$

and we have that

$$\begin{aligned}\mathscr{I}_4 &\leq (1-\iota_0)e^{-\nu(v)(t-t_1)}\omega_1(v)|f(t_1,x_1,\mathcal{V}_{x_1}v)| + e^{-\nu(v)(t-t_1)}\left|\widetilde{f}(t_1,x_1)\right| \\ &\leq (1-\iota_0)\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + \mathscr{I}^D,\end{aligned}$$

where we have used that $\omega_1\mathscr{M} \leq 1$ and we have defined $\mathscr{I}^D := e^{-\nu(v)(t-t_1)}\left|\widetilde{f}(t_1,x_1)\right|$.

Step 2. (Control of the diffusive reflection) To control the diffusive boundary condition we first use (K4) so that for every $\lambda_2 > 0$ there are $M, \eta > 0$ such that

$$\mathscr{I}^D \leq \lambda_2 e^{-\nu_0 t}\sup_{s\in(0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + \mathscr{I}_0^D,$$

where we have used that $\omega_1 \geq 1$ and we have defined

$$\mathscr{I}_0^D := e^{-\nu(v)(t-t_1)}\int_{|u|\leq M,\,|n(x_1)\cdot u|>\eta}\omega_1(u)|f(t_1,x_1,u)|(n(x_1)\cdot u)\mathrm{d}u.$$

Arguing as during the Step 3 of the proof of Proposition 3.10 we have that S_{x_1} has zero Lebesgue measure, thus $\mathscr{I}_0^D$ can be integrated on the set

$$\{|u|\leq M, |n(x_1)\cdot u|\geq\eta\}\setminus S_{x_1}.$$

We apply then Lemma 3.4, thus there is $\varepsilon_3^2 := \varepsilon_R(\eta, M, T)$ such that for every $\varepsilon \in (0, \min(\varepsilon_3^1, \varepsilon_3^2))$ there is no more bounce against the boundary through the backwards trajectory. We use then the Duhamel formula once again and we have that

$$f(t_1,x_1,u) = S_{\mathscr{T}}(t_1)f_0(x_1,u) + S_{\mathscr{T}} * Kf(t_1,x_1,u) + S_{\mathscr{T}} * G(t_1,x_1,u).$$

Hence

$$\mathscr{I}_0^D \leq \lambda_2 e^{-\nu_0 t}\sup_{s\in(0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + \mathscr{I}_1^D + \mathscr{I}_2^D + \mathscr{I}_3^D,$$

where

$$\begin{aligned}\mathscr{I}_1^D &:= e^{-\nu(v)(t-t_1)}\int_{|u|\leq M,\,|n(x_1)\cdot u|>\eta}\omega_1(u)|S_{\mathscr{T}}(t_1)f_0(x_1,u)|(n_x\cdot u)du, \\ \mathscr{I}_2^D &:= e^{-\nu(v)(t-t_1)}\int_{|u|\leq M,\,|n(x_1)\cdot u|>\eta}\omega_1(u)|S_{\mathscr{T}} * Kf(t_1,x_1,u)|(n_{x_1}\cdot u)du, \\ \mathscr{I}_3^D &:= e^{-\nu(v)(t-t_1)}\int_{|u|\leq M,\,|n(x_1)\cdot u|>\eta}\omega_1(u)|S_{\mathscr{T}} * G(t_1,x_1,u)|(n_{x_1}\cdot u)du,\end{aligned}$$

and we will control each of these terms separately. We observe first that

$$\mathscr{I}_1^D \leq DM^4 e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}, \tag{3.3.18}$$

and using (K2) we have that for every $\nu_2 \in (0, \nu_0)$ there holds

$$\begin{aligned}
\mathscr{I}_3^D &= e^{-\nu(v)(t-t_1)} \int_{|u|\leq M,\, |n(x_1)\cdot u|>\eta} \omega_1(u) |S_{\mathscr{T}} * G(t_1, x_1, u)| (n_{x_1}\cdot u) du \\
&\leq \frac{\nu_1}{\nu_1 - \nu_0} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] e^{-\nu_0 (t-t_1)} e^{-\nu_2 t_1} \int_{\{|u_*|\leq M\}} |u| du \\
&\leq DM^4 \frac{\nu_1}{\nu_1 - \nu_0} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].
\end{aligned}$$

To control the remaining term $\mathscr{I}_3^D$ we apply Proposition 3.10 and we have that for every $\lambda_3 > 0$ there is $\varepsilon_3^3 := \varepsilon_3(\lambda_3, T)$ such that for every $\varepsilon \in (0, \min(\varepsilon_3^1, \varepsilon_3^2, \varepsilon_3^3))$ there holds that for every $\nu_2 \in (0, \nu_0)$ there is a constant $C_3(\lambda_3) > 0$ such that

$$\begin{aligned}
\mathscr{I}_2^D &\leq DM^4 \lambda_3 (t + t^2) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + DM^4 C_3 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\
&+ DM^4 C_3 t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s} \|f_s\|_{\mathcal{H}}] + DM^4 C_3 (1+t) e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right],
\end{aligned} \tag{3.3.19}$$

furthermore there are $c_3, p > 0$ such that $C \leq c_3 \lambda_3^{-p}$.

Step 3. (Choice of the parameters) We deduce from Steps 2 and 3 that for every $\varepsilon \in (0, \min(\varepsilon_3^1, \varepsilon_3^2, \varepsilon_3^3))$ there holds

$$\begin{aligned}
\mathscr{I}_4 &\leq (1 - \iota_0 + \lambda_2) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + \mathscr{I}_1^D + \mathscr{I}_2^D + \mathscr{I}_3^D \\
&\leq \left[(1-\iota_0) + \left(\lambda_2 + DM^4 \lambda_3 \right) (1 + t + t^2) \right] e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\
&+ \left(DM^4 + DM^4 C_3 \right) (1+t) e^{-\nu_0 t} \|f_0\|_{L^\infty(\mathcal{O}^\varepsilon)} + DM^4 C_3 t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s} \|f_s\|_{\mathcal{H}}] \\
&+ \left(DM^4 \frac{\nu_1}{\nu_1 - \nu_0} + DM^4 C_3 \right) (1+t) e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].
\end{aligned}$$

Altogether we have that

$$\begin{aligned}
\omega_1(v) |f(t,x,v)| &\leq (1-\iota_0) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + \mathfrak{I}_1 (1+t) e^{\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\
&+ \left(\lambda_1 + \lambda_2 + DM^4 \lambda_3 \right) (1 + t + t^2) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\
&+ \mathfrak{I}_2 t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s} \|f_s\|_{\mathcal{H}}] + \mathfrak{I}_3 (1+t) e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right],
\end{aligned}$$

with the constants

$$\mathfrak{I}_1 = 1 + C_1 + DM^4 + DM^4 C_3, \qquad \mathfrak{I}_2 = C_1 + DM^4 C_3,$$

and

$$\mathfrak{I}_3 = \left(1 + DM^4 \right) \frac{\nu_1}{\nu_0 - \nu_2} + C_1 + DM^4 C_3.$$

We choose then $\lambda_1 = \lambda_2 = \lambda/3$, $\lambda_3 = \lambda/(3DM^4)$, we take $\varepsilon_3 = \min(\varepsilon_3^1, \varepsilon_3^2, \varepsilon_3^3)$, we define $C(\lambda) = \max(\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3)$ thus $C(\lambda) \lesssim \lambda^{-p}$, for some $p > 0$. This concludes the proof. $\square$

3.3.6 Weighted L^∞ control for solutions of Equation (3.3.2)

In this subsection we use the estimate given by Proposition 3.13 to deduce a weighted L^∞ control on the solutions of Equation (3.3.2).

Proposition 3.14. *Consider Assumption (RH1) to hold, $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function, and let f be a solution of Equation* (3.3.2). *There is $\varepsilon_4 = \varepsilon_4(T) > 0$ such that for every $\varepsilon \in (0, \varepsilon_4)$ there holds*

$$\|f_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \le C(1+T)(1+T+T^2)^p e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C(1+T+T^2)^p T^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s}\|f_s\|_{\mathcal{H}}]$$
$$+ C(1+T+T^2)^p(1+T)e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right].$$

for every $t \in [0,T]$, and some universal constants $C, p > 0$.

Proof. We consider $(t,x,v) \in \mathcal{U}_T^\varepsilon$ such that $v \notin S_x$, then Proposition 3.13 implies that for every $\lambda > 0$ there is $\varepsilon_4 := \varepsilon_3(\lambda, T)$ such that for every $\varepsilon \in (0, \varepsilon_4)$ and every $\nu_2 \in (0, \nu_0)$ there holds

$$\omega_1(v)|f(t,x,v)| \le \left[1 - \iota_0 + \lambda(1+t+t^2)\right] e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + c\,(1+t)e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}$$
$$+ c\,t^2 e^{-\nu_0 t}\sup_{s\in[0,t]}[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}] + c\,(1+t)e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right],$$

for every $t \in [0,T]$ and some constant $c > 0$ such that $c \le C_0\lambda^{-p}$ for some constants $C_0, p > 0$.

We recall [108, Lemma 17] implies that S_x has Lebesgue measure zero. Therefore we may take the $L^\infty(\mathbb{R}^d)$ norm first and then the $L^\infty(\Omega^\varepsilon)$ norm on the above estimate and we have that

$$\|f_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \le \left[1 - \iota_0 + \lambda(1+t+t^2)\right] e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + c\,(1+t)e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}$$
$$+ c\,t^2 e^{-\nu_0 t}\sup_{s\in[0,t]}[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}] + c\,(1+t)e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right],$$

for every $t \in [0,T]$. We then choose $\lambda = \iota_0(2(1+T+T^2))^{-1}$, we absorb the small contributions, and we conclude by setting $C = 2C_0(2-\iota_0)^{-1}$, and using the fact that $c \le C_0\lambda^{-p}$. □

3.3.7 Proof of Proposition 3.1

We split the proof into two steps.

Step 1. (Choice of parameters and L^2 estimate) We choose $T > 0$ large enough such that

$$C_1(1+T)(1+T+T^2)^p e^{-T\nu_0/2} \le 1,$$

where $C_1 > 0$ is given by Proposition 3.14, and we set $\varepsilon_1^1 = \min(\varepsilon_4(T), 1/2, \sqrt{\nu_0/\kappa^\star})$, where $\varepsilon_4(T) > 0$ is given by Proposition 3.14 and we recall that $\kappa^\star > 0$ is given by Theorem 3.2.1.

We remark that since $\omega_1 \in \mathfrak{W}_1$, then $\omega_1(v) = e^{\zeta|v|^2}$ with $\zeta \in (1/4, 1/2)$, and a direct computation implies that $L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon) \subset \mathcal{H}$.

We recall the hypocoercivity norm $|||\cdot|||$ given by Theorem 3.2.1 and the equivalency relation (3.2.2). We now denote by $S_{\mathscr{L}}$ the semigroup generated by the solutions of Equation (3.1.13), which is given by Theorem 3.6.8 and Remark 3.9, and using the Duhamel formula we have that

$$f_t = S_{\mathscr{L}}(t) f_0 + \int_0^t S_{\mathscr{L}}(t-s) G_s \, \mathrm{d}s \qquad \forall t \geq 0.$$

Using the fact that $\langle\langle G_t \rangle\rangle_{\mathcal{O}^\varepsilon} = 0$ for every $t \geq 0$, the decay estimate (3.2.3), and the equivalency relation (3.2.2), we then have that

$$|||f_t||| \leq e^{-\kappa^\star \varepsilon^2 t} |||f_0||| + t \int_0^t e^{-\kappa^\star \varepsilon^2 (t-s)} |||G_s||| \, \mathrm{d}s \leq e^{-\kappa^\star \varepsilon^2 t} |||f_0||| + ct \int_0^t e^{-\kappa^\star \varepsilon^2 (t-s)} \|G_s\|_{\mathcal{H}} \, \mathrm{d}s,$$

where the constant $c > 0$ is given by Theorem 3.2.1. Using now the fact that $L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon) \subset \mathcal{H}$, we remark that

$$\|G\|_{\mathcal{H}} \lesssim \left(\int_{\mathcal{O}^\varepsilon} \mathrm{d}x \right)^{1/2} \|G\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \lesssim \varepsilon^{-3/2} \|G\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)}, \tag{3.3.20}$$

thus we deduce that there is a constant $C_2 > 0$ such that

$$|||f_t||| \leq e^{-\kappa^\star \varepsilon^2 t} |||f_0||| + C_2 \, t^2 \, \varepsilon^{-3/2} \, e^{-\kappa^\star \varepsilon^2 t} \sup_{s \in [0,t]} \left[e^{\kappa^\star \varepsilon^2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right], \tag{3.3.21}$$

where $\kappa^\star > 0$ is given by Theorem 3.2.1. Putting together the estimate given by Proposition 3.14 and (3.3.21) we have that for every $\varepsilon \in (0, \varepsilon_1^1)$, every $\nu_2 \in (0, \nu_0)$, and $\kappa_0 \in (0, \kappa^\star)$ there holds

$$\begin{aligned} \|f_t\|_{L^\infty_{\omega_1}(\bar{\mathcal{O}}^\varepsilon)} \leq \; & C_1 (1+T)(1+T+T^2)^p e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_1 T^2 (1+T+T^2)^p e^{-\kappa^\star \varepsilon^2 t} |||f_0||| \\ & + C_1 (1+T+T^2)^p (1+T) e^{-\nu_2 t} \sup_{s \in [0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \\ & + C_1 C_4 \, T^4 (1+T+T^2)^p \, \varepsilon^{-3/2} \, e^{-\kappa_0 \varepsilon^2 t} \sup_{s \in [0,t]} \left[e^{\kappa_0 \varepsilon^2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right], \end{aligned}$$

where we have used the fact that $\nu_0 - \kappa^\star (\varepsilon)^2 \geq 0$ due to our choice of ε_1^1.

We then set $\theta = \min(\nu_0, \kappa^\star)/8$, $\nu_2 = \varepsilon^2 \theta$ and $\kappa_0 = \theta$. We remark that these choices are possible due to the fact that $\varepsilon \leq 1/2$ and $\theta \leq \nu_0/2$, thus $\varepsilon^2 \theta \in (0, \nu_0)$ and to the fact that $\theta \in (0, \kappa^\star)$.

Putting everything together we have that

$$\begin{aligned} \|f_T\|_{L^\infty_{\omega_1}(\bar{\mathcal{O}}^\varepsilon)} \leq \; & e^{-2\theta \varepsilon^2 T} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_T e^{-2\theta \varepsilon^2 T} |||f_0||| \\ & + \varepsilon^{-3/2} C_T e^{-\theta \varepsilon^2 T} \sup_{s \in [0,T]} \left[e^{\theta \varepsilon^2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right], \end{aligned} \tag{3.3.22}$$

and

$$\begin{aligned} \|f_t\|_{L^\infty_{\omega_1}(\bar{\mathcal{O}}^\varepsilon)} \leq \; & C_T e^{-2\theta \varepsilon^2 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_T e^{-2\theta \varepsilon^2 t} |||f_0||| \\ & + \varepsilon^{-3/2} C_T e^{-\theta \varepsilon^2 t} \sup_{s \in [0,t]} \left[e^{\theta \varepsilon^2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right], \end{aligned} \tag{3.3.23}$$

for some constant $C_T > 0$ and where (3.3.23) holds for all $t \in [0, T]$.

Step 2. (Decay estimate) We now set

$$X_t := e^{\theta\varepsilon^2 t}\|f_t\|_{L^\infty_{\omega_1}(\bar{\mathcal{O}}^\varepsilon)}, \quad Y_t := e^{\theta\varepsilon^2 t}\|\!| f_t |\!\|, \quad \text{and} \quad \Phi_{t_0,t_1} := \sup_{s\in[t_0,t_1]}\left[e^{\theta\varepsilon^2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right].$$

Translating (3.3.21) with $t = T$, and (3.3.22) into this new notations we observe that

$$Y_T \le e^{-\theta\varepsilon^2 T}Y_0 + C_2\,\varepsilon^{-3/2}T\,\Phi_{0,T}, \quad \text{and} \quad X_T \le e^{-\theta\varepsilon^2 T}X_0 + C_T e^{-\theta\varepsilon^2 T}Y_0 + \varepsilon^{-3/2}C_T\Phi_{0,T}.$$

We define $\varepsilon_1 = \min(\varepsilon_1^1, (\theta T)^{-1}\log 2)$ so that $e^{\theta\varepsilon^2 T} - 1 \le 1$, and we introduce a constant $\beta > 0$ defined by

$$\beta = \frac{1}{2C_T}\left(e^{\theta\varepsilon^2 T} - 1\right),$$

so that, due to our choice of ε_1, there holds $\beta C_T \le 1/2$ and simultaneously

$$\vartheta := e^{-\theta\varepsilon^2 T}(1 + \beta C_T) = \frac{1}{2}\left(1 + e^{-\theta\varepsilon^2 T}\right) < 1, \quad \forall\varepsilon > 0.$$

We then have that

$$\begin{aligned} Z_T := Y_T + \beta X_T &\le (1+\beta C_T)e^{-\theta\varepsilon^2 T}(Y_0 + \beta X_0) + \varepsilon^{-3/2}(\beta C_T + C_2 T)\Phi_{0,T} \\ &\le \vartheta Z_0 + \varepsilon^{-3/2}\left(\frac{1}{2} + C_2 T\right)\Phi_{0,T}, \end{aligned} \tag{3.3.24}$$

where we have used the fact that $\beta C_T \le 1/2$ due to the choice of β. On the other hand, using (3.3.23) and again our choice of β we have that

$$Z_t \le \frac{3}{2}e^{-\theta\varepsilon^2 t}Z_0 + \varepsilon^{-3/2}\left(\frac{1}{2} + C_2 T\right)\Phi_{0,t} \qquad \forall t \in [0,T]. \tag{3.3.25}$$

Then for any $\bar t \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that $\bar t \in [nT, (n+1)T)$ and iterating first (3.3.24) we have

$$\begin{aligned} Z_{\bar t} &\le \vartheta^n Z_{\bar t - nT} + \varepsilon^{-3/2}\left(\frac{1}{2} + C_2 T\right)\Phi_{\bar t - nT, \bar t}\left(\sum_{k=0}^{n-1}\vartheta^k\right) \\ &\le \frac{3}{2}\vartheta^n Z_0 + \varepsilon^{-3/2}\left(\frac{1}{2} + C_2 T\right)\left(\sum_{k=0}^{n-1}\vartheta^k\right)\Phi_{0,\bar t} \end{aligned}$$

where we have used (3.3.25) on the second line. Using the previous estimate we deduce that

$$\beta\, e^{\theta\varepsilon^2 t}\|f_t\|_{L^\infty_{\omega_1}(\bar{\mathcal{O}}^\varepsilon)} \le Z_t \le \frac{3}{2}Z_0 + \tilde C \sup_{s\in[0,t]}\left[e^{\theta\varepsilon^2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right] \quad \forall t > 0,$$

where

$$\tilde C = \varepsilon^{-3/2}\left(\frac{1}{2} + C_2 T\right)\left(\sum_{k=0}^{\infty}\vartheta^k\right) = \varepsilon^{-3/2}\frac{1}{1-\vartheta}\left(\frac{1}{2} + C_2 T\right) = \frac{\varepsilon^{-3/2}}{1 - e^{-\theta\varepsilon^2 T}}\left(\frac{1}{2} + C_2 T\right).$$

Finally, we observe that (3.2.2) and (3.3.20) imply together that

$$Z_0 \lesssim \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \|f_0\|_{\mathcal{H}} \lesssim \varepsilon^{-3/2}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

Altogether, using the definition of β and ϑ we have that

$$e^{\theta\varepsilon^2 t}\|f_t\|_{L^\infty_{\omega_1}(\bar{\mathcal{O}}^\varepsilon)} \le \frac{\varepsilon^{-3/2}C'}{e^{\theta\varepsilon^2 T} - 1}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \frac{\varepsilon^{-3/2}C'}{\left(e^{\theta\varepsilon^2 T} - 1\right)\left(1 - e^{-\theta\varepsilon^2 T}\right)}\sup_{s\in[0,t]}\left[e^{\theta\varepsilon^2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right],$$

for some constant $C' > 0$ independent of ε. So we conclude by taking

$$C(\varepsilon) = \frac{\varepsilon^{-3/2}C'}{\left(e^{\theta\varepsilon^2 T} - 1\right)\left(1 - e^{-\theta\varepsilon^2 T}\right)},$$

and remarking that $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$. □

3.4 Stretching method for strongly confining weights in cylindrical domains

Throughout this section we assume Assumption (RH2) to hold and we will prove the same result as in Proposition 3.1 within this framework.

Proposition 3.1. *Consider Assumption (RH2) to hold, $\omega_1 \in \mathfrak{W}_1$ a strongly confining weight function, and let f be a solution of Equation* (3.1.21)*. There are constructive constants $\varepsilon_5, \theta > 0$ such that for every $\varepsilon \in (0, \varepsilon_5)$ there holds*

$$\|f_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq C\, e^{-\theta\varepsilon^2 t} \left(\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \sup_{s\in[0,t]} \left[e^{\theta\varepsilon^2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \right) \qquad \forall t \geq 0, \quad (3.4.1)$$

for some constant $C = C(\varepsilon) > 0$ such that $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

We remark that the ideas developed during Section 3.3 do not immediately apply to cylindrical domains due to the presence of irregularities at the boundary. However, we can still provide a weighted L^∞ control for the solutions of Equation (3.1.21) by using a more delicate control on the trajectories of particles within the cylinder.

To do this we will exploit the geometrical properties of the domain described in Assumption (RH2), which will be expressed through a series of preliminary lemmas in the next subsection. Furthermore, we remark that the rest of this section is structured as Section 3.3.

3.4.1 Preliminary lemmas

We present now the coordinates of multiple consecutive collisions along the boundary set presenting specular reflections, Λ_3^ε. To be more precise, we consider $(t_0, x_0, v_0) \in \mathcal{U}^\varepsilon$ and we define the sequence (t_N, x_N, v_N), as long as it makes sense, of specular collisions following the backwards trajectories as follows

$$\begin{array}{rcl} t_b(x_{N-1}, V_{N-1}) &=& \inf\{s > 0;\ X(-s, 0, x_{N-1}, v_{N-1}) \notin \Omega^\varepsilon\}, \\ t_N(t_{N-1}, x_{N-1}, v_{N-1}) &=& t_{N-1} - t_b(x_{N-1}, v_{N-1}), \\ x_N(t_{N-1}, x_{N-1}, v_{N-1}) &=& x_{N-1} - v_{N-1}(t_{N-1} - t_N), \\ v_N(_{N-1}, x_{N-1}, v_{N-1}) &=& \mathcal{V}_{x_N}(v_{N-1}). \end{array} \qquad (3.4.2)$$

It is worth remarking that, when we say that we define (t_N, x_N, v_N) as long as it makes sense, we mean that for every $j \in [\![1, N]\!]$, there holds $x_j \in \Lambda_3^\varepsilon$.

In the sequel, we will be denoting the points $z \in \mathbb{R}^3$ as $z = (z^1, z^2, z^3)$ and we define $\widehat{z} = (z^2, z^3)$. We then remark that, in particular, if $x \in \Omega^\varepsilon$ then $\widehat{x} \in \Omega_0^\varepsilon$, the 2−dimensional ball of radius $\varepsilon^{-1}\mathfrak{R}$, and $x^1 \in (-\varepsilon^{-1}L, \varepsilon^{-1}L)$. Moreover, we remark that if $x \in \Lambda_3^\varepsilon$ then

$$n_x = \frac{1}{|\widehat{x}|}(0, x^2, x^3) = \varepsilon\mathfrak{R}^{-1}(0, x^2, x^3).$$

Similarly, we explicitly have that the normal on the surface $\partial\Omega_0^\varepsilon \subset \mathbb{R}^2$, at the point $\widehat{x} \in \partial\Omega_0^\varepsilon$ is given by $\widehat{n}_{\widehat{x}} := \varepsilon\mathfrak{R}^{-1}\widehat{x}$

We also define the set of non-smooth points of the boundary

$$\begin{aligned} \mathfrak{S}^\varepsilon &:= (\overline{\Lambda_1^\varepsilon} \cap \overline{\Lambda_3^\varepsilon}) \cup (\overline{\Lambda_2^\varepsilon} \cap \overline{\Lambda_3^\varepsilon}) \\ &= \left\{ x = (x^1, x^2, x^3) \in \mathbb{R}^3,\ x^1 = \pm L^\varepsilon, \text{ and } (x^2)^2 + (x^3)^2 = \varepsilon^{-2}\mathfrak{R}^2 \right\}, \end{aligned} \qquad (3.4.3)$$

and the set of singular velocities across multiple specular reflections through the backwards trajectories

$$\begin{aligned} W_{t,x} := \{v \in \mathbb{R}^3;\ &\text{such that starting at } (t,x,v) \text{ there is } N \in \mathbb{N} \text{ with } ((t_j,x_j,v_j))_{j=1}^N \\ &\text{as defined in (3.4.2), with } t_j > 0 \text{ and } x_j \in \Lambda_3^\varepsilon \text{ for every } j \in [\![1,N]\!], \\ &\text{and such that if } (t_{N+1},x_{N+1},v_{N+1}) \text{ is the first collision against} \\ &\text{the backwards trajectory starting at } (t_N,x_N,v_N), \\ &\text{given by (3.1.25) is such that } t_{N+1} > 0 \text{ and } x_{N+1} \in \mathfrak{S}\}. \end{aligned} \quad (3.4.4)$$

In words, for any fixed t,x we have that $v \in W_{t,x}$, when starting from the point (t,x,v) there are a certain number of consecutive purely specular collisions against Λ_3^ε followed by a collision against the singular set $\mathfrak{S}$.

We now prove a first result on the number of possible consecutive collisions through the specular reflection on Λ_3^ε.

Lemma 3.2. *For any fixed $M,T>0$ and any point $(t,x,v) \in [0,T]\times(\Omega^\varepsilon \cup \Lambda_3^\varepsilon)\times\{|v| \le M\}$ there is a constant $\mathfrak{N} = \mathfrak{N}(T,x,M)$ such that there are not more than $\mathfrak{N}$ consecutive specular collisions against the boundary through the backwards trajectory starting from (t,x,v), given by* (3.4.2).

This result is an inmediate consequence of the following geometrical lemma.

Lemma 3.3. *Consider a point $(t_0,x_0,v_0) \in (0,T)\times\Lambda_3^\varepsilon\times\{|v| \le M\}$ such that $n(x_0)\cdot v = \eta$ for some constants $\eta, M > 0$. Assume the two points of collision against the boundary through the backwards trajectories are $(t_1,x_1,v_1) \in (0,T)\times\Lambda_3^\varepsilon\times\mathbb{R}^3$ and $(t_2,x_2,v_2) \in (0,T)\times\Lambda_3^\varepsilon\times\mathbb{R}^3$. Then $t-t_1 = t_1-t_2 \ge \mathcal{T}$ for some $\mathcal{T} = \mathcal{T}(M,\eta) > 0$, $|v_0|=|v_1|=|v_2|$, and*

$$n_{x_1}\cdot v_1 = n_{x_2}\cdot v_2 = n_{x_0}\cdot v_0 = \eta. \quad (3.4.5)$$

Proof. We first observe that we elementary have that

$$|v_3|=|\mathcal{V}_{x_3}v_2|=|v_2|=|\mathcal{V}_{x_2}v_1|=|\mathcal{V}_{x_1}v_0|=|v_0|.$$

Moreover, there also holds

$$|\widehat{v}_3|=|\widehat{\mathcal{V}}_{\widehat{x}_3}\widehat{v}_2|=|\widehat{v}_2|=|\widehat{\mathcal{V}}_{\widehat{x}_2}\widehat{v}_1|=|\widehat{\mathcal{V}}_{\widehat{x}_1}\widehat{v}_0|=|\widehat{v}_0|, \quad (3.4.6)$$

where we have defined $\widehat{\mathcal{V}}_{\widehat{z}}\widehat{w} = \widehat{w} - (\widehat{n}_{\widehat{z}}\cdot\widehat{w})\widehat{w}$, for any $z \in \partial\Omega_0^\varepsilon$ and $w \in \mathbb{R}^3$.

We now denote A_j the line perpendicular to $\widehat{n}_{\widehat{x}_j}$ passing through the point $\widehat{x}_j$. We also denote $\alpha := \angle(\widehat{x}_0, v_0) \in (0,\pi/2)$ and we remark that $\alpha \ne \pi/2$ due to the fact that there are collisions through the backwards trajectories, and $\alpha = \pi/2$ would imply that $\widehat{v}_0$ is a tangent velocity to the circumference in $\widehat{x}_0$.

We then observe that, from the tangent theorem for the circle, the arc $\widehat{\widehat{x}_0\widehat{x}_1} = \pi_2 - \alpha =: \beta$. Moreover, and using the same argument we have that $\angle(A_1, v_0) = \beta$, thus $\angle(v_0,x_1) = \alpha + \pi/2$. We now bserve that from the very definition of the specular boundary conditions there holds $x_1\cdot v_0 = -x_1\cdot v_1$. This together with (3.4.6) implies that $\pi - \angle(v_1,x_1) = \angle(v_0,x_1)$. We deduce then that $\angle(v_1,x_1) = \alpha$, and in particular that $\widehat{\widehat{x}_1\widehat{x}_2} = \beta$. The fact that $\angle(v_0,x_0) = \angle(v_1,x_1)$ and (3.4.6) imply (3.4.5). Moreover, the fact that $\widehat{\widehat{x}_0\widehat{x}_1} = \widehat{\widehat{x}_1\widehat{x}_2}$ imply that the $\overline{\widehat{x}_0\widehat{x}_1} = \overline{\widehat{x}_1\widehat{x}_2}$. This information together with (3.4.6) and the very definition of the trajectories imply that $t-t_1 = t_1-t_2$. This concludes the proof. □

We present now some results regarding the singular sets in the framework of cylindrical domains.

Lemma 3.4. *Let $x \in \Omega^\varepsilon \cup \Lambda_3^\varepsilon$, the set S_x, that we recall was defined in* (3.3.3)*, has Lebesgue measure zero.*

Proof. We consider first $x \in \Omega^\varepsilon$ and assume there is $v \in \mathbb{R}^d$ and $t > 0$ such that $x_1 = x - tv \in \Lambda_3^\varepsilon$ with $n_{x_1} \cdot v = 0$. However, we recall that $n_{x_1} = (0, x_1^2, x_1^3)/|\widehat{x}_1|$, therefore $n_{x_1} \cdot v = x_1^2 v^2 + x_1^3 v^3$ has to be equal to zero. This then implies that $\widehat{n}_{\widehat{x}} \perp \widehat{v}$, and if we now try to go back to the initial position we observe that $\widehat{x}_1 + t\widehat{v} \notin \Omega_0^\varepsilon$ which is in contradiction with the fact that $x \in \Omega^\varepsilon$. We thus deduce that $S_x = \emptyset$ for any $x \in \Omega^\varepsilon$.

Taking then $x \in \Lambda_3^\varepsilon$, we immediately observe that

$$S_x = \{v \in \mathbb{R}^d, \, v \cdot n_x = 0\} = \mathbb{R} \times \{(a, b) \in \mathbb{R}^2, \, ax^2 + bx^3 = 0\},$$

which has codimension 1, thus Lebesgue measure zero. □

Lemma 3.5. *Consider some arbitrary $M, T > 0$ and a point $(t, x) \in [0, T] \times (\Omega \cup \Lambda_3^\varepsilon)$, the set $W_{t,x} \cap \{v \in \mathbb{R}^3, \, |v| \leq M\} \setminus S_x$ has Lebesgue measure zero in $\mathbb{R}^3$.*

Proof. This result is classical in the study of the ergodic properties of *dynamical billiards*, i.e particles moving by the dynamics (3.1.24) and colliding against the boundary following specular (also called elastic in the framework of billiards) reflections. We refer the interest reader towards [37, 59, 60, 159, 161] and the references therein for more information in this kind of systems and the current known results. However, even if it is a well accepted result within this field, we weren't able to find explicit references regarding the measure preservation of this flow, except in some two dimensional cases (see for instance in the previous references). Therefore, for the sake of completeness, we provide now a sketch of a proof.

We define the Hamiltonian energy function $H(v) := |v|^2/2$ and the flow Φ^t generated by the system

$$\dot{x} = -\frac{\partial H}{\partial v}(v) = -v, \qquad \dot{v} = \frac{\partial H}{\partial x}(v) = 0, \tag{3.4.7}$$

complemented with the specular boundary reflection, that we recall is given by

$$(x, v) = (x, \mathcal{V}_x v) \quad \text{for every } x \in \Lambda_3^\varepsilon, \text{ and } v \in \mathbb{R}^3 \text{ such that } n(x) \cdot v > 0. \tag{3.4.8}$$

We will then prove that the flow Φ^t defined this way is measure preserving, that is for every Borel set $A \subset (\Omega^\varepsilon \cup \bar{\Lambda}_3^\varepsilon) \times \mathbb{R}^3$ there holds

$$\mu_L(\Phi^t(A)) = \mu_L(A) = \mu_L((\Phi^t)^{-1}(A)), \tag{3.4.9}$$

for every $t \geq 0$, and where μ_L is the Lebesgue measure in phase space $\mathrm{d}x\mathrm{d}v$.

To do this, we remark first that during free flight, i.e between collisions with the boundary, the flow Φ^t corresponds to a Hamiltonian flow, therefore Liouville's theorem [11, Part II, Chapter 3, Theorem 16.1] implies that this map is measure preserving with respect to μ_L. We prove then that the specular reflection map given by (3.4.8) is a measure preserving map.

To prove this we observe that

$$\text{a map } \psi \text{ is measure preserving iff } \int_{\Lambda_3^\varepsilon \times \mathbb{R}^3} F(x, v) \mathrm{d}x\mathrm{d}v = \int_{\Lambda_3^\varepsilon \times \mathbb{R}^3} F \circ \psi(x, v) \mathrm{d}x\mathrm{d}v, \tag{3.4.10}$$

for every integrable function $F \in L^1(\Lambda_3^\varepsilon \times \mathbb{R}^3)$. Indeed, on the one hand, the left direction of the equivalency is evident by taking indicator functions. On the other hand, let ψ be a measure preserving map in the sense of (3.4.9), when $F := \mathbf{1}_A$, for a Borel set $A \in \bar{\Lambda}_3^\varepsilon \times \mathbb{R}^3$, we have that

$$\int_{\Lambda_3^\varepsilon \times \mathbb{R}^3} |F(x,v)| \mathrm{d}x\mathrm{d}v = \mu_L(A) = \mu_L(\psi^{-1}(A)) = \int_{\Lambda_3^\varepsilon \times \mathbb{R}^3} |F \circ \psi(x,v)| \mathrm{d}x\mathrm{d}v,$$

and we obtain the validity of (3.4.10) by using standard measure theory arguments.

In particular, we remark that if $\psi(x,v) = \mathcal{V}_x v$, then (3.4.10) holds by using the change of variables $v \mapsto \mathcal{V}_x v$.

We have then proved that the flow Φ^t is measure preserving and the conclusion follows by using this and the fact that $\mu_L(\mathfrak{S}) = 0$. □

We provide now a control on the angle of reflections against the normal on Λ_3^ε after a diffusive collision. To do this, we introduce the vector field $\mathfrak{n} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $\mathfrak{n}(x) = (0, x^2, x^3)/|\widehat{x}|$, and we remark that $\mathfrak{n}(x) = n_x$ for every $x \in \Lambda_3^\varepsilon$.

Lemma 3.6. *Assume $t > 0$, $x \in \Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon$, $v \in \mathbb{R}^3$, such that $(x,v) \in \Sigma_+^\varepsilon$, and $|n(x) \cdot v| > \eta_0$ for some $\eta_0 > 0$. Consider (t_1, x_1, v_1) the point of collision against the boundary through the backwards trajectory starting at (t,x,v) given by (3.1.25), and assume $x_1 \in \Lambda_3^\varepsilon$.*

For every $\eta > 0$ there is a constructive constant $A > 0$ such that if $|\mathfrak{n}(x) \cdot v| > \eta$ then $|n(x_1) \cdot v_1| \geq A$, uniformly in x.

Proof. This result is a consequence of elementary geometrical properties of the circumference, therefore we will only sketch it.

We place our framework in $\bar{\Omega}_0^\varepsilon$, the circle of radious $\varepsilon^{-1}\mathfrak{R}$, and we assume without loss of generality that $\widehat{x} = (0, x_0)$ for some $x_0 > 0$. Moreover, we assume that $\widehat{v} = (v^2, v^3)$ with $v^2, v^3 \leq 0$, otherwise the proof follows similarly.

In particular, this implies that $\widehat{x}_1 = (x_1^2, x_1^3)$ with $x_1^2, x_1^3 \geq 0$. We denote r_0 the line parallel to the x-axis passing through $\widehat{x}$ and r_1 the line passing through $\widehat{x}$ and with $\widehat{v}$ as the vector indicating its direction.

We now remark that the condition $|\mathfrak{n}(x) \cdot v| > \eta$ implies that there is $\gamma \in (0, \pi/2)$, which is the angle formed by the intersection between the lines r_0 and r_1, thus also the angle of intersection between r_1 and the x-axis.

We denote $A_1 = (\varepsilon^{-1}\mathfrak{R}, 0)$, A_2 the intersection of r_0 with $\partial\Omega_0^\varepsilon$ in the first quadrant, A_3 the intersection of r_1 with $\partial\Omega_0^\varepsilon$ in the first quadrant, A_4 the intersection of r_0 with $\partial\Omega_0^\varepsilon$ in the second quadrant, A_5 the second intersection of r_1 with $\partial\Omega_0^\varepsilon$, $A_6 = (-\varepsilon^{-1}\mathfrak{R}, 0)$, and A_7 the intersection of r_1 with the x-axis.

Furthermore, for any two point P_1, D_2 in $\partial\Omega_0^\varepsilon$, we denote $\widehat{P_1P_2}$ as the arc in counter clockwise sense with endpoints P_1 and P_2.

We now remark that, to conclude, we want to prove that there is a lower bound (only depending on γ) for the arc $\beta := \widehat{A_3A_5}$.

We then have that, on the one hand, if $|A_7| < \varepsilon^{-1}\mathfrak{R}$ then we obviously have that $\eta > \pi/2$. On the other hand, we denote $\beta_1 = \widehat{A_1A_3}$, $\beta_2 = \widehat{A_5A_6}$, and if γ is such that $|A_7| \geq \varepsilon^{-1}\mathfrak{R}$, then there holds

$$\beta + \beta_1 + \beta_2 = \pi, \qquad \text{and} \qquad \gamma = \frac{\beta_1 - \beta_2}{2}.$$

This implies that $\beta = \pi + 2\gamma - 2\beta_1$, and we conclude the proof by remarking that $\beta_1 < \pi/2$, thus $\beta \geq 2\gamma$. □

Finally, we present the following stretching lemmas for the cylindrical framework.

Lemma 3.7. *Assume (RH2) to hold, and take* $(t,x,v) \in [0,T] \times \Lambda_3^\varepsilon \times \{|v| \leq M, |n(x_1)\cdot v| > \eta\}$ *for some* $M, T, \eta > 0$*, and where* $(x,v) \in \Sigma_+^\varepsilon$*. There exists* $\varepsilon_S = \varepsilon_S(\eta, M, T) := 2\mathfrak{R} M^{-2} T^{-1} \eta$ *such that for every* $\varepsilon \in (0, \varepsilon_S)$ *there is no more specular reflection along the backwards trajectory.*

Proof. We consider the point (t_1, x_1, v_1) as the point of collision against the boundary through the backwards trajectory starting from (t,x,v) and which is given by (3.1.25). If we assume that $x \in \Lambda_3^\varepsilon$, that means that $\widehat{x}_1 \in \partial\Omega_0^\varepsilon$ and $x_1^1 \in (-\varepsilon^{-1}L, \varepsilon^{-1}L)$. In particular, we remark that this means that the time of the trajectory from x towards x_1, $t - t_1$, is equal to the time of the backwards trajectory from $\widehat{x}$ towards $\widehat{x}_1$ with velocity $\widehat{v}$, within Ω_0^ε.

We call now $d := |\widehat{x} - \widehat{x}_1|$, the length of the chord formed by the points $\widehat{x}$ and $\widehat{x}_1$ within $\bar{\Omega}_0^\varepsilon$. We call then $\mathscr{O}$ the center point of the circumference $\partial\Omega_0^\varepsilon$, and we observe that the angle

$$\angle(\widehat{x}\mathscr{O}\widehat{x}_1) = \pi - 2\angle(\widehat{n}_{\widehat{x}}, \widehat{v}),$$

where we remark that the first one is interpreted as the angle between the segments $\overline{\widehat{x}\mathscr{O}}$ and $\overline{\mathscr{O}\widehat{x}_1}$, and the second is interpreted as the angle between the two vectors $\widehat{n}_{\widehat{x}}$ and $\widehat{v}$.

Using then the law of cosines we elementary deduce that

$$\begin{aligned} d^2 &= 2\mathfrak{R}^2\varepsilon^{-2}\left(1 - \cos\left(\angle(\widehat{x}, \mathscr{O}, \widehat{x}_1)\right)\right) = 2\mathfrak{R}^2\varepsilon^{-2}\left(1 - \cos\left(\pi - 2\angle\left(\widehat{n}_{\widehat{x}}, \widehat{v}\right)\right)\right) \\ &= 2\mathfrak{R}^2\varepsilon^{-2}\left(1 + \cos\left(2\angle\left(\widehat{n}_{\widehat{x}}, \widehat{v}\right)\right)\right) = 4\mathfrak{R}^2\varepsilon^{-2}\left|\cos\left(\angle\left(\widehat{n}_{\widehat{x}}, \widehat{v}\right)\right)\right|^2 \geq 4\mathfrak{R}^2\varepsilon^{-2}\frac{\eta^2}{M^2} \end{aligned}$$

Therefore

$$t - t_1 = \frac{|\widehat{x} - \widehat{x}_1|}{|v|} \geq 2\mathfrak{R}\varepsilon^{-1}\frac{\eta}{M^2},$$

and we conclude by remarking that if $\varepsilon \in (0, \varepsilon_S)$, then $t - t_1 > T$ which is a contradiction with the fact that, from its very definition $0 \geq t - t_1 \leq T$. □

Lemma 3.8. *Assume (RH2) to hold, and* $(t,x,v) \in [0,T] \times (\Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon) \times \{|v| \leq M, |n(x)\cdot v| > \eta\}$ *for some* $M, T, \eta > 0$*, and where* $(x,v) \in \Sigma_+^\varepsilon$*. There exists* $\varepsilon_D = \varepsilon_D(M,T) := 2LM^{-1}T^{-1}$ *such that for every* $\varepsilon \in (0, \varepsilon_D)$ *there is no more diffussive reflection along the backwards trajectory.*

Proof. We will follow the same ideas as those exposed during Lemma 3.7. We assume, without loss of generality that $x \in \Lambda_1^\varepsilon$, i.e $x^1 = -L\varepsilon^{-1}$.

We denote (t_1, x_1, v_1) as the point of collision against the boundary through the backwards trajectory starting from (t,x,v) and which is given by (3.1.25).

We recall now that for every $z \in \Lambda_3^\varepsilon$ and $w \in \mathbb{R}^3$ there holds that $|\mathcal{V}_z w| = |w|$ and $(\mathcal{V}_z w)^1 = w^1$. Therefore, and because $|n(x)\cdot v| > \eta$, we deduce that if there is a diffusive reflection through the backwards trajectory it has to be because the particle eventually arrives at Λ_2^ε, even after several possible specular reflections against Λ_3^ε of the form (3.4.2). Moreover, we also observe in an elementary way that the trajectory that minimizes the time to arrive to Λ_2^ε is when $x_1^1 = L\varepsilon^{-1}$.

Due to this analysis, if we assume that $x_1^1 = L\varepsilon^{-1}$ and we have that this collision cannot happen in the time interval $[0,T]$ then this will conclude the proof. Finally, we just need to observe that, for every $\varepsilon \in (0, \varepsilon_D)$, there holds

$$t - t_1 = \frac{|x^1 - x_1^1|}{|v^1|} \geq \frac{2L\varepsilon^{-1}}{M} > T,$$

which concludes the proof. □

3.4.2 Regularizing effect of K

We extend in this subsection the regularizing effect generated by the interplay between the free transport semigroup and the non-local operator K to the cylindrical framework.

Proposition 3.9. *Consider Assumption (RH2) to hold, $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function and let f be a solution of Equation (3.3.2). For every $\lambda > 0$ there is $\varepsilon_6 = \varepsilon_6(\lambda, T)$ such that for every $\varepsilon \in (0, \varepsilon_6)$ and every $\nu_2 \in (0, \nu_0)$ we have that for every point $(t, x, v) \in \mathcal{U}_T^\varepsilon$ with $v \notin S_x \cup W_x$ there holds*

$$\omega_1(v)|S_{\mathscr{T}} *_\sigma Kf(t,x,v)| \leq \lambda(t+t^2)e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + Cte^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}$$
$$+ Ct^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s}\|f_s\|_{\mathcal{H}}] + C(1+t)e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right],$$

for a constant $C = C(\lambda) > 0$ and any $\sigma \in [0, T]$ such that $x - v(t-s) \in \bar{\Omega}^\varepsilon$. Furthermore, there holds $C(\lambda) \lesssim \lambda^{-p}$ for some constant $p > 0$.

Proof. We define

$$\mathfrak{I} := \omega_1(v)|S_{\mathscr{T}} *_\sigma Kf(t,x,v)| = \omega_1(v)\left|\int_\sigma^t S_{\mathscr{T}}(t-s)Kf(s,x,v)\mathrm{d}s\right|,$$

and arguing as during the Step 1 of the proof of Proposition 3.10 we have that by using (K1) we deduce that for every $N_1 > 0$ there is $m_1(N_1) > 0$ such that

$$\mathfrak{I} \leq \frac{1}{N_1} te^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + S_{\mathscr{T}} *_\sigma K_{m_1}(\omega_1(v)|f(t,x,v)|), \quad (3.4.11)$$

where we recall that

$$S_{\mathscr{T}} *_\sigma K_{m_1}(\omega_1(v)|f(t,x,v)|) = \int_\sigma^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} k_{m_1}(v,v_*)\omega_1(v_*)|f(t,x_s,v_*)|\mathrm{d}v_*\mathrm{d}s$$

with $x_s = x - v(t-s)$, and we remark that $|v| \leq m_1$ from the very definition of k_{m_1} as given by (3.3.5).

Using then Lemma 3.4 and Lemma 3.5 we deduce that we can take the previous integral over the set $\mathbb{R}^3 \setminus (S_{x_s} \cup W_{t-s,x_s})$, therefore we can define (s_1, x_1, v_1) as the point of the first bounce against the boundary through the backwards trajectory starting at (s, x_s, v_*), with $(x_1, v_1) \in \Sigma_+$, $x_1 \notin \mathfrak{S}$, and whose formulas are given by (3.1.25). In particular, we remark that

$$x_1 = x_s - (s - s_1)v_*.$$

For any $\alpha_0 > 0$ we define now $k_{0,m_1} = \mathbf{1}_{\{|n(x_1)\cdot v_*| \geq \alpha_0\}} k_{m_1}$, and the operator K_{0,m_1} as in (3.3.10). Repeating again the arguments from the Step 1 of the proof of Proposition 3.10 we have that

$$\mathfrak{I} \leq \left(\frac{1}{N_1} + 2C_k D m_1^4 \alpha_0\right) te^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + \mathfrak{I}_0,$$

with

$$\mathfrak{I}_0 := S_{\mathscr{T}} *_\sigma K_{0,m_1}(\omega_1(v)|f(t,x,v)|) = \int_\sigma^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*)\omega_1(v_*)|f(s,x_s,v_*)|\mathrm{d}v_*\mathrm{d}s.$$

Using the Duhamel formula we then have that

$$f(s,x_s,v_*) = \mathbf{1}_{\{s_1\le 0\}} S_{\mathscr{T}}(s) f_0(x_s,v_*) + \int_{\max\{0,s_1\}}^{s} S_{\mathscr{T}}(s-r) K f(r,x_s,v_*) \mathrm{d}r \\ + \int_{\max\{0,s_1\}}^{s} S_{\mathscr{T}}(s-r) G(r,x_s,v_*) \mathrm{d}r + \mathbf{1}_{\{s_1>0\}} S_{\mathscr{T}}(s-s_1) f(s_1,x_s,v_*), \quad (3.4.12)$$

and repeating one more time the arguments from the Step 1 of the proof of the Proposition 3.10, we further deduce that for every $N_2, \alpha_2 > 0$ there is $m_2(N_2) > 0$ such that for every $\nu_2 \in (0,\nu_0)$ there holds

$$\begin{aligned} \mathfrak{I} \ \le\ & \left(\frac{1}{N_1} + 2C_k D m_1^4 \alpha_0 + \frac{C_k D m_1^4}{N_2} + C_k^2 D^2 m_1^4 m_2^4 \alpha_1 \right) t e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\ & + C_k D m_1^4 t e^{-\nu t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ & + C_k D m_1^4 \frac{\nu_1 t}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \\ & + C_k^2 D m_1^{1+3/2} m_2^{1+3/2} \omega_1(m_2) \alpha_1^{-3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right] + \mathfrak{I}_4, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{I}_4 \ :=\ & \int_\sigma^t e^{-\nu(v)(t-s)} \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*) \omega_1(v_*) |f(s_1,x_1,v_*)| \, \mathrm{d}v_* \mathrm{d}s \\ =\ & \int_\sigma^t e^{-\nu(v)(t-s)} \int_{|v_*|\le m_1,\, |n(x_1)\cdot v_*|\ge \alpha_0} k_{0,m_1}(v,v_*) \omega_1(v_*) |f(s_1,x_1,v_*)| \, \mathrm{d}v_* \mathrm{d}s, \end{aligned}$$

and we recall that the expression on the second line is due to the very definition of k_{0,m_1}. Using the Maxwell boundary conditions we have that

$$\begin{aligned} f(s_1,x_1,v_*) \ &=\ (1-\iota^\varepsilon(x_1)) \mathscr{S} f(s_1,x_1,v_*) + \iota^\varepsilon(x_1) \mathscr{D} f(s_1,x_1,v_*) \\ &=\ (1-\iota^\varepsilon(x_1)) f(s_1,x_1,\mathcal{V}_{x_1} v_*) + \iota^\varepsilon(x_1) \mathscr{M}(v_*) \widetilde{f}(s_1,x_1), \end{aligned}$$

thus we deduce that $\mathfrak{I}_4 \le \mathfrak{I}^S + \mathfrak{I}^D$ where $\mathfrak{I}^S$ and $\mathfrak{I}^D$ are given by (3.3.13), (3.3.14) respectively.

Step 1. (Control of the diffusive term $\mathfrak{I}^D$) We observe first that, since $\mathscr{M}\omega_1 \le 1$, there holds

$$\mathfrak{I}^D \le \int_\sigma^t \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{\mathbb{R}^3} |f(s_1,x_1,u_*)| \, (n(x_1)\cdot u_*)_+ \, \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s,$$

Using then (K4) we have that for every $\lambda_1 > 0$ there are $M_1, \eta > 0$ such that

$$\begin{aligned} \mathfrak{I}^D \ \le\ & \int_\sigma^t \int_{\mathbb{R}^3} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)} \\ & \times \left[\lambda_1 \|f_{s_1}\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \int_{|u_*|\le M_1,\, |n_{x_1}\cdot u|>\eta} |f(s_1,x_1,u_*)| \, (n(x_1)\cdot u_*)_+ \mathrm{d}u_* \right] \mathrm{d}v_* \mathrm{d}s \\ \le\ & \lambda_1 C_k D m_1^4 t e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + \mathfrak{I}_0^D \end{aligned}$$

where we have used the properties of k_{m_1} as exposed in (K1), and we have defined

$$\mathfrak{I}_0^D = \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{|u_*|\le M_1,\, |n_{x_1}\cdot u_*|>\eta} |f(s_1,x_1,u_*)| \, (n(x_1)\cdot u_*)_+ \, \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s.$$

Arguing now as during the Step 1 we deduce that we may rewrite the previous integral as being integrated over the set

$$\{|u_*| \le M_1, |n(x_1)\cdot u_*| > \eta\} \setminus (S_{x_1} \cup W_{t_1,x_1}).$$

Using then Lemma 3.8 there is $\varepsilon_6^1 = \varepsilon_D(\eta, M_1, T)$ such that for every $\varepsilon \in (0, \varepsilon_6^1)$ there is no more collision against the diffusive boundary through the backwards trajectory starting at (s_1, x_1, u_*). We then denote (s_2, x_2, u_2) as the point of collision against the boundary through the backwards trajectory starting at (s_1, x_1, u_*), moreover we remark that $(x_2, u_*) \in \Sigma_-^\varepsilon$ and due to the previous choice of ε we also have that

$$x_2 = x_1 - u_*(s_1 - s_2) \in \Lambda_3^\varepsilon.$$

Then, for any $\eta_1 > 0$ small, we define $\mathscr{U}_1 = \{|u_*| \le M_1,\ |n_{x_1}\cdot u_*| > \eta, |\mathfrak{n}(x_1)\cdot u_*| \le \eta_1\}$, $\mathscr{U}_2 = \{|u_*| \le M_1,\ |n_{x_1}\cdot u_*| > \eta, |\mathfrak{n}(x_1)\cdot u_*| > \eta_1\}$, and we have that $\mathfrak{I}_0^D \le \mathfrak{I}_1^D + \mathfrak{I}_2^D$ with

$$\begin{aligned}\mathfrak{I}_1^D &:= \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v, v_*) e^{-\nu_0(t-s_1)} \int_{\mathscr{U}_1} |f(s_1, x_1, u_*)|\, (n(x_1)\cdot u_*)_+ \,\mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s,\\ \mathfrak{I}_2^D &:= \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v, v_*) e^{-\nu_0(t-s_1)} \int_{\mathscr{U}_1} |f(s_1, x_1, u_*)|\, (n(x_1)\cdot u_*)_+ \,\mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s.\end{aligned}$$

In order to control $\mathfrak{I}_1^D$ we perform the change of variable $u_*^{par} = (\mathfrak{n}(x_1)\cdot u_*)\mathfrak{n}(x_1)$ and its perpendicular direction $u_*^\perp = u_* - u_*^{par}$ such that $u_* = u_*^{par} + u_*^\perp$, and we obtain that

$$\begin{aligned}\mathfrak{I}_1^D &\le M_1 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v, v_*) \int_{|u_*^\perp| \le M_1} \mathrm{d}u_*^\perp \int_{-\eta_1}^{\eta_1} \mathrm{d}u_*^{par} \mathrm{d}v_* \mathrm{d}s\\ &\le 2\eta_1\, t M_1^3 D^2 C_k m_1^3 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right],\end{aligned}$$

where we have used the bounds on k_{m_1} as exposed during (K1).

Step 1.1. Using then the Duhamel decomposition we have that

$$\begin{aligned}f(s_1, x_1, u_*) = \mathbf{1}_{\{s_2 \le 0\}} S_{\mathscr{T}}(s_1) f_0(x_1, u_*) + \int_{\max\{0,s_2\}}^{s_1} S_{\mathscr{T}}(s_1 - r) K f(r, x_1, u_*) \mathrm{d}r\\ + \int_{\max\{0,s_2\}}^{s_1} S_{\mathscr{T}}(s_1 - r) G(r, x_1, u_*) \mathrm{d}r + \mathbf{1}_{\{s_2 > 0\}} S_{\mathscr{T}}(s_1 - s_2) f(s_2, x_1, u_*),\end{aligned}$$

and we remark that, since $x_2 \in \Lambda_3^\varepsilon$, then

$$S_{\mathscr{T}}(s_1 - s_2) f(s_2, x_1, u_*) = e^{-\nu(u_*)(s_1 - s_2)} f(s_2, x_2, \mathcal{V}_{x_2} u_*).$$

By arguing now as during the Step 3 of Proposition 3.10 and we deduce that for every $N_3, \alpha_3 > 0$ there is $m_3(N_3) > 0$ such that there holds

$$\begin{aligned}\mathfrak{I}_0^D \le &\left(\lambda_1 C_k D m_1^4 t + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_3} + C_k^2 D^3 m_1^4 M_1^4 m_3^4 \alpha_3 t + 2\eta_1\, t M_1^3 D^2 C_k m_1^3 \right)\\ &\times e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + C_k D^2 m_1^4 M_1^4 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\\ &+ \frac{\nu_1 C_k D^2 m_1^4 M_1^4 t}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right]\\ &+ C_k^2 D^2 m_1^4 m_4^{1+3/2} M_1^{1+3/2} \omega(m_3) \alpha_3^{-3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s} \|f_s\|_{\mathcal{H}}] + \mathfrak{I}_S^D\end{aligned}$$

for every $\nu_2 \in (0,\nu_0)$, and where we have defined

$$\mathfrak{I}_S^D = \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_2)} \int_{\mathscr{U}_2} |f(s_2,x_2,\mathcal{V}_{x_2}u_*)| \, (n(x_1)\cdot u_*)_+ \, \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s.$$

Using then Lemma 3.6, we deduce that there is a constructive constant $A_1 = A_1(\eta) > 0$ such that $|n(x_2)\cdot \mathcal{V}_{x_2}| \geq A_1$ uniformly in x. Applying now Lemma 3.7 we have that there is $\varepsilon_6^2 = \varepsilon_S(A_1, M_1, T)$, such that for every $\varepsilon \in (0, \min(\varepsilon_6^1, \varepsilon_6^2))$ there is no more bounce against the specular reflection. Therefore the Duhamel formula gives

$$f(s_2,x_2,\mathcal{V}_{x_2}u_*) = S_{\mathscr{T}}(s_2) f_0(x_2,\mathcal{V}_{x_2}u_*) + S_{\mathscr{T}} * Kf(s_2,x_2,\mathcal{V}_{x_2}u_*) + S_{\mathscr{T}} * G(s_2,x_2,\mathcal{V}_{x_2}u_*).$$

We then have that

$$\mathfrak{I}_S^D \leq \mathfrak{I}_{S,1}^D + \mathfrak{I}_{S,2}^D + \mathfrak{I}_{S,3}^D,$$

with

$$\begin{aligned}
\mathfrak{I}_{S,1}^D &= \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_2)} \int_{\mathscr{U}_2} |S_{\mathscr{T}}(s_2) f_0(x_2,\mathcal{V}_{x_2}u_*)| \, (n(x_1)\cdot u_*)_+, \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s, \\
\mathfrak{I}_{S,2}^D &= \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_2)} \int_{\mathscr{U}_2} |S_{\mathscr{T}} * Kf(s_2,x_2,\mathcal{V}_{x_2}u_*)| \, (n(x_1)\cdot u_*)_+, \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s, \\
\mathfrak{I}_{S,3}^D &= \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_2)} \int_{\mathscr{U}_2} |S_{\mathscr{T}} * G(s_2,x_2,\mathcal{V}_{x_2}u_*)| \, (n(x_1)\cdot u_*)_+, \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s,
\end{aligned}$$

and we will control each of these terms separately. We first compute in a similar way as during the Step 3 of the proof of Proposition 3.10,

$$\mathfrak{I}_{S,1}^D \leq C_k D^2 m_1^4 M_1^4 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and we also have that, for every $\nu_2 \in (0,\nu_0)$, there holds

$$\begin{aligned}
\mathfrak{I}_{S,3}^D &\leq \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{\mathscr{U}_2} |\omega_1(\mathcal{V}_{x_2}u_*) S_{\mathscr{T}} * G(s_2,x_2,\mathcal{V}_{x_2}u_*)| \, |u_*| \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s \\
&\leq \frac{\nu_1}{\nu_0-\nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \int_0^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) \int_{\{|u_*|\leq M_1\}} |u_*| \mathrm{d}r \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s \\
&\leq \frac{\nu_1 C_k D^2 m_1^4 M_1^4 t}{\nu_0-\nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)} \right],
\end{aligned}$$

where we have used that $\omega_1 \geq 1$ on the first inequality, (K2) in the second, and the very definition of k_{m_1} given by (3.3.5) on the third.

To analyze $\mathfrak{I}_{S,2}^D$ we use (K1) so that for every $N_4 > 0$ there is $m_4(N_4) > 0$ for which there holds

$$\begin{aligned}
\mathfrak{I}_{S,2}^D &\leq \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{\mathscr{U}_2} \omega \, |S_{\mathscr{T}} * Kf(s_2,x_2,\mathcal{V}_{x_2}u_*)| \, (n_{x_1}\cdot u_*)_+ \mathrm{d}u_* \mathrm{d}v_* \mathrm{d}s \\
&\leq \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{\{|u_*|\leq M_1\}} |u_*| \int_0^{s_2} e^{\nu_0(s_2-r)} \\
&\quad \times \left(\frac{1}{N_4} \|f_s\|_{L^\infty_{\omega\infty}(\mathcal{O}^\varepsilon)} + \int_{\mathbb{R}^3} k_{m_4}(\mathcal{V}_{x_2}u_*, u') \omega_1(u') |f(u')| du' \right) \\
&\leq \frac{t^2 D^2 m_1^4 M_1^4}{N_4} e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\
&\quad + \int_\sigma^t \int_{\mathbb{R}^3} k_{m_1}(v,v_*) \int_{\{|u_*|\leq M_1\}} |u_*| \int_0^{s_2} e^{\nu_0(t-r)} \int_{\mathbb{R}^3} k_{m_4}(\mathcal{V}_{x_2}u_*, u') \omega_1(u') |f(u')| \\
&\leq \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_4} e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + C_k^2 m_1 m_4 M_1 \, \mathfrak{I}_{S,2,0}^D
\end{aligned}$$

where we have used the fact that $\omega_1 \geq 1$ and $s_1 \geq s_2$ in the first inequality, (K1) on the second inequality, (3.3.5) on the third and fourth inequalities, and we have defined

$$\mathfrak{I}^D_{S,2,0} = \int_\sigma^t \int_{|v_*|\leq m_1} \int_{\{|u_*|\leq M_1\}} \int_0^{s_2} e^{\nu_0(t-r)} \int_{|u'|\leq m_4} \omega(u')|f(r, x_2 - \mathcal{V}_{x_2}u_*(s_1-r), u')|.$$

Arguing then exactly as during the proof of Lemma 3.12 and using Lemma 3.8 we have that for every $\alpha_4 > 0$ there is $\varepsilon_6^3 = \varepsilon_U(\alpha_4) > 0$ such that for every $\varepsilon \in (0, \min(\varepsilon_6^1, \varepsilon_6^2, \varepsilon_6^3))$ there holds

$$\begin{aligned}
\mathfrak{I}^D_{S,2,0} &\leq Dm_1^3 \sup_{|v_*|\leq m_1} \left[\int_\sigma^t \int_{\{|u_*|\leq M_1\}} \int_0^{s_2} e^{\nu_0(t-r)} \int_{|u'|\leq m_4} \omega(u')|f(r, x_2 - \mathcal{V}_{x_2}u_*(s_1-r), u')| \right] \\
&\leq D^3\, m_1^3\, M_1^3\, m_4^3\, \alpha_4\, t\, e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{-\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\
&\quad + D^2 2^{1/2} m_1^3 M_1^{3/2} m_4^{3/2}\, \omega_1(m_4)\, \alpha_4^{-3/2}\, t^2\, e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right].
\end{aligned}$$

Putting together the above estimates controlling the diffusion term we have that

$$\begin{aligned}
\mathfrak{I}^D \leq\ & \Bigg(\lambda_1 C_k D m_1^4 t + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_3} + C_k^2 D^3 m_1^4 M_1^4 m_3^4 \alpha_3 t + 2\eta_1\, t M_1^3 D^2 C_k m_1^3 \\
& + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_4} + C_k^2 D^3 m_1^4 M_1^4 m_4^4 \alpha_4 t \Bigg) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\
& + 2 C_k D^2 m_1^4 M_1^4 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + 2\frac{\nu_1 C_k D^2 m_1^4 M_1^4 t}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \\
& + C_k^2 D^2 m_1^4 m_3^{1+3/2} M_1^{1+3/2} \omega(m_3) \alpha_3^{-3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right] \\
& + C_k^2 D^2 2^{1/2} m_1^4 m_4^{1+3/2} M_1^{1+3/2} \omega(m_4) \alpha_4^{-3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right]
\end{aligned}$$

for every $\nu_2 \in (0, \nu_0)$.

Step 2. (Control of the specular term $\mathfrak{I}^S$) We recall now $\mathfrak{I}^S$ defined in (3.3.13) and we recall that $x_1 \in \Lambda_3^\varepsilon$. Using then the fact that

$$|\mathcal{V}_{x_1} v_*| = |v_*|, \qquad |n(x_1) \cdot \mathcal{V}_{x_1} v_*| = |n(x_1) \cdot v_*|,$$

and that we integrate on the velocity set $\{|v_*| \leq m_1,\, |n(x_1) \cdot v_*| \geq \alpha_0\}$ we deduce that it is equivalent to integrate on the set

$$\{|\mathcal{V}_{x_1} v_*| \leq m_1, |n(x_1) \cdot \mathcal{V}_{x_1} v_*| \geq \alpha_0\} \cap \{\mathcal{V}_{x_1} v_* \notin S_{x_1}\} \cap \{\mathcal{V}_{x_1} v_* \notin W_{t_1,x_1}\}.$$

where we have used again that the sets S_{x_1} and $W_{t_1,x_1} \cap \{|v_*| \leq m_1\} \setminus S_{x_1}$ have Lebesgue measure zero, due to Lemmas 3.4 and 3.5. We denote (s_2, x_2, v_2) the first bounce against the boundary through the backwards trajectory starting at $(s_1, x_1, \mathcal{V}_{x_1} v_*)$ given by (3.1.25) and we remark that

$$x_2 = x_1 - \mathcal{V}_{x_1} v_* (s_1 - s_2) \notin \mathfrak{S} \quad \text{and} \quad (x_2, \mathcal{V}_{x_1} v_*) \in \Sigma_+^\varepsilon.$$

We define the sets

$$\begin{aligned}
\mathscr{A}_S &:= \{v^* \in \mathbb{R}^3,\, |v_*| \leq m_1,\, |n(x_s) \cdot v_*| \geq \alpha_0 \text{ and } x_2 \in \Lambda_3^\varepsilon\}, \\
\mathscr{A}_D &:= \{v^* \in \mathbb{R}^3,\, |v_*| \leq m_1,\, |n(x_s) \cdot v_*| \geq \alpha_0 \text{ and } x_2 \in \Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon\},
\end{aligned}$$

so we have that $\mathfrak{I}^S \leq \mathfrak{I}^S_S + \mathfrak{I}^S_D$, where

$$\begin{aligned}
\mathfrak{I}^S_S &= \int_0^t e^{-\nu(v)(t-s)} \int_{\mathscr{A}_S} k_{0,m_1}(v,v_*) e^{-\nu(v)(s-s_1)} \omega_1(v_*)(1-\iota^\varepsilon(x_1)) \left| f(s_1,x_1,\mathcal{V}_{x_1}v_*)\right| \mathrm{d}v_* \mathrm{d}s, \\
\mathfrak{I}^S_D &= \int_0^t e^{-\nu(v)(t-s)} \int_{\mathscr{A}_D} k_{0,m_1}(v,v_*) e^{-\nu(v)(s-s_1)} \omega_1(v_*)(1-\iota^\varepsilon(x_1)) \left| f(s_1,x_1,\mathcal{V}_{x_1}v_*)\right| \mathrm{d}v_* \mathrm{d}s,
\end{aligned}$$

and we study each one separately.

By repeating the analysis performed on the Step 2 of the proof of the Proposition 3.10, using Lemma 3.7 instead of Lemma 3.4, we deduce that there is $\varepsilon_6^4 > 0$ such that for every $\varepsilon \in (0, \min(\varepsilon_6^1, \varepsilon_6^2, \varepsilon_6^3, \varepsilon_6^4))$ and every $N_6, \alpha_6 > 0$ there is $m_6(N_6) > 0$ such that there holds

$$\begin{aligned}
\mathfrak{I}^S_S \leq\ & \left(C_k D m_1^4 \frac{t}{N_6} + C_k^2 D^2 m_1^4 m_6^4 \alpha_6 t \right) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\
& + C_k D m_1^4 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_k D m_1^4 \frac{\nu_1 t}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \\
& + 2^{1/2} \omega_1(m_6) D C_k^2 m_1^{1+3/2} m_6^{1+3/2} \alpha_6^{-3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right],
\end{aligned}$$

for every $\nu_2 \in (0,\nu_0)$. On the other hand, to control $\mathfrak{I}^S_D$ we use again the Duhamel formula and we have that

$$\begin{aligned}
f(s_1,x_1,\mathcal{V}_{x_1}v_*) = \mathbf{1}_{\{s_2\leq 0\}} S_{\mathscr{T}}(s_1) f_0(x_1,\mathcal{V}_{x_1}v_*) + \int_{\max\{0,s_2\}}^{s_1} S_{\mathscr{T}}(s_1-r) K f(r,x_1,\mathcal{V}_{x_1}v_*) \mathrm{d}r \\
+ \int_{\max\{0,s_2\}}^{s_1} S_{\mathscr{T}}(s_1-r) G(r,x_1,\mathcal{V}_{x_1}v_*) \mathrm{d}r + \mathbf{1}_{\{s_2>0\}} S_{\mathscr{T}}(s_1-s_2) f(s_2,x_2,\mathcal{V}_{x_1}v_*).
\end{aligned}$$

We have then that $\mathfrak{I}^S_D \leq \mathfrak{I}^S_{D,1} + \mathfrak{I}^S_{D,2} + \mathfrak{I}^S_{D,3} + \mathfrak{I}^S_{D,4}$ with

$$\begin{aligned}
\mathfrak{I}^S_{D,1} &= \int_0^t \int_{\mathscr{A}_D} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)} \omega_1(v_*) \left| S_{\mathscr{T}}(s_1) f_0(x_1,\mathcal{V}_{x_1}v_*)\right| \mathrm{d}v_* \mathrm{d}s, \\
\mathfrak{I}^S_{D,2} &= \int_0^t \int_{\mathscr{A}_D} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)} \omega_1(v_*) \left| \int_{\max\{0,s_2\}}^{s_1} S_{\mathscr{T}}(s_1-r) K f(r,x_1,\mathcal{V}_{x_1}v_*) \mathrm{d}r \right| \mathrm{d}v_* \mathrm{d}s, \\
\mathfrak{I}^S_{D,3} &= \int_0^t \int_{\mathscr{A}_D} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)} \omega_1(v_*) \left| \int_{\max\{0,s_2\}}^{s_1} S_{\mathscr{T}}(s_1-r) G(r,x_1,\mathcal{V}_{x_1}v_*) \mathrm{d}r \right| \mathrm{d}v_* \mathrm{d}s, \\
\mathfrak{I}^S_{D,4} &= \int_0^t \int_{\mathscr{A}_D} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)} \omega_1(v_*) \left| S_{\mathscr{T}}(s_1-s_2) f(s_2,x_2,\mathcal{V}_{x_1}v_*)\right| \mathrm{d}v_* \mathrm{d}s.
\end{aligned}$$

We observe then that the first three terms can be controlled by repeating the same computations as during the Step 1 of the proof of the Proposition 3.10, using Lemma 3.7 instead of Lemma 3.4. Therefore, there is $\varepsilon_6^5 > 0$ such that for every $\varepsilon \in (0, \min(\varepsilon_6^1, \varepsilon_6^2, \varepsilon_6^3, \varepsilon_6^4, \varepsilon_6^5))$ and every $N_6, \alpha_6 > 0$ there is $m_6(N_6) > 0$ such that there holds

$$\begin{aligned}
\mathfrak{I}^S_D \leq\ & \left(C_k D m_1^4 \frac{t}{N_6} + C_k^2 D^2 m_1^4 m_6^4 \alpha_6 t \right) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\
& + C_k D m_1^4 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_k D m_1^4 \frac{\nu_1 t}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \\
& + 2^{3/2} \omega_1(m_6) D C_k^2 m_1^{1+3/2} m_6^{1+3/2} \alpha_6^{-3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right] + \mathfrak{I}^S_{D,4}.
\end{aligned}$$

We proceed then to control the remaining term $\mathfrak{I}^S_{D,4}$, and using the boundary conditions of Equation (3.3.2) we first have that

$$\mathfrak{I}^S_{D,4} \leq \int_0^t \int_{\mathscr{A}_D} k_{0,m_1}(v,v_*) e^{-\nu_0(t-s_1)} \int_{\mathbb{R}^3} |f(s_2,x_2,u')| \, (n(x_2)\cdot u')_+ \, \mathrm{d}u' \mathrm{d}v_* \mathrm{d}s.$$

Repeating then exactly the same computations performed during the Step 1 of this proof we deduce that there are $\varepsilon_6^6, \varepsilon_6^7 > 0$ such that for every $\varepsilon \in (0, \min(\varepsilon_6^1, \varepsilon_6^2, \varepsilon_6^3, \varepsilon_6^4, \varepsilon_4^5, \varepsilon_6^6, \varepsilon_6^7))$, there holds that for every $\lambda_2, \eta_2, N_7, N_8, \alpha_7, \alpha_8 > 0$, there are constants $m_6(N_6), m_7(N_7) > 0$ such that

$$\begin{aligned} \mathfrak{I}^S_{D,4} \leq & \left(\lambda_2 C_k D m_1^4 t + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_7} + C_k^2 D^3 m_1^4 M_1^4 m_7^4 \alpha_7 t + 2\eta_2 \, t M_1^3 D^2 C_k m_1^3 \right. \\ & \left. + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_8} + C_k^2 D^3 m_1^4 M_1^4 m_8^4 \alpha_8 t \right) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\ & + 2 C_k D^2 m_1^4 M_1^4 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + 2 \frac{\nu_1 C_k D^2 m_1^4 M_1^4 t}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \\ & + \left(\sum_{j=7}^8 \omega_1(m_j) m_j^{1+3/2} \alpha_j^{-3/2} \right) 2^{1/2} D^2 M_1^{1+3/2} C_k^2 m_1^{1+3/2} t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right], \end{aligned}$$

for every $\nu_2 \in (0,\nu_0)$. We then conclude this step by putting together the previous informations and we have obtained that

$$\begin{aligned} \mathfrak{I}^S \leq C_1^S e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + 2 C_k D m_1^4 (1 + D M_1^4) \frac{\nu_1 t}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \\ + 2 C_k D m_1^4 (1 + D M_1^4) t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_2^S t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right], \end{aligned}$$

with

$$\begin{aligned} C_1^S = & C_k D m_1^4 \frac{t}{N_6} + C_k^2 D^2 m_1^4 m_6^4 \alpha_6 t + C_k D m_1^4 \frac{t}{N_6} + C_k^2 D^2 m_1^4 m_6^4 \alpha_6 t + \lambda_2 C_k D m_1^4 t \\ & + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_7} + C_k^2 D^3 m_1^4 M_1^4 m_7^4 \alpha_7 t + 2\eta_2 \, t M_1^3 D^2 C_k m_1^3 + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_8} + C_k^2 D^3 m_1^4 M_1^4 m_8^4 \alpha_8 t, \end{aligned}$$

and

$$C_2^S = 2^{1/2} D C_k^2 m_1^{1+3/2} \left[\left(\sum_{j=5}^6 \omega_1(m_j) m_j^{1+3/2} \alpha_j^{-3/2} \right) + \left(\sum_{j=7}^8 \omega_1(m_j) m_j^{1+3/2} \alpha_j^{-3/2} \right) D M_1^{1+3/2} \right].$$

Step 3. (Choice of the parameters) Putting together the estimates obtained from Steps 1, 2 we get that for every $\varepsilon \in (0, \min(\varepsilon_6^1, \varepsilon_6^2, \varepsilon_6^3, \varepsilon_6^4, \varepsilon_4^5, \varepsilon_6^6, \varepsilon_6^7))$ and $\nu_2 \in (0,\nu_0)$ there holds

$$\begin{aligned} \mathfrak{I} \leq C_1 \left(t + t^2 \right) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + C_2 t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ + C_3 t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right] + C_4 t e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right], \end{aligned}$$

with

$$\begin{aligned} C_1 = & \frac{1}{N_1} + 2 C_k D m_1^4 \alpha_0 + \frac{C_k D m_1^4}{N_2} + C_k^2 D^2 m_1^4 m_2^4 \alpha_1 + \lambda_1 C_k D m_1^4 t + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_3} \\ & + C_k^2 D^3 m_1^4 M_1^4 m_3^4 \alpha_3 t + \frac{t^2 C_k^2 D^2 m_1^4 M_1^4}{N_4} + C_k^2 D^3 m_1^4 M_1^4 m_4^4 \alpha_4 t + C_1^S, \end{aligned}$$

and

$$\begin{aligned} C_2 &= 3C_k D m_1^4 + 2C_k D m_1^4 (1 + D M_1^4), \qquad C_3 = C_3^0 + C_2^S, \\ C_3^0 &= 2^{1/2} D C_k^2 m_1^{1+3/2} \left[\left(\sum_{j=1}^{2} \omega_1(m_j) m_j^{1+3/2} \alpha_j^{-3/2} \right) + D M_1^{1+3/2} \left(\sum_{j=3}^{4} \omega_1(m_j) m_j^{1+3/2} \alpha_j^{-3/2} \right) \right], \\ C_4 &= \frac{\nu_1}{\nu_0 - \nu_2} \left(3C_k D m_1^4 + 4 C_k D^2 m_1^4 M_1^4 \right). \end{aligned}$$

We then set the constants in a similar way as in the Step 4 of the proof of the Proposition 3.10 so that $C_1 \leq \lambda$. We define $C = \max(C_2, C_3, C_4)$ and we observe that, since the constants that define C come from (K1), (K3) and (K4), there is a constant $p > 0$ such that $C \lesssim \lambda^{-p}$. We conclude by setting $\varepsilon_6 = \min(\varepsilon_6^1, \varepsilon_6^2, \varepsilon_6^3, \varepsilon_6^4, \varepsilon_6^5, \varepsilon_6^6, \varepsilon_6^7)$. □

3.4.3 Estimate on the trajectories

In this subsection we use then the regularization property given by Proposition 3.9 to proof a L^∞ estimate of the solutions of the equation following a similar argument as during the proof of Proposition 3.13.

Proposition 3.10. *Consider Assumption (RH2) to hold, $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function, and let f be a solution to Equation* (3.3.2)*. For every $t \in [0, T]$ and every $\lambda > 0$ there is $\varepsilon_7 = \varepsilon_7(\lambda, T) > 0$ such that for every $\varepsilon \in (0, \varepsilon_7)$ and every $\nu_2 \in (0, \nu_0)$, we have that for every $(x, v) \in \mathcal{O}^\varepsilon$, with $|v| \leq \mathfrak{M}$ for any arbitrary $\mathfrak{M} > 0$, and $v \notin S_x \cup W_{t,x}$ there holds*

$$\begin{aligned} \omega_1(v)|f(t,x,v)| \leq \lambda(1 + t + t^2) e^{-\nu_0 t} \sup_{s \in [0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + C(1+t) e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ + C t^2 e^{-\nu_0 t} \sup_{s \in [0,t]} \left[e^{\nu_0 s} \|f_s\|_{\mathcal{H}} \right] + C(1+t) e^{-\nu_2 t} \sup_{s \in [0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right], \end{aligned}$$

uniformly in $\mathfrak{M}$. Moreover, $C = C(\lambda) > 0$ and there is $p > 0$ such that $C \lesssim \lambda^{-p}$.

Proof. Starting from (t, x, v), and since $|v| \leq \mathfrak{M}$ then Lemma 3.2 implies that there are only two possible scenarios to consider:

- *Case 1.* There are only specular reflections through the backwards trajectories. In particular, there is $\mathfrak{N} = \mathfrak{N}(T, x, \mathfrak{M}) < \infty$, given by Lemma 3.2, such that there are no more than $\mathfrak{N}$ consecutive specular reflections through the backwards trajectories.
- *Case 2.* There is a reflection against the diffusive boundary subsets before $\mathfrak{N}$ specular reflections.

We treat now each case separately.

Case 1. (Purely specular reflection) We recall that there are at most $\mathfrak{N}$ collisions against the boundary through the specular reflection boundary condition starting at (t, x, v), where $\mathfrak{N}$ is given by Lemma 3.2. Then the iterated Duhamel formula gives

$$\begin{aligned} f(t,x,v) = e^{-\nu(v)(t - t_{\mathfrak{N}})} f(t_{\mathfrak{N}}, x_{\mathfrak{N}}, v_{\mathfrak{N}}) + \int_0^t \sum_{j=1}^{\mathfrak{N}} \mathbf{1}_{t_j \leq s \leq t_{j-1}}(s) S_{\mathscr{T}}(t_{j-1} - s) K f(s, x_{j-1}, v_{j-1}) \mathrm{d}s \\ + \int_0^t \sum_{j=1}^{\mathfrak{N}} \mathbf{1}_{t_j \leq s \leq t_{j-1}}(s) S_{\mathscr{T}}(t_{j-1} - s) G(s, x_{j-1}, v_{j-1}) \mathrm{d}s, \end{aligned}$$

where we have set $(t_0, x_0, v_0) = (t, x, v)$ and we have defined (t_j, x_j, v_j) as in (3.4.2) for $j \in [\![1, \mathfrak{N}]\!]$.

We then define $(\bar{t}, \bar{x}, \bar{v})$ as the point of collision with the boundary set Λ_3^ε through the backwards trajectory starting at $(t_{\mathfrak{N}}, x_{\mathfrak{N}}, v_{\mathfrak{N}})$. From the fact that there have already been $\mathfrak{N}$ specular reflections we remark that $\bar{t} < 0$, and using one more time the Duhamel formula we obtain

$$\begin{aligned} f(t,x,v) &= e^{-\nu(v)(t-t_{\mathfrak{N}})} S_{\mathscr{T}}(t_{\mathfrak{N}}) f_0(x_{\mathfrak{N}}, v_{\mathfrak{N}}) + S_{\mathscr{T}} * Kf(t_{\mathfrak{N}}, x_{\mathfrak{N}}, v_{\mathfrak{N}}) + S_{\mathscr{T}} * G(t_{\mathfrak{N}}, x_{\mathfrak{N}}, v_{\mathfrak{N}}) \\ &\quad + \int_0^t \sum_{j=1}^{\mathfrak{N}+1} \mathbf{1}_{t_j \le s \le t_{j-1}}(s) S_{\mathscr{T}}(t_{j-1} - s) Kf(s, x_{j-1}, v_{j-1}) ds \\ &\quad + \int_0^t \sum_{j=1}^{\mathfrak{N}} \mathbf{1}_{t_j \le s \le t_{j-1}}(s) S_{\mathscr{T}}(t_{j-1} - s) G(s, x_{j-1}, v_{j-1}) ds, \end{aligned}$$

where we have denoted $t_{\mathfrak{N}+1} = 0$. Using now the fact that (K2) and Proposition 3.9 are uniform in $(t, x, v) \in \mathcal{U}_T^\varepsilon$, $v \notin S_x \cup W_{t,x}$, and the fact that

$$\int_0^t \sum_{j=1}^{\mathfrak{N}+1} \mathbf{1}_{t_j \le s \le t_{j-1}}(s)\, \mathrm{d}s = t,$$

we deduce that, by arguing as during the proof of Proposition 3.13, there is $\varepsilon_7^1 = \varepsilon_6(\lambda, T)$, where $\varepsilon_6(\lambda, T)$ is given by Proposition 3.9, such that for every $\varepsilon \in (0, \varepsilon_7^1)$ there holds

$$\begin{aligned} \omega_1(v)|f(t,x,v)| &\le \lambda(t + t^2) e^{-\nu_0 t} \sup_{s \in [0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + C_S(1+t) e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ &\quad + C_S t^2 e^{-\nu_0 t} \sup_{s \in [0,t]} [e^{\nu_0 s} \|f_s\|_{\mathcal{H}}] + C_S(1+t) e^{-\nu_2 t} \sup_{s \in [0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right], \end{aligned}$$

for some constant $C_S \lesssim \lambda^{-p}$ with $p > 0$. We conclude this case by emphasizing that ε_7^1 and the constant C_S, are independent of $\mathfrak{N}$ and $\mathfrak{M}$ due to the above arguments.

Case 2. (Possible diffussive reflection) We assume without loss of generality that the diffusive collision against the diffusive boundary through the backwards trajectory happens at the first collision. Otherwise, we just repeat the Duhamel formulation through the specular reflections like performed on the Step 1, until the diffusive reflection happens and then we proceed as follows.

We denote (t_1, x_1, v_1) the first collision through the backwards trajectory as given by (3.1.25), and we remark that $x_1 = x - v(t - t_1)$. Using then the Duhamel formula we obtain

$$\begin{aligned} f(t,x,v) = \mathbf{1}_{\{t_1 \le 0\}} S_{\mathscr{T}}(t) f_0(x,v) &+ \int_{\max\{0,t_1\}}^t S_{\mathscr{T}}(t-s) Kf(s,x,v) \mathrm{d}s \\ &+ \int_{\max\{0,t_1\}}^t S_{\mathscr{T}}(t-s) G(s,x,v) \mathrm{d}s + e^{-\nu(v)(t-t_1)} f(t_1, x_1, v). \end{aligned}$$

We multiply the above expression by ω_1 to get

$$\begin{aligned} |\omega_1(v) f(t,x,v)| &\le \omega_1(v) S_{\mathscr{T}}(t) \left(|f_0(x,v)| \right) + \omega_1(v) |S_{\mathscr{T}} * Kf(t,x,v)| + \omega_1(v) |S_{\mathscr{T}} * G(t,x,v)| \\ &\quad + e^{-\nu(v)(t-t_1)} \omega_1(v) |f(t_1, x_1, v)| \\ &=: \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4, \end{aligned}$$

and we proceed as during the proof of Proposition 3.13. On the one hand, we bound

$$\mathscr{I}_1 = S_{\mathscr{T}}(t)\left(\omega_1(v)|f_0(x,v)|\right) \leq e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and on the other hand, by using (K2) we have that for every $\nu_2 \in (0,\nu_0)$ there holds

$$\mathscr{I}_3 = \omega_1(v)|S_{\mathscr{T}} * G(t,x,v)| \leq \frac{\nu_1}{\nu_0-\nu_2}e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right].$$

Moreover, using now Proposition 3.9 we have that for every $\lambda_1 > 0$ there is $\varepsilon_7^2 = \varepsilon_6(\lambda_1, T) > 0$ and a constant $C_1 = C_1(\lambda_1) > 0$ such that for every $\varepsilon \in (0,\varepsilon_7^2)$ and every $\nu_2 \in (0,\nu_0)$ there holds

$$\mathscr{I}_2 \leq \lambda_1(t+t^2)e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + C_1 t e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}$$
$$+ C_1 t^2 e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{\mathcal{H}}\right] + C_1(1+t)e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right],$$

furthermore, there are $c_1, p_1 > 0$ such that $C_1 \leq c_1\lambda_1^{-p_1}$.

We then control the boundary term $\mathscr{I}_4$, and since we have assumed the reflection at the boundary to be diffusive, then using (K4) we have that for every $\lambda_2 > 0$ there are $M, \eta > 0$ such that

$$\begin{aligned}\mathscr{I}_4 &\leq e^{-\nu(v)(t-t_1)}\int_{\mathbb{R}^3}|f(t_1,x_1,u)|(n_{x_1}\cdot u)_+ du\\ &\leq \lambda_2 e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + \mathscr{I}_0^D,\end{aligned}$$

where we have used that $\omega_1\mathscr{M} \leq 1$, and we have defined

$$\mathscr{I}_0^D := e^{-\nu(v)(t-t_1)}\int_{\{|u|\leq M,\, |n(x_1)\cdot u|>\eta\}}|f(t_1,x_1,u)|\,(n(x_1)\cdot u)_+\mathrm{d}u.$$

We then use Lemma 3.8 and we obtain that there is $\varepsilon_7^3 = \varepsilon_D(\eta, M, T) = 2L(MT)^{-1}$ such that for every $\varepsilon \in (0, \min(\varepsilon_7^2, \varepsilon_7^3))$ there are no more diffusion collisions through the backwards trajectory.

For any $\eta_1 > 0$, we define now the sets $\mathscr{U}_1 = \{|u_*| \leq M,\, |n_{x_1}\cdot u_*| > \eta, |\mathfrak{n}(x_1)\cdot u_*| \leq \eta_1\}$, $\mathscr{U}_2 = \{|u_*| \leq M,\, |n_{x_1}\cdot u_*| > \eta, |\mathfrak{n}(x_1)\cdot u_*| > \eta_1\}$, and we have that $\mathscr{I}_0^D = \mathscr{I}^{D,1} + \mathscr{I}^{D,2}$ with

$$\begin{aligned}\mathscr{I}^{D,1} &= e^{-\nu(v)(t-t_1)}\int_{\mathscr{U}_1}|f(t_1,x_1,u)|\,(n(x_1)\cdot u)_+\mathrm{d}u,\\ \mathscr{I}^{D,2} &= e^{-\nu(v)(t-t_1)}\int_{\mathscr{U}_2}|f(t_1,x_1,u)|\,(n(x_1)\cdot u)_+\mathrm{d}u.\end{aligned}$$

By arguing then as during the Step 1 of the proof of Proposition 3.9, we perform the change of variable $u^{\mathrm{par}} = (\mathfrak{n}(x_1)\cdot u)\mathfrak{n}(x_1)$ and its perpendicular direction $u^\perp = u - u^{\mathrm{par}}$ such that $u = u^{\mathrm{par}} + u^\perp$, and we obtain that

$$\begin{aligned}\mathscr{I}^{D,1} &\leq Me^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right]\int_{|u^\perp|\leq M}\mathrm{d}u^\perp\int_{-\eta_1}^{\eta_1}\mathrm{d}u^{\mathrm{par}}\\ &\leq 2\eta_1 M^3 D e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right].\end{aligned}$$

We denote then (t_2, x_2, v_2) as the first collision against the boundary starting at (t_1, x_1, u) through the backwards trajectory as defined in (3.1.25). We emphasize that due to the previous choice of ε_7^3, we have that $x_2 \in \Lambda_3^\varepsilon$.

Moreover, from the fact that $|\mathfrak{n}(x_1) \cdot u| > \eta_1$, Lemma 3.8 implies that there is $A_1 > 0$ independent from x_1 such that

$$|n(x_2) \cdot \mathcal{V}_{x_2} u| = |n(x_2) \cdot u| \geq A_1.$$

Using Lemma 3.7 there is $\varepsilon_7^4 = \varepsilon_S(A_1, M, T)$ such that for every $\varepsilon \in (0, \min(\varepsilon_7^2, \varepsilon_7^3, \varepsilon_7^4))$ there is no more collision against the specular boundary condition through the specular reflection. In particular, this means that, in the time interval $[0, T]$ there are no more collision against the boundary through the backwards trajectory after the collision in x_2, with velocity $\mathcal{V}_{x_2} u_*$, with $|u_*| \leq M$, $|n_{x_1} \cdot u_*| > \eta$, $|\mathfrak{n}(x_1) \cdot u_*| > \eta_1$.

Using then the Duhamel formula we have that

$$\begin{aligned} f(t_1, x_1, u) = \mathbf{1}_{\{t_2 \leq 0\}} S_{\mathscr{T}}(t_1) f_0(x_1, u) + \int_{\max\{0, t_2\}}^{t_1} S_{\mathscr{T}}(t_1 - s) K f(s, x_1, u) \mathrm{d}s \\ + \int_{\max\{0, t_2\}}^{t_1} S_{\mathscr{T}}(t_1 - s) G(s, x_1, u) \mathrm{d}s + e^{-\nu(v)(t_1 - t_2)} f(t_2, x_2, \mathcal{V}_{x_2} u). \end{aligned}$$

Therefore $\mathscr{I}^{D,2} \leq \mathscr{I}_1^D + \mathscr{I}_2^D + \mathscr{I}_3^D + \mathscr{I}_4^D$ with

$$\begin{aligned} \mathscr{I}_1^D &= e^{-\nu(v)(t-t_1)} \int_{\mathscr{U}_2} |S_{\mathscr{T}}(t_1) f_0(x_1, u)| \, (n(x_1) \cdot u)_+ \mathrm{d}u, \\ \mathscr{I}_2^D &= e^{-\nu(v)(t-t_1)} \int_{\mathscr{U}_2} \int_{\max\{0, t_2\}}^{t_1} |S_{\mathscr{T}}(t_1 - s) K f(s, x_1, u)| \mathrm{d}s \, (n(x_1) \cdot u)_+ \mathrm{d}u, \\ \mathscr{I}_3^D &= e^{-\nu(v)(t-t_1)} \int_{\mathscr{U}_2} \int_{\max\{0, t_2\}}^{t_1} |S_{\mathscr{T}}(t_1 - s) G(s, x_1, u)| \mathrm{d}s \, (n(x_1) \cdot u)_+ \mathrm{d}u, \\ \mathscr{I}_4^D &= e^{-\nu(v)(t-t_1)} \int_{\mathscr{U}_2} e^{-\nu(v)(t_1 - t_2)} |f(t_2, x_2, \mathcal{V}_{x_2} u)| \, (n(x_1) \cdot u)_+ \mathrm{d}u. \end{aligned}$$

On the one hand, we bound

$$\mathscr{I}_1^D \leq M^4 D \, e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and on the other hand, by using (K2) we have that for every $\nu_2 \in (0, \nu_0)$ there holds

$$\mathscr{I}_3^D \leq M^4 D \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s \in [0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].$$

Moreover, using Proposition 3.9 we have that for every $\lambda_3 > 0$ there is $\varepsilon_7^5 = \varepsilon_6(\lambda_3, T) > 0$ and a constant $C_3 = C_3(\lambda_3) > 0$, such that for every $\varepsilon \in (0, \min(\varepsilon_7^2, \varepsilon_7^3, \varepsilon_7^4, \varepsilon_7^5))$ and every $\nu_2 \in (0, \nu_0)$ there holds

$$\begin{aligned} \mathscr{I}_2^D \leq \lambda_3 M^4 D (t + t^2) e^{-\nu_0 t} \sup_{s \in [0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + C_3 M^4 D \, t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ + C_3 M^4 D \, t^2 e^{-\nu_0 t} \sup_{s \in [0,t]} [e^{\nu_0 s} \|f_s\|_{\mathcal{H}}] + C_3 M^4 D (1 + t) e^{-\nu_2 t} \sup_{s \in [0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right], \end{aligned}$$

furthermore, there are $c_3, p_3 > 0$ such that $C_3 \leq c_3 \lambda_3^{-p_3}$.

To control the boundary term $\mathscr{I}_4^D$ we use the boundary conditions of Equation (3.3.2), the Duhamel formula again, and we recall that there is no more collision through the boundary trajectory. We have then that

$$f(t_2, x_2, \mathcal{V}_{x_2}u) = S_{\mathscr{T}}(t_2) f_0(x_2, \mathcal{V}_{x_2}u) + \int_0^{t_2} S_{\mathscr{T}}(t_2 - s) K f(s, x_2, \mathcal{V}_{x_2}u) \mathrm{d}s \\ + \int_0^{t_2} S_{\mathscr{T}}(t_2 - s) G(s, x_2, \mathcal{V}_{x_2}u) \mathrm{d}s,$$

thus $\mathscr{I}_4^D \leq \mathscr{I}_1^{D,S} + \mathscr{I}_2^{D,S} + \mathscr{I}_3^{D,S}$ with

$$\begin{aligned} \mathscr{I}_1^{D,S} &= \int_{\mathscr{U}_2} e^{-\nu_0(t-t_2)} |S_{\mathscr{T}}(t_2) f_0(x_2, \mathcal{V}_{x_2}u)| \, (n(x_1)\cdot u)_+ \mathrm{d}u, \\ \mathscr{I}_2^{D,S} &= \int_{\mathscr{U}_2} e^{-\nu_0(t-t_2)} \int_0^{t_2} |S_{\mathscr{T}}(t_2 - s) K f(s, x_2, \mathcal{V}_{x_2}u)| \mathrm{d}s \, (n(x_1)\cdot u)_+ \mathrm{d}u, \\ \mathscr{I}_3^{D,S} &= \int_{\mathscr{U}_2} e^{-\nu_0(t-t_2)} \int_0^{t_2} |S_{\mathscr{T}}(t_2 - s) G(s, x_2, \mathcal{V}_{x_2}u)| \mathrm{d}s \, (n(x_1)\cdot u)_+ \mathrm{d}u. \end{aligned}$$

Proceeding as above we have that

$$\mathscr{I}_1^{D,S} \leq M^4 D \, e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and using (K2) we have that for every $\nu_2 \in (0, \nu_0)$ there holds

$$\mathscr{I}_3^{D,S} \leq M^4 D \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].$$

Moreover, using Proposition 3.9 we have that for every $\lambda_4 > 0$ there is $\varepsilon_7^6 = \varepsilon_6(\lambda_4, T) > 0$ and a constant $C_4 = C_4(\lambda_4) > 0$, such that for every $\varepsilon \in (0, \min(\varepsilon_7^2, \varepsilon_7^3, \varepsilon_7^4, \varepsilon_7^5, \varepsilon_7^6))$ and every $\nu_2 \in (0, \nu_0)$ there holds

$$\mathscr{I}_2^D \leq \lambda_4 M^4 D (t + t^2) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + C_4 M^4 D \, t e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ + C_4 M^4 D \, t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s} \|f_s\|_{\mathcal{H}}] + C_4 M^4 D (1+t) e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right],$$

furthermore, there are $c_4, p_4 > 0$ such that $C_4 \leq c_4 \lambda_4^{-p_4}$. Altogether we have obtained that

$$\omega_1 |f(t,x,v)| \leq \Big(\lambda_1 + \lambda_2 + 2\eta_1 M^3 D + \lambda_3 M^4 D + \lambda_4 M^4 D \Big) (t+t^2) e^{-\nu_0 t} \sup_{s\in(0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] \\ + \Big(1 + C_1 + M^4 D + C_3 M^4 D + M^4 D + C_4 M^4 D \Big) (1+t) e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ + \Big(C_1 + C_3 M^4 D + C_4 M^4 D \Big) t^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s} \|f_s\|_{\mathcal{H}}] \\ + \Big(1 + C_1 + M^4 D + C_3 M^4 D + M^4 D + C_4 M^4 D \Big) (1+t) \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].$$

We take then $\lambda_1 = \lambda_2 = \lambda/5$, $\eta_1 = \lambda/(10M^3 D)$ and $\lambda_3 = \lambda_4 = \lambda/(5M^4 D)$, we define

$$C_D = 1 + C_1 + M^4 D + C_3 M^4 D + M^4 D + C_4 M^4 D,$$

thus there is $p' > 0$ such that $C_D \lesssim \lambda^{-p'}$.

We set $\varepsilon_7 = \min(\varepsilon_7^1, \varepsilon_7^2, \varepsilon_7^3, \varepsilon_7^4, \varepsilon_7^5, \varepsilon_7^6)$ and we conclude the proof by putting together the estimates from both of the possible scenarios. $\square$

3.4.4 Weighted L^∞ control for solutions of Equation (3.3.2)

In this subsection we use the estimate obtained in Proposition 3.10 to deduce a weighted L^∞ control on the solutions of Equation (3.3.2).

Proposition 3.11. *Consider Assumption (RH2) to hold, $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function, and let f be a solution of Equation* (3.3.2)*. There is $\varepsilon_7 = \varepsilon_7(T) > 0$ such that for every $\varepsilon \in (0, \varepsilon_7)$ there holds*

$$\|f_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq C(1+T)(1+T+T^2)^p e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C(1+T+T^2)^p T^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s}\|f_s\|_{\mathcal{H}}]$$
$$+ C(1+T+T^2)^p(1+T)e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right],$$

for every $t \in [0,T]$, and some universal constants $C, p > 0$.

Proof. We follow the proof of Proposition 3.14. Let $\mathfrak{M} > 0$ and consider a point

$$(t,x,v) \in (0,T) \times \Omega^\varepsilon \times \{v \in \mathbb{R}^3,\, |v| \leq \mathfrak{M}\}$$

such that $v \notin S_x \cup W_{t,x}$. Applying then Proposition 3.10 we have that for every $\lambda > 0$ there is $\varepsilon_7 = \varepsilon_6(\lambda, T)$ such that for every $\varepsilon \in (0, \varepsilon_7)$ and every $\nu_2 \in (0, \nu_0)$ there holds

$$\omega_1(v)|f(t,x,v)| \leq \lambda(1+t+t^2)e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + C(1+t)e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}$$
$$+ Ct^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s}\|f_s\|_{\mathcal{H}}] + C(1+t)e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right],$$

for some constant $C > 0$ such that $C \lesssim \lambda^{-p}$ for some $p > 0$.

Using now Lemmas 3.4 and 3.5 we have that the sets S_x and $W_{t,x}$ have Lebesgue measure zero, therefore we may take the $L^\infty_v(B_{\mathfrak{M}})$ norm in the previous inequality followed by the supremum in $\mathfrak{M}$ due to the fact that ε_7 and the constants given by Proposition 3.10 do not depend on $\mathfrak{M}$, and finally taking the $L^\infty_x(\Omega^\varepsilon)$ norm on the above estimate we obtain

$$\|f_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq \lambda(1+t+t^2)e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + C(1+t)e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}$$
$$+ Ct^2 e^{-\nu_0 t} \sup_{s\in[0,t]} [e^{\nu_0 s}\|f_s\|_{\mathcal{H}}] + C(1+t)e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|G_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right].$$

We conclude by choosing $\lambda = (2(1+t+t^2))^{-1}$ and following the ideas from the proof of Proposition 3.14. □

3.4.5 Proof of Proposition 3.1

The proof follows exactly the proof of Proposition 3.1 by using Proposition 3.11 in the place of Proposition 3.14. □

3.5 A priori estimates for weakly confining weights

We consider the function $G : \mathcal{U}^\varepsilon \to \mathbb{R}$, and during this section we study the following evolution equation

$$\begin{cases} \partial_t f_1 &= \mathscr{T} f_1 + \mathcal{A}_\delta f_1 + G & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_1 &= \mathscr{R}\gamma_+ f_1 & \text{on } \Gamma^\varepsilon_- \\ f_{1,t=0} &= f_0 & \text{in } \mathcal{O}^\varepsilon, \end{cases} \tag{3.5.1}$$

where we recall that $\mathscr{T}$ is given by (3.1.20) and $\mathcal{A}_\delta$ is defined in (3.1.26). We dedicate this section to prove the following result.

Proposition 3.1. *Consider either Assumption (RH1) or (RH2) to hold, $\omega_0 \in \mathfrak{W}_0$ a weakly confining admissible weight function, and let f_1 be a solution of Equation* (3.5.1)*. There are constructive constants $\varepsilon_8, \delta_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_8)$ and every $\delta \in (0, \delta_0)$ there holds*

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \le Ce^{-\frac{\nu_0}{2}t}\left(\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + \sup_{s\in[0,t]}\left[e^{\frac{\nu_0}{2}s}\|G_s\|_{L^\infty_{\omega_0\nu^{-1}}(\mathcal{O}^\varepsilon)}\right]\right) \qquad \forall t \ge 0, \quad (3.5.2)$$

for some universal constant $C > 0$, independent of ε.

Remark 3.2. The proof of Proposition 3.1 follows the arguments developed during Sections 3.3 and 3.4. We pay particular attention to polynomial weights in order to determine the validity of the lower bounds $q^\star_\iota$ on their degree.

3.5.1 Preliminary lemmas

We present in this subsection [102, Lemma 4.12] providing dissipative estimates on the operator $\mathcal{A}_\delta$.

Lemma 3.3. *Let ω be either a polynomial or a stretched exponential admissible weight function. Assume furthermore that $h \in L^r_{\nu\omega}(\mathcal{O}^\varepsilon)$ for any $r \in [1, \infty]$. There holds*

$$\|\mathcal{A}_\delta h\|_{L^r_\omega(\mathcal{O}^\varepsilon)} \le \varpi_{\delta,\omega}\|h\|_{L^r_{\nu\omega}(\mathcal{O}^\varepsilon)}, \qquad (3.5.3)$$

for a constructive constant $\varpi_{\delta,\omega} > 0$ where

- *$\varpi_{\delta,\omega} \to 0$ as $\delta \to 0$, when ω is a stretched exponential admissible weight function,*
- *$\varpi_{\delta,\omega} \to \left(\frac{4}{q+2}\right)^{1/r}\left(\frac{4}{q-1}\right)^{1-1/r} < 1$ as $\delta \to 0$, when $\omega(v) = \langle v\rangle^q$ for $q > q^*_r$ and where $q^*_r := (3 + \sqrt{49 - 48/r})/2$. Furthermore, we use the convention $1/\infty = 0$.*

3.5.2 Weighted L^∞ estimate in smooth domains

We prove the following long-time behavior result under Assumption (RH1).

Proposition 3.4. *We consider Assumption (RH1) to hold, $\omega_0 \in \mathfrak{W}_0$ a weakly confining admissible weight function, and let f_1 be a solution to Equation* (3.5.1)*. There are constructive constants $\varepsilon_9 > 0$ and $\delta_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_9)$ and every $\delta \in (0, \delta_1)$ there holds*

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \le e^{-\frac{\nu_0}{2}t}\left(\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + C\sup_{s\in[0,t]}\left[e^{\frac{\nu_0}{2}s}\|G_s\|_{L^\infty_{\omega_0\nu^{-1}}(\mathcal{O}^\varepsilon)}\right]\right) \qquad \forall t \ \ge 0,$$

for some constant $C > 0$, independent ε.

Proof. We define $\widetilde{G} = \mathcal{A}_\delta f_1 + G$, we take an arbitrary $T > 0$ to be defined later, and we proceed in several steps by following the ideas of Proposition 3.13, Proposition 3.14 and Proposition 3.1.

Step 1. Let $(t,x,v) \in \mathcal{U}_T^\varepsilon$ such that $v \notin S_x(v)$, we denote (t_1,x_1,v_1) the first collision through the backwards trajectory starting at (t,x,v) as defined in (3.1.25). The Duhamel formula then gives

$$f_1(t,x,v) = \mathbf{1}_{\{t_1\le 0\}} S_{\mathscr{T}}(t) f_0(x,v) + \int_{\max\{0,t_1\}}^t S_{\mathscr{T}}(t-s)\widetilde{G}_s\,\mathrm{d}s + \mathbf{1}_{\{t_1>0\}} e^{-\nu(v)(t-t_1)} f_1(t_1,x_1,v).$$

We multiply both sides of the previous equality by $\omega_0 = \omega_0(v)$, we define $\sigma = \max(0,t_1)$ and we get that

$$\begin{aligned} \omega_0|f_1(t,x,v)| &\le \omega_0|S_{\mathscr{T}}(t)f_0(x,v)| + \omega_0|S_{\mathscr{T}} *_\sigma \widetilde{G}(t,x,v)| + \omega_0|e^{-\nu(v)(t-t_1)}f_1(t_1,x_1,v)| \\ &=: \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3, \end{aligned}$$

and we will control each term separately. We first have that

$$\mathscr{I}_1 = \omega_0(v)|S_{\mathscr{T}}(t)f_0(x,v)| \le e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)},$$

and using (K2) we further deduce that

$$\mathscr{I}_2 = \omega_0(v)|S_{\mathscr{T}} *_\sigma \widetilde{G}(t,x,v)| \le \frac{\nu_1}{\nu_0-\nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|\widetilde{G}_s\|_{L^\infty_{\omega_0\nu^{-1}}(\mathcal{O}^\varepsilon)} \right],$$

for every $\nu_2 \in (0,\nu_0)$. To control then the boundary term $\mathscr{I}_3$ we remark that $(x_1,v) \in \Sigma_-^\varepsilon$, and $x_1 \notin \mathfrak{S}$, thus the Maxwell boundary condition holds and

$$\begin{aligned} f_1(t_1,x_1,v) &= (1-\iota^\varepsilon(x_1))\mathscr{S} f_1(t_1,x_1,v) + \iota^\varepsilon(x_1)\mathscr{D} f_1(t_1,x_1,v) \\ &= (1-\iota^\varepsilon(x_1)) f_1(t_1,x_1,\mathcal{V}_{x_1}v) + \iota^\varepsilon(x_1)\mathscr{M}(v)\widetilde{f}_1(t_1,x_1). \end{aligned}$$

Using then that $\omega_0\mathscr{M} \le \mathfrak{C}_0$ for some constant $\mathfrak{C}_0 > 0$, there holds

$$\begin{aligned} \mathscr{I}_3 &\le (1-\iota_0)\, e^{-\nu(v)(t-t_1)}\omega_0(v)|f(t_1,x_1,\mathcal{V}_{x_1}v)| + e^{-\nu(v)(t-t_1)}\mathfrak{C}_0 \left|\widetilde{f}_1(t_1,x_1)\right| \\ &\le (1-\iota_0)e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right] + \mathfrak{C}_0\mathscr{I}^D \end{aligned}$$

where we have defined $\mathscr{I}^D := e^{-\nu(v)(t-t_1)}|\widetilde{f}_1(t_1,x_1)|$.

Step 2. We use now (K4) on the diffusive reflection term $\mathscr{I}^D$ and we have that for every $\lambda > 0$ there are $M, \eta > 0$ such that

$$\begin{aligned} \mathscr{I}^D \le \lambda e^{-\nu_0 t} &\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right] \\ &+ e^{-\nu(v)(t-t_1)}\int_{|u|\le M,\, |n(x_1)\cdot u|>\eta} |f_1(t_1,x_1,u)|(n(x_1)\cdot u)_+ \mathrm{d}u. \end{aligned} \quad (3.5.4)$$

Furthermore, by arguing as during the Step 2 of Proposition 3.13 we deduce that we may apply Lemma 3.4 and there is $\varepsilon_9^1 = \varepsilon_R(\eta, M, T)$ such that for every $\varepsilon \in (0,\varepsilon_9^1)$ there is no bounce against the boundary through the backwards trajectory starting at (t_1,x_1,u), and given by (3.1.25). Therefore the Duhamel formula gives that

$$f_1(t_1,x_1,u) = S_{\mathscr{T}}(t_1)f_0(x_1,u) + S_{\mathscr{T}} *_{t_1} \widetilde{G}(x_1,u). \quad (3.5.5)$$

Combining (3.5.4) and (3.5.5) we have that

$$\mathscr{I}^D \le \lambda e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_{1,s}\|_{L^\infty_{\omega}(\mathcal{O}^\varepsilon)}\right] + \mathscr{I}_1^D + \mathscr{I}_2^D,$$

with

$$\begin{aligned}\mathscr{I}_1^D &= e^{-\nu(v)(t-t_1)}\int_{|u|\le M,\,|n(x_1)\cdot u|>\eta}|S_{\mathscr{T}}(t_1)f_0(x_1,u)|(n_x\cdot u)_+\mathrm{d}u,\\ \mathscr{I}_2^D &= e^{-\nu(v)(t-t_1)}\int_{|u|\le M,\,|n(x_1)\cdot u|>\eta}|S_{\mathcal{B}}*_{t_1}\widetilde{G}(x_1,u)|(n_{x_1}\cdot u)_+\mathrm{d}u.\end{aligned}$$

On the one hand, using the fact that $\omega_0\ge 1$ we have that

$$\mathscr{I}_1^D \le e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\int_{|u|\le M}|u|\mathrm{d}u \le DM^4e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}.$$

On the other hand, to control $\mathscr{I}_3$ we use (K2) so that for every $\nu_2\in(0,\nu_0)$ there holds

$$\mathscr{I}_2^D\le\frac{\nu_1}{\nu_1-\nu_0}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|\widetilde{G}_s\|_{L^\infty_{\omega_0\nu^{-1}}(\mathcal{O}^\varepsilon)}\right]e^{-\nu_0(t-t_1)}e^{-\nu_2 t_1}\int_{|u_*|\le M}\omega_0^{-1}(u)|u|du,$$

and we will provide a precise control on the last integral.

First, when ω_0 is an admissible polynomial weight function as considered in Subsection 3.1.2, then $\omega_0(v)=\langle v\rangle^q$ and we compute

$$\begin{aligned}\int_{|u_*|\le M}\omega_0^{-1}(u)|u|du &= \int_{|u_*|\le M}\langle u\rangle^{-q}|u|\mathrm{d}u=\int_0^{2\pi}\int_0^{\pi}\int_0^{M}\frac{r^3}{(1+r^2)^{q/2}}\sin(\beta)\mathrm{d}r\mathrm{d}\beta\mathrm{d}\alpha\\ &= 4\pi\int_0^M\frac{r(1+r^2-1)}{(1+r^2)^{q/2}}\mathrm{d}r\le 4\pi\int_0^M\frac{r}{(1+r^2)^{-1+q/2}}\mathrm{d}r\\ &\le \frac{4\pi}{q-4}\left[-(1+r^2)^{2-q/2}\right]_0^M\le\frac{4\pi}{q-4}\end{aligned}$$

where we have used the spherical change of variables on the first line and we have used that $q>q^\star_\iota\ge 5$ on the last line.

Second, when ω_0 is a weakly confining admissible weight function, which is not a polynome, we easily deduce that there is a constant $C_{\omega_0}>1$ such that

$$\int_{|u_*|\le M}\omega_0^{-1}(u)|u|du\le C_{\omega_0}.$$

We then have obtained that for any weakly confining admissible weight function ω_0 there is a constant $C_{\omega_0}>0$ such that

$$\mathscr{I}_2^D\le C_{\omega_0}\left(\frac{\nu_1}{\nu_1-\nu_0}\right)e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|\widetilde{G}_s\|_{L^\infty_{\omega_0\nu^{-1}}(\mathcal{O}^\varepsilon)}\right],$$

where $C_{\omega_0}=4\pi/(q-4)$ when $\omega_0(v)=\langle v\rangle^q$, an polynomial admissible weight function. Altogether we have obtained that for every $\varepsilon\in(0,\varepsilon_8^1)$ there holds

$$\begin{aligned}\mathscr{I}_3 &\le (1-\iota_0+\mathfrak{C}_0\lambda)e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right]+\mathfrak{C}_0\mathscr{I}_1^D+\mathfrak{C}_0\mathscr{I}_2^D\\ &\le (1-\iota_0+\lambda)e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right]+D\mathfrak{C}_0M^4e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\\ &\quad +C_{\omega_0}\mathfrak{C}_0\frac{\nu_1}{\nu_0-\nu_2}e^{-\nu_2 t}\sup_{s\in[0,t]}\left[e^{\nu_2 s}\|\widetilde{G}_s\|_{L^\infty_{\omega_0\nu^{-1}}(\mathcal{O}^\varepsilon)}\right],\end{aligned}$$

for every $\nu_2\in(0,\nu_0)$.

Step 3. Putting together all the previous estimates we have that $\varepsilon \in (0, \varepsilon_9^1)$ there holds

$$\begin{aligned}
\omega_0(v)|f_1(t,x,v)| &\leq \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 \\
&\leq (1 - \iota_0 + \mathfrak{C}_0\lambda)e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right] + (1 + \mathfrak{C}_0 D M^4)e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \\
&\quad + \frac{\nu_1}{\nu_0 - \nu_2}\left(1 + \mathfrak{C}_0 C_{\omega_0}\right) e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s}\|\widetilde{G}_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right].
\end{aligned}$$

We choose then $\varepsilon_9 = \varepsilon_9^1$ and $\nu_2 = \nu_0/2$, therefore we obtain that

$$\begin{aligned}
\omega_0(v)|f_1(t,x,v)| &\leq \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 \\
&\leq (1 - \iota_0 + \mathfrak{C}_0\lambda)e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right] + (1 + \mathfrak{C}_0 D M^4)e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \\
&\quad + 2\frac{\nu_1}{\nu_0}\left(1 + \mathfrak{C}_0 C_{\omega_0}\right) e^{-\frac{\nu_0}{2} t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2} s}\|\widetilde{G}_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right],
\end{aligned}$$

We observe then that, since $S_x(v)$ is a set of zero Lebesgue measure as discussed on Proposition 3.13, we can take the $L^\infty(\mathcal{O}^\varepsilon)$ norm on both sides of the estimate above and we deduce that

$$\begin{aligned}
\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} &\leq (1 - \iota_0 + \mathfrak{C}_0\lambda)e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right] + (1 + \mathfrak{C}_0 D M^4)e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \\
&\quad + 2\frac{\nu_1}{\nu_0}\left(1 + \mathfrak{C}_0 C_{\omega_0}\right) e^{-\frac{\nu_0}{2} t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2} s}\|\widetilde{G}_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right].
\end{aligned}$$

Step 4. Using that $\widetilde{G} = \mathcal{A}_\delta f_1 + G$ on the estimate obtained on the Step 3 of this proof we have that

$$\begin{aligned}
\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} &\leq (1 - \iota_0 + \mathfrak{C}_0\lambda)e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s}\|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right] + (1 + \mathfrak{C}_0 D M^4)e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \\
&\quad + 2\frac{\nu_1}{\nu_0}\left(1 + \mathfrak{C}_0 C_{\omega_0}\right) e^{-\frac{\nu_0}{2} t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2} s}\|\mathcal{A}_\delta f_{1,s}\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right] \\
&\quad + 2\frac{\nu_1}{\nu_0}\left(1 + \mathfrak{C}_0 C_{\omega_0}\right) e^{-\frac{\nu_0}{2} t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2} s}\|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right],
\end{aligned}$$

and we proceed to use the dissipative properties of $\mathcal{A}_\delta$ and to fix the constants for each type of weight function.

- **Weak inverse gaussian weights:** If $\omega_0(v) = e^{\zeta|v|^2}$, with $\zeta \in (0, 1/4]$, we recall that $\mathcal{A}_\delta f_1 = K((1 - \chi_\delta)f)$ as defined in (3.1.26), therefore arguing as during the proof of (K1) we deduce that for every $N > 0$ there is $\delta_1^1 = m(N)^{-1}$, where $m(N)$ is given by the Step 2 of the proof of (K1) in Lemma 3.6, such that for every $\delta \in (0, \delta_1^1)$ there holds

$$\|\mathcal{A}_\delta f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq \frac{1}{N}\|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}.$$

We then choose

$$N = 8\frac{\nu_1}{\nu_0 \iota_0}\left(1 + \mathfrak{C}_0 C_{\omega_0}\right), \qquad \lambda = \iota_0/(4\mathfrak{C}_0),$$

and we deduce that

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq C e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + C e^{-\frac{\nu_0}{2} t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2} s}\|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right].$$

for some constant $C > 0$.

- **Stretched exponential weights:** If $\omega_0(v) = e^{\zeta \langle v \rangle^s}$, with $s \in (0,2)$ and $\zeta > 0$, we observe that Lemma 3.3 implies that

$$\|\mathcal{A}_\delta f_{1,s}\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \leq \varpi_{\delta,\omega_0} \|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}, \tag{3.5.6}$$

and we further have that we may choose $\delta_1^2 > 0$ such that for every $\delta \in (0, \delta_1^2)$ there holds

$$2\frac{\nu_1}{\nu_0} \left(1 + \mathfrak{C}_0 C_{\omega_0}\right) \varpi_{\delta,\omega_0} \leq \frac{\iota_0}{4},$$

and choosing $\lambda = \iota_0/(4\mathfrak{C}_0)$ we deduce that

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq Ce^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + Ce^{-\frac{\nu_0}{2}t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2}s} \|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].$$

for some constant $C > 0$.

- **Polynomial weights:** If $\omega_0 = \langle v \rangle^q$, with $q > q_\iota^\star$ as defined in (3.1.7), using again Lemma 3.3, we obtain (3.5.6) with $\varpi_{\delta,\omega_0} \to 4/(q-1)$ as $\delta \to 0$. We observe then that since $q > q_\iota^\star$, there holds

$$2\frac{\nu_1}{\nu_0} \left(1 + \frac{4\pi}{q-4}\mathfrak{C}_0\right) \frac{4}{q-1} < \iota_0.$$

We may choose then $\delta_1^3 > 0$ small enough such that for every $\delta \in (0, \delta_1^3)$ there holds

$$2\frac{\nu_1}{\nu_0} \left(1 + C_{\omega_0}\frac{4\pi}{q-4}\right) \varpi_{\delta,\omega_0} \leq \alpha\iota_0,$$

for some $\alpha \in (0,1)$. Taking then $\lambda = (1-\alpha)\iota_0/2$ we deduce again that

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq Ce^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + Ce^{-\frac{\nu_0}{2}t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2}s} \|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right],$$

for some constant $C > 0$.

We then have obtained that

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq C \left(e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + e^{-\frac{\nu_0}{2}t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2}s} \|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \right) \qquad \forall t \in [0,T],$$

when choosing $\delta_1 = \delta_1^1$ in the case of weak inverse gaussian weight functions, $\delta_1 = \delta_1^2$ in the case of stretched exponential weight functions, and $\delta_1 = \delta_1^3$ in the case of polynomial weight functions.

Step 5. We conclude the proof by taking $T > 0$ such that $Ce^{-\frac{\nu_0}{2}T} \leq 1/2$ and extending the result for all time by repeating the analysis from the Step 3 of the proof of Proposition 3.1 (in a much simpler setting). □

3.5.3 Weighted L^∞ estimate in cylindrical domains

We will now prove a similar long-time behavior result to the one on the previous section under Assumption (RH2).

Proposition 3.5. *Consider Assumption (RH2) to hold, $\omega_0 \in \mathfrak{W}_0$ a weakly confining admissible weight function, and let f_1 be a solution to Equation* (3.5.1)*. There are constructive constants $\varepsilon_{10} > 0$ and $\delta_2 > 0$ such that for every $\varepsilon \in (0, \varepsilon_{10})$ and every $\delta \in (0, \delta_2)$ there holds*

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \le e^{-\frac{\nu_0}{2}t} \left(\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + C \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2}s} \|G_s\|_{L^\infty_{\omega_0\nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \right) \qquad \forall t \ge 0,$$

for some constant $C > 0$, independent of ε.

Proof. The proof is a repetition of the proof of Proposition 3.4, by using the stretching method argument for cylindrical domains as during the proof of Proposition 3.10, Proposition 3.11 and Proposition 3.1.

Step 1. (Pointwise estimate) We define $\widetilde{G} = \mathcal{A}_\delta f_1 + G$, and we fix $(t, x, v) \in [0, T] \times \Omega^\varepsilon \times \mathbb{R}^3$, such that $v \notin S_x$, $v \notin W_{t,x}$ and $|v| \le \mathfrak{M}$ for some arbitrary $T, \mathfrak{M} > 0$. We remark then that Lemma 3.2 implies that there are two possible scenarios to consider.

- *Case 1.* There are only specular reflections through the backwards trajectories within the time interval $[0, T]$. In particular, there is $\mathfrak{N} = \mathfrak{N}(T, x, \mathfrak{M}) < \infty$, given by Lemma 3.2, such that there are no more than $\mathfrak{N}$ consecutive specular reflections through the backwards trajectories.
- *Case 2.* There is at least one reflection against the diffusive boundary subsets before completing $\mathfrak{N}$ specular reflections.

We treat now each case separately.

Case 1. (Purely specular reflection) We recall that there are at most $\mathfrak{N}$ collisions against the boundary through the specular reflection boundary condition starting at (t, x, v), where $\mathfrak{N}$ is given by Lemma 3.2. Then the iterated Duhamel formula gives

$$f(t, x, v) = e^{-\nu(v)(t-t_\mathfrak{N})} f(t_\mathfrak{N}, x_\mathfrak{N}, v_\mathfrak{N}) + \int_0^t \sum_{j=1}^{\mathfrak{N}} \mathbf{1}_{t_j \le s \le t_{j-1}}(s) S_\mathscr{T}(t_j - s) \widetilde{G}(s, x_{j-1}, v_{j-1}) \mathrm{d}s$$

where we have set $(t_0, x_0, v_0) = (t, x, v)$ and we have defined (t_j, x_j, v_j) as in (3.4.2) for $j \in [\![1, \mathfrak{N}]\!]$.

Using that there are no more collision against the boundary, we use again the Duhamel formula and we obtain

$$f(t, x, v) = e^{-\nu(v)(t-t_\mathfrak{N})} S_\mathscr{T}(t_\mathfrak{N}) f_0(x_\mathfrak{N}, v_\mathfrak{N}) + \int_0^t \sum_{j=1}^{\mathfrak{N}} \mathbf{1}_{t_j \le s \le t_{j-1}}(s) S_\mathscr{T}(t_{j-1} - s) \widetilde{G}(s, x_{j-1}, v_{j-1}) ds.$$

Arguing now as during the proof of (K2), and using that $\sum_{j=1}^{\mathfrak{N}} \mathbf{1}_{t_j \le s \le t_{j-1}}(s) = \mathbf{1}_{[0,t]}$, we deduce that

$$\omega_1(v)|f(t, x, v)| \le e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + t \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|\widetilde{G}_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)} \right],$$

We conclude this case by emphasizing that the above estimate is independent of the choices of $T > 0$ and $\mathfrak{M} > 0$.

Case 2. (Diffussive reflection) We assume without loss of generality that the diffusive collision against the diffusive boundary through the backwards trajectory happens at the first collision. Otherwise, we just repeat the Duhamel formulation through the specular reflections like performed on the Step 1, until the diffusive reflection happens and then we proceed as follows.

We denote (t_1, x_1, v_1) the first collision through the backwards trajectory as given by (3.1.25), and we remark that $x_1 = x - v(t-t_1) \in \Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon$. Using then the Duhamel formula we obtain

$$f(t,x,v) = \mathbf{1}_{\{t_1 \le 0\}} S_{\mathscr{T}}(t) f_0(x,v) + \int_{\max\{0,t_1\}}^t S_{\mathscr{T}}(t-s)\widetilde{G}(s,x,v)\mathrm{d}s + e^{-\nu(v)(t-t_1)} f(t_1,x_1,v).$$

We multiply the above expression by ω_1 to get

$$\begin{aligned} |\omega_1(v) f(t,x,v)| &\le \omega_1(v) S_{\mathscr{T}}(t)\left(|f_0(x,v)|\right) + \omega_1(v)|S_{\mathscr{T}} * \widetilde{G}(t,x,v)| + e^{-\nu(v)(t-t_1)}\omega_1(v)|f(t_1,x_1,v)| \\ &=: \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3, \end{aligned}$$

and we proceed to control each of these terms separately. On the one hand, we bound

$$\mathscr{I}_1 = S_{\mathscr{T}}(t)\left(\omega_1(v)|f_0(x,v)|\right) \le e^{-\nu_0 t}\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and on the other hand, by using (K2) we have that for every $\nu_2 \in (0,\nu_0)$ there holds

$$\mathscr{I}_2 = \omega_1(v)|S_{\mathscr{T}} * \widetilde{G}(t,x,v)| \le \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]}\left[e^{\nu_2 s}\|\widetilde{G}_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)}\right].$$

We then control the boundary term $\mathscr{I}_3$, and since we have assumed the reflection at the boundary to be diffusive, then using (K4) we have that for every $\lambda_1 > 0$ there are $M, \eta > 0$ such that

$$\begin{aligned} \mathscr{I}_3 &\le e^{-\nu(v)(t-t_1)}\mathfrak{C}_0 \int_{\mathbb{R}^3} |f(t_1,x_1,u)|(n_{x_1}\cdot u)_+ du \\ &\le \lambda_1 \mathfrak{C}_0 e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right] + \mathfrak{C}_0 \mathscr{I}_0^D, \end{aligned}$$

where we have used that $\omega_1 \mathscr{M} \le \mathfrak{C}_0$, for some constant $\mathfrak{C}_0 > 0$, and we have defined

$$\mathscr{I}_0^D := e^{-\nu(v)(t-t_1)}\int_{\{|u|\le M,\, |n(x_1)\cdot u| > \eta\}} |f(t_1,x_1,u)|\,(n(x_1)\cdot u)_+ \mathrm{d}u.$$

We then use Lemma 3.8 and we obtain that there is $\varepsilon_{10}^1 = \varepsilon_D(\eta, M, T) = 2L(MT)^{-1}$ such that for every $\varepsilon \in (0, \varepsilon_{10}^1)$ there are no more diffusion collisions through the backwards trajectory.

For any $\eta_1 > 0$, we define now the sets $\mathscr{U}_1 = \{|u_*| \le M,\, |n_{x_1}\cdot u_*| > \eta, |\mathfrak{n}(x_1)\cdot u_*| \le \eta_1\}$, $\mathscr{U}_2 = \{|u_*| \le M,\, |n_{x_1}\cdot u_*| > \eta, |\mathfrak{n}(x_1)\cdot u_*| > \eta_1\}$, and we have that $\mathscr{I}_0^D = \mathscr{I}^{D,1} + \mathscr{I}^{D,2}$ with

$$\begin{aligned} \mathscr{I}^{D,1} &= e^{-\nu(v)(t-t_1)}\int_{\mathscr{U}_1} |f(t_1,x_1,u)|\,(n(x_1)\cdot u)_+ \mathrm{d}u, \\ \mathscr{I}^{D,2} &= e^{-\nu(v)(t-t_1)}\int_{\mathscr{U}_2} |f(t_1,x_1,u)|\,(n(x_1)\cdot u)_+ \mathrm{d}u. \end{aligned}$$

By arguing as during the Step 1 of the proof of Proposition 3.9, we perform the change of variable $u^{\mathrm{par}} = (\mathfrak{n}(x_1)\cdot u)\mathfrak{n}(x_1)$ and its perpendicular direction $u^\perp = u - u^{par}$ such that $u = u^{\mathrm{par}} + u^\perp$, and we obtain that

$$\begin{aligned} \mathscr{I}^{D,1} &\le M e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right]\int_{|u^\perp|\le M}\mathrm{d}u^\perp \int_{-\eta_1}^{\eta_1}\mathrm{d}u^{\mathrm{par}} \\ &\le 2\eta_1 M^3 D e^{-\nu_0 t}\sup_{s\in[0,t]}\left[e^{\nu_0 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right]. \end{aligned}$$

We denote then (t_2, x_2, v_2) as the first collision against the boundary starting at (t_1, x_1, u) through the backwards trajectory as defined in (3.1.25). We emphasize that due to the previous choice of ε_{10}^1, we have that $x_2 \in \Lambda_3^\varepsilon$.

Moreover, from the fact that $|\mathfrak{n}(x_1) \cdot u| > \eta_1$, Lemma 3.8 implies that there is $A_1 > 0$ independent from x_1 such that

$$|n(x_2) \cdot \mathcal{V}_{x_2} u| = |n(x_2) \cdot u| \geq A_1.$$

Using Lemma 3.7 there is $\varepsilon_{10}^2 = \varepsilon_S(A_1, M, T)$ such that for every $\varepsilon \in (0, \min(\varepsilon_{10}^1, \varepsilon_{10}^2))$ there is no more collision against the specular boundary condition through the specular reflection. In particular, we have that there are no more collision against the boundary through the backwards trajectory after the collision in x_2.

Using the Duhamel formula we have that

$$f(t_1, x_1, u) = \mathbf{1}_{\{t_2 \leq 0\}} S_{\mathscr{T}}(t_1) f_0(x_1, u) + \int_{\max\{0, t_2\}}^{t_1} S_{\mathscr{T}}(t_1 - s) \widetilde{G}(s, x_1, u) \mathrm{d}s \\ + e^{-\nu(v)(t_1 - t_2)} f(t_2, x_2, \mathcal{V}_{x_2} u).$$

Therefore $\mathscr{I}^{D,2} \leq \mathscr{I}_1^D + \mathscr{I}_2^D + \mathscr{I}_3^D$ with

$$\begin{aligned} \mathscr{I}_1^D &= e^{-\nu(v)(t-t_1)} \int_{\mathscr{U}_2} |S_{\mathscr{T}}(t_1) f_0(x_1, u)| \, (n(x_1) \cdot u)_+ \mathrm{d}u, \\ \mathscr{I}_2^D &= e^{-\nu(v)(t-t_1)} \int_{\mathscr{U}_2} \int_{\max\{0,t_2\}}^{t_1} |S_{\mathscr{T}}(t_1 - s) \widetilde{G}(s, x_1, u)| \mathrm{d}s \, (n(x_1) \cdot u)_+ \mathrm{d}u, \\ \mathscr{I}_3^D &= e^{-\nu(v)(t-t_1)} \int_{\mathscr{U}_2} e^{-\nu(v)(t_1 - t_2)} |f(t_2, x_2, \mathcal{V}_{x_2} u)| \, (n(x_1) \cdot u)_+ \mathrm{d}u. \end{aligned}$$

On the one hand, we bound

$$\mathscr{I}_1^D \leq M^4 D \, e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and on the other hand, by arguing as during the Step 2 of the proof of Proposition 3.4 we have that for every $\nu_2 \in (0, \nu_0)$ there holds

$$\mathscr{I}_2^D \leq C_{\omega_0} \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s \in [0,t]} \left[e^{\nu_2 s} \|\widetilde{G}_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].$$

for some constant $C_{\omega_0} > 0$, where $C_{\omega_0} = 4\pi/(q-4)$, when $\omega_0(v) = \langle v \rangle^q$, is a polynomial admissible weight function.

To control the boundary term $\mathscr{I}_3^D$ we use the boundary conditions of Equation (3.3.2), the Duhamel formula again, and we recall that there is no more collision through the boundary trajectory. We have then that

$$f(t_2, x_2, \mathcal{V}_{x_2} u) = S_{\mathscr{T}}(t_2) f_0(x_2, \mathcal{V}_{x_2} u) + \int_0^{t_2} S_{\mathscr{T}}(t_2 - s) \widetilde{G}(s, x_2, \mathcal{V}_{x_2} u) \mathrm{d}s,$$

thus $\mathscr{I}_3^D \leq \mathscr{I}_1^{D,S} + \mathscr{I}_2^{D,S}$ with

$$\begin{aligned} \mathscr{I}_1^{D,S} &= \int_{\mathscr{U}_2} e^{-\nu_0(t-t_2)} |S_{\mathscr{T}}(t_2) f_0(x_2, \mathcal{V}_{x_2} u)| \, (n(x_1) \cdot u)_+ \mathrm{d}u, \\ \mathscr{I}_2^{D,S} &= \int_{\mathscr{U}_2} e^{-\nu_0(t-t_2)} \int_0^{t_2} |S_{\mathscr{T}}(t_2 - s) \widetilde{G}(s, x_2, \mathcal{V}_{x_2} u)| \mathrm{d}s \, (n(x_1) \cdot u)_+ \mathrm{d}u. \end{aligned}$$

Proceeding as above we have that

$$\mathscr{I}_1^{D,S} \leq M^4 D\, e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)},$$

and for every $\nu_2 \in (0, \nu_0)$ there holds

$$\mathscr{I}_3^{D,S} \leq C_{\omega_0} \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|\widetilde{G}_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].$$

Altogether we have obtained that

$$\begin{aligned} \omega_0 |f(t,x,v)| \leq \Big(\lambda_1 \mathfrak{C}_0 + 2\eta_1 M^3 D\Big)\, e^{-\nu_0 t} \sup_{s\in(0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + \Big(1 + 2\mathfrak{C}_0 M^4 D\Big)\, e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ + (1 + 2\mathfrak{C}_0 C_{\omega_0})\, \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|\widetilde{G}_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right]. \end{aligned}$$

Step 2. (Choice of the parameters) We put together now both estimates from each of the cases treated as during the Step 1 and proceding exactly as during the proof of Proposition 3.11 we obtain that

$$\begin{aligned} \|f_t\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq \Big(\lambda_1 \mathfrak{C}_0 + 2\eta_1 M^3 D\Big)\, e^{-\nu_0 t} \sup_{s\in(0,t]} \left[e^{\nu_0 s} \|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \right] + \Big(2 + 2\mathfrak{C}_0 M^4 D\Big)\, e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \\ + (2 + 2\mathfrak{C}_0 C_{\omega_0})\, \frac{\nu_1}{\nu_0 - \nu_2} e^{-\nu_2 t} \sup_{s\in[0,t]} \left[e^{\nu_2 s} \|\widetilde{G}_s\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right]. \end{aligned}$$

Choosing $\nu_2 = \nu_0/2$ and recalling that $\widetilde{G} = \mathcal{A}_\delta f_1 + G$, the above estimate transforms into

$$\begin{aligned} \|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq\ & (\lambda_1 \mathfrak{C}_0 + 2\eta_1 M^3 D) e^{-\nu_0 t} \sup_{s\in[0,t]} \left[e^{\nu_0 s} \|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \right] + 2(1 + \mathfrak{C}_0 D M^4) e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \\ & + 4\frac{\nu_1}{\nu_0} (1 + \mathfrak{C}_0 C_{\omega_0})\, e^{-\frac{\nu_0}{2} t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2} s} \|\mathcal{A}_\delta f_{1,s}\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \\ & + 4\frac{\nu_1}{\nu_0} (1 + \mathfrak{C}_0 C_{\omega_0})\, e^{-\frac{\nu_0}{2} t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2} s} \|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right], \end{aligned}$$

and we proceed to use the dissipative properties of $\mathcal{A}_\delta$ and to fix the parameters for each type of weight function as during the proof of Proposition 3.4.

- **Weak inverse gaussian weights:** If $\omega_0(v) = e^{\zeta |v|^2}$, with $\zeta \in (0, 1/4]$, we recall that $\mathcal{A}_\delta f_1 = K((1 - \chi_\delta) f)$ as defined in (3.1.26), therefore arguing as during the proof of (K1) we deduce that for every $N > 0$ there is $\delta_2^1 = m(N)^{-1}$, where $m(N)$ is given by the Step 2 of the proof of (K1) in Lemma 3.6, such that for every $\delta \in (0, \delta_2^1)$ there holds

$$\|\mathcal{A}_\delta f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq \frac{1}{N} \|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}.$$

We then choose

$$N = 16 \frac{\nu_1}{\nu_0} (1 + \mathfrak{C}_0 C_{\omega_0}), \quad \lambda_1 = 1/(4\mathfrak{C}_0), \quad \eta_1 = 1(8M^3 D),$$

and we deduce that

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq C e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + C e^{-\frac{\nu_0}{2} t} \sup_{s\in[0,t]} \left[e^{\frac{\nu_0}{2} s} \|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].$$

for some constant $C > 0$.

- **Stretched exponential weights:** If $\omega_0(v) = e^{\zeta \langle v \rangle^s}$, with $s \in (0,2)$ and $\zeta > 0$, we observe that Lemma 3.3 implies that

$$\|\mathcal{A}_\delta f_{1,s}\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \leq \varpi_{\delta,\omega_0} \|f_{1,s}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}, \tag{3.5.7}$$

and we further have that we may choose $\delta_2^2 > 0$ such that for every $\delta \in (0, \delta_2^2)$ there holds

$$4\frac{\nu_1}{\nu_0}\left(1 + \mathfrak{C}_0 C_{\omega_0}\right) \varpi_{\delta,\omega_0} \leq \frac{1}{4},$$

and choosing $\lambda_1 = 1/(4\mathfrak{C}_0)$, and $\eta_1 = 1/(8M^3 D)$ we deduce that

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq C e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + C e^{-\frac{\nu_0}{2} t} \sup_{s \in [0,t]} \left[e^{\frac{\nu_0}{2} s} \|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].$$

for some constant $C > 0$.

- **Polynomial weights:** If $\omega_0 = \langle v \rangle^q$, with $q > q_\iota^\star$ as defined in (3.1.7), we recall that we have $C_{\omega_0} = 4\pi/(q-4)$. Using again Lemma 3.3, we obtain (3.5.7) with $\varpi_{\delta,\omega_0} \to 4/(q-1)$ as $\delta \to 0$. We observe then that since $q > q_\iota^\star$, there holds

$$4\frac{\nu_1}{\nu_0}\left(1 + \frac{4\pi}{q-4}\mathfrak{C}_0\right)\frac{4}{q-1} < 1.$$

We may choose then $\delta_2^3 > 0$ small enough such that for every $\delta \in (0, \delta_2^3)$ there holds

$$2\frac{\nu_1}{\nu_0}\left(1 + \mathfrak{C}_0 \frac{4\pi}{q-4}\right) \varpi_{\delta,\omega_0} \leq \alpha,$$

for some $\alpha \in (0,1)$. Taking then $\lambda = (1-\alpha)/(4\mathfrak{C}_0)$ and $\eta_1 = (1-\alpha)/(8M^3 D)$ we deduce again that

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq C e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + C e^{-\frac{\nu_0}{2} t} \sup_{s \in [0,t]} \left[e^{\frac{\nu_0}{2} s} \|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right].$$

for some constant $C > 0$.

We have then obtained that

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq C \left(e^{-\nu_0 t} \|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + e^{-\frac{\nu_0}{2} t} \sup_{s \in [0,t]} \left[e^{\frac{\nu_0}{2} s} \|G_s\|_{L^\infty_{\omega_0 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \right) \qquad \forall t \in [0,T],$$

when choosing $\delta_2 = \delta_2^1$ in the case of weak inverse gaussian weight functions, $\delta_2 = \delta_2^2$ in the case of stretched exponential weight functions, and $\delta_2 = \delta_2^3$ in the case of polynomial weight functions.

Step 5. We conclude the proof by taking $T > 0$ such that $Ce^{-\frac{\nu_0}{2} T} \leq 1/2$ and extending the result for all time by repeating the analysis from the Step 3 of the proof of Proposition 3.1 (in a much simpler setting). □

3.5.4 Proof of Proposition 3.1

We conclude Proposition 3.1 by choosing $\varepsilon_8 = \varepsilon_9$ and $\delta_0 = \delta_1$, where $\varepsilon_9, \delta_1 > 0$ are given by Proposition 3.4, if Assumption (RH1) holds, or $\varepsilon_8 = \varepsilon_{10}$ and $\delta_0 = \delta_2$, where $\varepsilon_{10}, \delta_2 > 0$ are given by Proposition 3.5, under Assumption (RH2). □

3.6 Well-posedness of transport equations with Maxwell boundary conditions

Before studying well-posedness results for transport equations, we summarize the results obtained during Section 3.3 and Section 3.4 for the solutions of Equation (3.1.21).

Proposition 3.1. *Consider either Assumption (RH1) or (RH2) to hold, $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function, $G : \mathcal{U}^\varepsilon \to \mathbb{R}$ satisfying $\langle\!\langle G_t \rangle\!\rangle_{\mathcal{O}^\varepsilon} = 0$ for every $t \geq 0$, and let f be a solution of Equation* (3.1.21)*. There are constructive constants $\varepsilon_{11}, \theta > 0$ such that for every $\varepsilon \in (0, \varepsilon_{11})$ there holds*

$$\|f_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq Ce^{-\theta\varepsilon^2 t} \left(\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \sup_{s\in[0,t]} \left[e^{\theta\varepsilon^2 s} \|G_s\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \right) \qquad \forall t \geq 0,$$

for some constant $C > 0$, independent of ε.

Proof. The proof is immediate from Proposition 3.1 and Proposition 3.1. We remark that we take $\varepsilon_{11} = \varepsilon_1$ if Assumption (RH1)holds, where ε_1 is given by Proposition 3.1, and $\varepsilon_{11} = \varepsilon_5$ if Assumption (RH2)holds, where ε_5 is given by Proposition 3.1. □

We dedicate then this section to prove the two following well-posedness results.

Theorem 3.6.2. *Assume either Assumption (RH1) or (RH2) to hold, consider $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function, let $f_0 \in L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)$, and $G \in L^\infty_{\omega_1\nu^{-1}}(\mathcal{U}^\varepsilon)$.*

For every $\varepsilon \in (0, \varepsilon_{11})$, where we recall that $\varepsilon_{11} > 0$ is given by Proposition 3.1, there is $f \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$, with a trace function $\gamma f \in L^\infty_{\omega_1}(\Gamma^\varepsilon)$, unique solution to Equation (3.1.21) *in the distributional sense, i.e for any $\varphi \in \mathcal{D}(\mathcal{U}^\varepsilon)$ there holds*

$$\int_{\mathcal{O}^\varepsilon} f(t,\cdot)\, \varphi(t,\cdot)\, \mathrm{d}x\mathrm{d}v - \int_0^t \int_{\mathcal{O}^\varepsilon} f\, K^*\varphi + f\,(\partial_t\varphi - v\cdot\nabla_x\varphi - \nu\,\varphi) + G\,\varphi\, \mathrm{d}x\mathrm{d}v\mathrm{d}s$$
$$= \int_{\mathcal{O}^\varepsilon} f_0(\cdot)\varphi(0,\cdot)\mathrm{d}x\mathrm{d}v + \int_0^t \int_{\Sigma^\varepsilon} \gamma f\, \varphi\, (n_x\cdot v)\mathrm{d}\sigma_x\mathrm{d}v, \quad (3.6.1)$$

for every $t \geq 0$, and where we have defined the formal adjoint operator

$$K^*\varphi(v) := \int_{\mathbb{R}^3} k(v_*, v)\, \varphi(v_*)\, \mathrm{d}v_*. \qquad (3.6.2)$$

Moreover, there holds the representation formula

$$f(t,x,v) = e^{-\nu(v)t} f_0(x,v)\, \mathbf{1}_{t_1\leq 0} + \int_{\max(0,t_1)}^t e^{-\nu(v)(t-s)}\, Kf(s,x,v)\, ds$$
$$+ \int_{\max(0,t_1)}^t e^{-\nu(v)(t-s)}\, G(s,x,v)\, ds + e^{-\nu(v)(t-t_1)} f(t_1,x_1,v)\, \mathbf{1}_{t_1>0}, \quad (3.6.3)$$

pointwise for every $t \geq 0$, almost every $(x,v) \in \mathcal{O}^\varepsilon$, and where $t_1 = t_1(t,x,v)$ and $x_1 = x_1(t,x,v)$ are given by (3.1.25)*. Furthermore, there holds the results from Proposition 3.1.*

Theorem 3.6.3. *Assume either Assumption (RH1) or (RH2) to hold, consider $\omega_0 \in \mathfrak{W}_0$ a weakly confining admissible weight function, let $f_0 \in L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)$, and $G \in L^\infty_{\omega_0\nu^{-1}}(\mathcal{U}^\varepsilon)$.*

For every $\varepsilon \in (0, \varepsilon_8)$ and every $\delta \in (0, \delta_0)$, where we recall that $\varepsilon_8, \delta_0 > 0$ are given by Proposition 3.1, there is $f_1 \in L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)$, with a trace function $\gamma f_1 \in L^\infty_{\omega_0}(\Gamma^\varepsilon)$, unique solution to Equation (3.5.1) *in the distributional sense, i.e for any $\varphi \in \mathcal{D}(\bar{\mathcal{U}}^\varepsilon)$ there holds*

$$\int_{\mathcal{O}^\varepsilon} f_1(t,\cdot)\,\varphi(t,\cdot)\,\mathrm{d}x\mathrm{d}v - \int_0^t \int_{\mathcal{O}^\varepsilon} f_1\,\mathcal{A}^*_\delta\,\varphi + f\,(\partial_t\varphi - v\cdot\nabla_x\varphi - \nu\,\varphi) + G\,\varphi\,\mathrm{d}x\mathrm{d}v\mathrm{d}s$$
$$= \int_{\mathcal{O}^\varepsilon} f_0(\cdot)\varphi(0,\cdot)\mathrm{d}x\mathrm{d}v + \int_0^t \int_{\Sigma^\varepsilon} \gamma f_1\,\varphi\,(n_x\cdot v)\mathrm{d}\sigma_x\mathrm{d}v, \quad (3.6.4)$$

for every $t \geq 0$, and where we have defined the formal adjoint operator

$$\mathcal{A}^*_\delta\,\varphi(v) := (1-\chi_\delta(v)) \int_{\mathbb{R}^3} k(v_*, v)\,\varphi(v_*)\,\mathrm{d}v_*, \quad (3.6.5)$$

where we recall that χ_δ is defined in Subsection 3.1.4. Moreover, there holds the representation formula

$$f_1(t,x,v) = e^{-\nu(v)t} f_0(x,v)\,\mathbf{1}_{t_1\leq 0} + \int_{\max(0,t_1)}^t e^{-\nu(v)(t-s)}\,\mathcal{A}^*_\delta\,f_1(s,x,v)\,ds$$
$$+ \int_{\max(0,t_1)}^t e^{-\nu(v)(t-s)}\,G(s,x,v)\,ds + e^{-\nu(v)(t-t_1)} f_1(t_1,x_1,v)\,\mathbf{1}_{t_1>0}, \quad (3.6.6)$$

pointwise for every $t \geq 0$, almost every $(x,v) \in \mathcal{O}^\varepsilon$, and where $t_1 = t_1(t,x,v)$ and $x_1 = x_1(t,x,v)$ are given by (3.1.25)*, Furthermore, there holds the result from Proposition 3.1.*

We structure this section as follows: In Subsection 3.6.1 we provide extra a priori estimates we will need in order to proof the previous theorems. In Subsection 3.6.2 we obtain the well-posedness of Equation (3.1.13) in a weighted L^2 framework in order to provide the validity of the hypocoercivity estimates of Section 3.2, and using this we proceed in Subsection 3.6.3 to prove Theorem 3.6.2. We complete this section with Subsection 3.6.4 where we prove Theorem 3.6.3.

3.6.1 Extra a priori estimates

We provide now extra a priori estimates needed for the well-posedness results of this section.

Proposition 3.4. *There is $\kappa \in \mathbb{R}$ such that for every f solution of Equation* (3.1.13) *there holds*

$$\|f_t\|_{\mathcal{H}} \leq e^{\kappa t}\|f_0\|_{\mathcal{H}}, \quad (3.6.7)$$

for every $t \geq 0$.

Proof. We first observe that [56, Theorem 7.2.4] implies that

$$\|Kh\|_{\mathcal{H}} \leq C_K\|h\|_{\mathcal{H}}, \quad (3.6.8)$$

for some constant $C_K > 0$. Arguing then at a formal level we have that if f is a solution of Equation (3.1.13) there holds

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}^\varepsilon} f_t^2\mathcal{M}^{-1} = \int_{\mathcal{O}^\varepsilon} f_t\,(-v\cdot\nabla_x f_t + Kf_t - \nu f_t)\,\mathcal{M}^{-1}$$
$$\leq -\frac{1}{2}\int_{\Sigma^\varepsilon} \gamma f_t^2\,(n_x\cdot v)\,\mathcal{M}^{-1} + (C_K - \nu_0)\int_{\mathcal{O}^\varepsilon} f_t^2\mathcal{M}^{-1}, \quad (3.6.9)$$

where we have used the Cauchy-Schwartz inequality and (3.6.8) to obtain the second line. We now observe that

$$-\int_{\Sigma^\varepsilon} \gamma f_t^2\,(n_x\cdot v)\,\mathcal{M}^{-1} = -\int_{\Sigma_+^\varepsilon} \gamma_+ f_t^2\,(n_x\cdot v)_+\,\mathcal{M}^{-1} + \int_{\Sigma_-^\varepsilon} \gamma_- f_t^2\,(n_x\cdot v)_-\,\mathcal{M}^{-1},$$

and using the Maxwell boundary conditions (3.1.9), we further have that

$$\begin{aligned}\int_{\Sigma_-^\varepsilon} \gamma_- f^2\,\mathcal{M}^{-1}\,(n_x\cdot v)_- &= \int_{\Sigma_-^\varepsilon} \left((1-\iota^\varepsilon)\mathscr{S}\gamma_+ f + \iota^\varepsilon \mathscr{D}\gamma_+ f\right)^2\,\mathcal{M}^{-1}\,(n_x\cdot v)_- \\ &\le \int_{\Sigma_-^\varepsilon} \left((1-\iota^\varepsilon)(\mathscr{S}\gamma_+ f)^2 + \iota^\varepsilon \mathscr{M}^2\,(\widetilde{\gamma}_+ f)^2\right)\,\mathcal{M}^{-1}\,(n_x\cdot v)_- \\ &\le \int_{\Sigma_+^\varepsilon} (1-\iota^\varepsilon)\gamma_+ f^2 \mathcal{M}^{-1}\,(n_x\cdot v)_+ + \int_{\Sigma_-^\varepsilon} \iota^\varepsilon (2\pi)\mathcal{M}\left(\widetilde{\gamma_+ f}\right)^2 (n_x\cdot v)_-,\end{aligned}$$

where we have used a convexity inequality with the function $x\mapsto |x|^2$ to obtain the second inequality. Moreover, we have used the change of variables $v\mapsto \mathcal{V}_x v$ in the integral with the specular boundary conditions, together with the fact that $\mathscr{M} = \sqrt{2\pi}\mathcal{M}$ to obtain the third inequality. We now remark that

$$\begin{aligned}\int_{\Sigma_-^\varepsilon} \iota^\varepsilon (2\pi)\mathcal{M}\left(\widetilde{\gamma_+ f}\right)^2 (n_x\cdot v)_- &= \int_{\partial\Omega^\varepsilon} \iota^\varepsilon\sqrt{2\pi}\left(\int_{\mathbb{R}^3} \mathscr{M}(n_x\cdot v)_-\right)\left(\int_{\mathbb{R}^3} \gamma_+ f(n_x\cdot v)_+\right) \\ &= \int_{\partial\Omega^\varepsilon} \iota^\varepsilon\sqrt{2\pi}\left(\int_{\mathbb{R}^3} \gamma_+ f(u)(n_x\cdot u)_+\right)^2 \\ &\le \int_{\Sigma_-^\varepsilon} \iota^\varepsilon\left(\int_{\mathbb{R}^3} \gamma_+ f^2\mathcal{M}^{-1}(n_x\cdot v)_+\right)\left(\int_{\mathbb{R}^3} \sqrt{2\pi}\mathcal{M}(n_x\cdot v)_+\right) \\ &\le \int_{\Sigma_+^\varepsilon} \iota^\varepsilon\gamma_+ f^2(n_x\cdot v)_+,\end{aligned}$$

where we have used the fact that $\widetilde{\mathscr{M}} = 1$ and the Cauchy-Schwartz inequality. Altogether we have obtained that

$$-\int_{\Sigma^\varepsilon} \gamma f_t^2\,\mathcal{M}^{-1}\,(n_x\cdot v) \le 0. \tag{3.6.10}$$

We conclude by putting together (3.6.9) with (3.6.10), using the Grönwall lemma and setting $\kappa = C_K - \nu_0 \in \mathbb{R}$. □

Moreover, we have now the following lemmas on the a priori estimates for the trace.

For a set $\mathcal{S}\in\{\bar\Lambda_1^\varepsilon\cup\bar\Lambda_2^\varepsilon,\ \bar\Lambda_3^\varepsilon,\ \mathfrak{S}^\varepsilon\}$, where we recall that $\mathfrak{S}^\varepsilon$ has been defined in (3.4.3), we define $\delta_{\mathcal{S}}$ as a smooth C^2 function coinciding, in a small neighborhood of $\mathcal{S}$, with the distance function to the compact set $\mathcal{S}$. We remark that regularity of $\delta_{\mathcal{S}}$ is justified since $\mathcal{C}$ is always a smooth compact submanifold (or the disjoint union of compact submanifolds) of $\mathbb{R}^3$ making the distance function to such set a smooth function in a small neighborhood around it, see for instance [92, 95, 121].

We define the function

$$\zeta_{\mathcal{S}}(x) := \begin{cases} 1 & \text{if Assumption (RH1) holds} \\[2ex] \dfrac{(\delta_{\mathcal{S}}(x))^2}{1+(\delta_{\mathcal{S}}(x))^2} & \text{if Assumption (RH2) holds.}\end{cases}$$

We observe that $\zeta_{\mathcal{S}}\in C^1(\bar\Omega^\varepsilon)$ for any $\mathcal{S}\in\{\bar\Lambda_1^\varepsilon\cup\bar\Lambda_2^\varepsilon,\ \bar\Lambda_3^\varepsilon,\ \mathfrak{S}^\varepsilon\}$, and we have then the following extra a priori estimate for the boundary term.

Proposition 3.5. *Assume there to hold either Assumption (RH1) or Assumption (RH2). There is $\kappa \in \mathbb{R}$ such that for every f solution of Equation* (3.1.13) *there holds*

$$\int_0^t \int_{\Sigma^\varepsilon} (\gamma f_s)^2 \, (n_x \cdot v)^2 \, \zeta_{\mathfrak{S}^\varepsilon}(x) \, \langle v \rangle^{-2} \, \mathcal{M}^{-1} \, \mathrm{d}v \mathrm{d}\sigma_x \mathrm{d}s \le e^{\kappa t} \|f_0\|_{\mathcal{H}}, \qquad \forall t \ge 0. \tag{3.6.11}$$

Remark 3.6. We note that on cylindrical domains (i.e under Assumption (RH2)) the estimates for the boundary are more degenerate than for smooth domains. The extra term $\zeta_{\mathfrak{S}^\varepsilon}$, making at all possible the control of the trace, serves to—in a sense—smooth out the normal vector when approaching the singular set $\mathfrak{S}^\varepsilon$.

Proof of Proposition 3.5. We divide the proof in two steps. First we obtain (3.6.11) for smooth domains, and after we repeat and adapt those computations for the setting of the cylinder.

Case 1. (Smooth domains—Assumption (RH1)) We recall that $\zeta_{\mathfrak{S}^\varepsilon} \equiv 1$ under Assumption (RH1), and that in this setting the normal vector is a C^2 vector field of $\mathbb{R}^3$. We then compute

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{O}^\varepsilon} (f_t)^2 \mathcal{M}^{-1} \langle v \rangle^{-2} (n_x \cdot v) &= \int_{\mathcal{O}^\varepsilon} f_t \left(-v \cdot \nabla_x f_t + K f_t - \nu f_t \right) \mathcal{M}^{-1} \langle v \rangle^{-2} (n_x \cdot v) \\ &= -\frac{1}{2} \int_{\mathcal{O}^\varepsilon} v \cdot \nabla_x (f_t^2) \mathcal{M}^{-1} (n_x \cdot v) \langle v \rangle^{-2} + C \|f_t\|^2_{\mathcal{H}}, \end{aligned} \tag{3.6.12}$$

for some constant $C > 0$, and we remark that we have used that $|\langle v \rangle^{-2} (n_x \cdot v)| \le 1$, the Cauchy-Schwartz inequality, and (3.6.8) to obtain the second line. This was in the same spirit as in the obtention of (3.6.9) during the proof of Proposition 3.4.

Using integration by parts we now have that

$$\begin{aligned} & - \int_{\mathcal{O}^\varepsilon} v \cdot \nabla_x (f_t^2) \mathcal{M}^{-1} (n_x \cdot v) \langle v \rangle^{-2} \mathrm{d}v \mathrm{d}x \\ &= - \int_{\Sigma^\varepsilon} f_t^2 \mathcal{M}^{-1} (n_x \cdot v)^2 \langle v \rangle^{-2} \mathrm{d}v \mathrm{d}\sigma_x + \int_{\mathcal{O}^\varepsilon} f_t^2 \mathcal{M}^{-1} \left(v \cdot \nabla_x (n_x \cdot v) \right) \langle v \rangle^{-2} \mathrm{d}v \mathrm{d}x \\ &\le - \int_{\Sigma^\varepsilon} f_t^2 \mathcal{M}^{-1} (n_x \cdot v)^2 \langle v \rangle^{-2} \mathrm{d}v \mathrm{d}\sigma_x + C \|f_t\|^2_{\mathcal{H}}, \end{aligned}$$

for some constant $C > 0$, and we remark that we have used the regularity of the normal vector to deduce the last line. Integrating in time in (3.6.12) we deduce that

$$\int_0^t \int_{\Sigma^\varepsilon} f_s^2 \mathcal{M}^{-1} (n_x \cdot v)^2 \langle v \rangle^{-2} \mathrm{d}v \mathrm{d}\sigma_x \mathrm{d}s \lesssim \|f_0\|^2_{\mathcal{H}} + \int_0^t \|f_s\|^2_{\mathcal{H}} \mathrm{d}s,$$

and we conclude (3.6.11) by using (3.6.7).

Case 2. (Cylindrical domains—Assumption (RH2)) During this proof we write any element $x \in \mathbb{R}^3$ by its components as $x = (x_1, x_2, x_3)$. We consider the vector fields $n_1, n_2 : \mathbb{R}^3 \to \mathbb{R}^3$ defined respectively by

$$n_1(x_1, x_2, x_3) := \frac{(x_1, 0, 0)}{\varepsilon^{-1} L} \qquad \text{and} \qquad n_2(x_1, x_2, x_3) := \frac{(0, x_2, x_3)}{\varepsilon^{-1} \mathfrak{R}}.$$

We then observe that each of them is smooth and we remark that $n_1(x) = n(x)$ for every $x \in \Lambda_1^\varepsilon \cup \Lambda_2^\varepsilon$ and $n_2(x) = n(x)$ for every $x \in \Lambda_3^\varepsilon$. Moreover, from their very definition there holds

$$|n_i(x)| \le 1 \qquad \forall x \in \bar{\Omega}^\varepsilon,$$

for any $i \in \{1,2\}$. We then compute

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}^\varepsilon}(f_t)^2\mathcal{M}^{-1}\langle v\rangle^{-2}\,\zeta_{\Lambda_3^\varepsilon}(x)\,(n_1(x)\cdot v)\,\mathrm{d}v\mathrm{d}x\\
&=\int_{\mathcal{O}^\varepsilon} f_t\,(-v\cdot\nabla_x f_t+Kf_t-\nu f_t)\,\mathcal{M}^{-1}\langle v\rangle^{-2}\,\zeta_{\Lambda_3^\varepsilon}(x)\,(n_1(x)\cdot v)\,\mathrm{d}v\mathrm{d}x\\
&=-\frac{1}{2}\int_{\mathcal{O}^\varepsilon} v\cdot\nabla_x(f_t^2)\mathcal{M}^{-1}\langle v\rangle^{-2}\,\zeta_{\Lambda_3^\varepsilon}(x)\,(n_1(x)\cdot v)\,\mathrm{d}v\mathrm{d}x+C\|f_t\|_{\mathcal{H}}^2,
\end{aligned}
\tag{3.6.13}
$$

for some constant $C>0$, and we remark that we have used that $|\langle v\rangle^{-2}(n_1(x)\cdot v)\ |\leq 1$, that $\zeta_{\Lambda_3^\varepsilon}$ is uniformly bounded from its very definition, the Cauchy-Schwartz inequality, and (3.6.8) to obtain the last line.

We then recall that by its very definition $\zeta_{\Lambda_3^\varepsilon}(x)=0$ for every $x\in\Lambda_3^\varepsilon$. We then have that

$$
\begin{aligned}
&-\int_{\mathcal{O}^\varepsilon} v\cdot\nabla_x(f_t^2)\mathcal{M}^{-1}\langle v\rangle^{-2}\,\zeta_{\Lambda_3}(x)\,(n_1(x)\cdot v)\,\mathrm{d}v\mathrm{d}x\\
&=-\int_{\Lambda_1^\varepsilon\cup\Lambda_2^\varepsilon} f_t^2\mathcal{M}^{-1}\zeta_{\Lambda_3}(x)\,(n_x\cdot v)^2\langle v\rangle^{-2}\mathrm{d}v\mathrm{d}\sigma_x+\int_{\mathcal{O}^\varepsilon} f_t^2\mathcal{M}^{-1}\,(v\cdot\nabla_x(n_1(x)\cdot v))\,\langle v\rangle^{-2}\mathrm{d}v\mathrm{d}x\\
&\qquad+\int_{\mathcal{O}^\varepsilon} f_t^2\mathcal{M}^{-1}\langle v\rangle^{-2}\,(v\cdot\nabla_x\zeta_{\Lambda_3}(x))\,(n_1(x)\cdot v)\,\mathrm{d}v\mathrm{d}x\\
&\leq-\int_{\Lambda_1^\varepsilon\cup\Lambda_2^\varepsilon} f_t^2\mathcal{M}^{-1}\zeta_{\Lambda_3}(x)\,(n_x\cdot v)^2\langle v\rangle^{-2}\mathrm{d}v\mathrm{d}\sigma_x+C\|f_t\|_{\mathcal{H}}^2,
\end{aligned}
$$

where we have simultaneously used the regularity and boundedness properties of $\zeta_{\Lambda_3^\varepsilon}$ and n_1, and the fact that $n_1(x)=n(x)$ for every $\Lambda_1^\varepsilon\cup\Lambda_2^\varepsilon$, and $\zeta_3(x)=0$ for every $x\in\Lambda_3^\varepsilon$.

We now observe that

$$
\zeta_{\Lambda_3}(x)=\zeta_{\mathfrak{S}^\varepsilon}(x)\qquad\forall x\in\Lambda_1^\varepsilon\cup\Lambda_2^\varepsilon,
$$

and integrating in time and arguing as during the Case 1 of this proof we deduce that there is a constant $\kappa>0$ such that

$$
\int_0^t\int_{\Lambda_1^\varepsilon\cup\Lambda_2^\varepsilon} f_s^2\mathcal{M}^{-1}\zeta_{\mathfrak{S}^\varepsilon}(x)\,(n_x\cdot v)^2\langle v\rangle^{-2}\mathrm{d}v\mathrm{d}\sigma_x\mathrm{d}s\lesssim e^{\kappa t}\|f_0\|_{\mathcal{H}}^2.
$$

Repeating this exact arguments but using $\zeta_{\Lambda_1^\varepsilon\cup\Lambda_2^\varepsilon}(x)\,(n_2(x)\cdot v)$ instead of $\zeta_{\Lambda_3^\varepsilon}(x)\,(n_1(x)\cdot v)$ in (3.6.13) we obtain that

$$
\int_0^t\int_{\Lambda_3^\varepsilon} f_s^2\mathcal{M}^{-1}\zeta_{\mathfrak{S}^\varepsilon}(x)\,(n_x\cdot v)^2\langle v\rangle^{-2}\mathrm{d}v\mathrm{d}\sigma_x\mathrm{d}s\lesssim e^{\kappa t}\|f_0\|_{\mathcal{H}}^2.
$$

We conclude this case by putting together the two last estimates. □

We define now certain notions before passing to the proof of the main of this section. First, we introduce the boundary measures

$$
\mathrm{d}\xi_1:=\mathcal{M}^{-1}(n_x\cdot v)\mathrm{d}v\mathrm{d}\sigma_x,\qquad\text{and}\qquad \mathrm{d}\xi_2:=\mathcal{M}^{-1}(n_x\cdot v)^2\,\zeta_{\mathfrak{S}^\varepsilon}(x)\,\langle v\rangle^{-2}\,\mathrm{d}v\mathrm{d}\sigma_x.
$$

Secondly, we define the space of renormalizing functions $C^1_{pw,*}(\mathbb{R})$ as the space of C^1 piecewise functions with finite limits at $\pm\infty$, and such that $s\mapsto\langle s\rangle\beta'(s)$ is bounded in $\mathbb{R}$.

Furthermore, we provide a priori estimates for the solutions of Equation (3.1.21) in weighted L^∞ spaces, when $\langle\!\langle G\rangle\!\rangle_{\mathcal{O}^\varepsilon}\neq 0$.

Proposition 3.7. *Consider either Assumption (RH1) or (RH2) to hold, $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function, $G : \mathcal{U}^\varepsilon \to \mathbb{R}$, and let f be a solution of Equation* (3.1.21)*. For every $\varepsilon \in (0, \varepsilon_{11})$, where we recall that $\varepsilon_{11} > 0$ is given by Proposition 3.1, there holds*

$$\|f_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq C \left(e^{-\theta\varepsilon^2 t} \|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \|G\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{U}^\varepsilon)} \right) \qquad \forall t \geq 0, \tag{3.6.14}$$

for some constant $C > 0$, independent of ε.

Proof. We emphasize that the only difference between Proposition 3.7 and Proposition 3.1 is the fact that G does not have total mass zero for all time. This implies that during the proof we will not be able to apply the hypocoercivity decay (3.2.1) to G, and in its place we will use Proposition 3.4. We then put the extra exponential growth in time given by (3.6.7) with the constants outside and shifting our estimate to control instead $\|G\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{U}^\varepsilon)}$, the proof of Proposition 3.7 follows exactly that of Proposition 3.1, thus we skip it. □

3.6.2 Well-posedness of Equation (3.1.13) in a weighted L^2 framework

We define now the boundary measures

$$\mathrm{d}\xi_1 := \mathcal{M}^{-1}(n_x \cdot v)\mathrm{d}v\mathrm{d}\sigma_x, \qquad \text{and} \qquad \mathrm{d}\xi_2 := \mathcal{M}^{-1}(n_x \cdot v)^2 \, \zeta_{\mathfrak{S}^\varepsilon}(x) \, \langle v \rangle^{-2} \, \mathrm{d}v\mathrm{d}\sigma_x,$$

and we define the space of renormalizing functions $C^1_{pw,*}(\mathbb{R})$ as the space of C^1 piecewise functions with finite limits at $\pm\infty$, and such that $s \mapsto \langle s \rangle \beta'(s)$ is bounded in $\mathbb{R}$.

We then prove the following well-posedness of Equation (3.1.13) in $\mathcal{H}$ in order to justify the computations from Section 3.2.

Theorem 3.6.8. *Assume there to hold either Assumption (RH1) or Assumption (RH2) and consider $f_0 \in \mathcal{H}$. There is $f \in C(\mathbb{R}_+, \mathcal{H})$ with an associated trace function $\gamma f \in L^2(\Gamma^\varepsilon; \mathrm{d}\xi_2 \mathrm{d}t)$, unique global solution to Equation* (3.1.13) *in the distributional sense (see for instance* (3.6.1)*).*

Remark 3.9. In particular, Theorem 3.6.8 implies the existence of a strongly continuous semigroup $S_{\mathscr{L}} : \mathcal{H} \to \mathcal{H}$ associated to the solutions of Equation (3.1.13).

Remark 3.10. We observe that, in the case of cylindrical domains (i.e under Assumption (RH2)), the functional space where the trace function is well defined is more singular than in the case of smooth domains. This is in particular reminiscent of our comments from Remark 3.6

The problem of well-posedness for transport equations with non-local terms presenting boundary conditions has been deeply addressed in the literature, see for instance [16, 17, 54, 77, 133, 137, 138, 157]. However, due to the lack of a precise reference, we provide a proof for Theorem 3.6.8.

Proof of Theorem 3.6.8. We divide the proof into three steps.

Step 1. (Auxiliary problem with inflow boundary conditions) We consider a function $\mathfrak{f} \in L^2(\Gamma^\varepsilon; \mathrm{d}t\mathrm{d}\xi_1)$, and we study the following evolution equation

$$\begin{cases} \partial_t f &= \mathscr{L} f & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f &= \mathfrak{f} & \text{on } \Gamma^\varepsilon_- \\ f_{t=0} &= f_0 & \text{in } \mathcal{O}^\varepsilon. \end{cases} \tag{3.6.15}$$

A direct application of [157, Proposition 8.16] gives the existence of $f \in C(\mathbb{R}_+, \mathcal{H})$, with a trace $\gamma f \in L^2(\Gamma^\varepsilon; \mathrm{d}\xi_1 \mathrm{d}t)$, unique renormalized solution to Equation (3.6.15), i.e for every test function $\varphi \in \mathcal{D}(\bar{\mathcal{U}}^\varepsilon)$ there holds

$$\int_{\mathcal{O}^\varepsilon} \beta(f)(t,\cdot)\,\varphi(t,\cdot) - \int_0^t \int_{\mathcal{O}^\varepsilon} \beta'(f)\varphi K f + \beta(f)\,(\partial_t \varphi - v\cdot\nabla_x \varphi) - \nu \beta'(f) f$$
$$- \int_0^t \int_{\Sigma_+^\varepsilon} \gamma_+ \beta(f)\,\varphi\,(n_x\cdot v)_+ = \int_{\mathcal{O}^\varepsilon} \beta(f_0)(\cdot)\varphi(0,\cdot) - \int_0^t \int_{\Sigma_-^\varepsilon} \beta(\mathfrak{f})\,\varphi\,(n_x\cdot v)_-, \quad (3.6.16)$$

for every $t \geq 0$, and every renormalizing function $\beta \in C^1_{pw,*}(\mathbb{R})$. Furthermore, there is a constant $\kappa > 0$ for which there holds the enegy estimate

$$\|f_t\|^2_{\mathcal{H}} + \int_0^t e^{2\kappa(t-s)} \|\gamma_+ f_s\|^2_{L^2(\Sigma_+^\varepsilon;\mathrm{d}\xi_1)} \mathrm{d}s \leq e^{2\kappa t}\|f_0\|^2_{\mathcal{H}} + \int_0^t e^{2\kappa(t-s)} \|\mathfrak{f}_s\|^2_{L^2(\Sigma_-^\varepsilon;\mathrm{d}\xi_1)} \mathrm{d}s, \quad (3.6.17)$$

for every $t \geq 0$.

Step 2. (Banach fixed point for modified Maxwell boundary conditions) We take an arbitrary $T > 0$ to be fixed later, a constant $\alpha \in (0,1)$, and we consider a function $h \in C(\mathbb{R}_+, \mathcal{H})$, with a trace $\gamma h \in L^2(\Gamma^\varepsilon; \mathrm{d}\xi_1 \mathrm{d}t)$, unique solution of Equation (3.6.15) given by Step 1. We study now the following kinetic equation

$$\begin{cases} \partial_t f &= -v\cdot\nabla_x f + Kf - \nu f & \text{in } \mathcal{U}_T^\varepsilon, \\ \gamma_- f &= \alpha \mathscr{R} \gamma_+ h & \text{on } \Gamma_{T,-}^\varepsilon, \\ f_{t=0} &= f_0 & \text{in } \mathcal{O}^\varepsilon. \end{cases} \quad (3.6.18)$$

We recall that during the computations performed on Step 1, we have proved that

$$\|\mathscr{R}\gamma_+ h\|_{L^2(\Sigma_-^\varepsilon,\mathrm{d}\xi_1)} \leq \|\gamma_+ h\|_{L^2(\Sigma_+^\varepsilon,\mathrm{d}\xi_1)}.$$

Moreover, Step 1 also implies that there is $f \in C(\mathbb{R}_+, \mathcal{H})$, with a trace $\gamma f \in L^2(\Gamma^\varepsilon; \mathrm{d}\xi_1 \mathrm{d}t)$, unique renormalized solution of Equation (3.6.18). Furthermore, (3.6.17) and the above boundary estimate give that

$$\|f_t\|^2_{\mathcal{H}} + \int_0^t e^{2\kappa(t-s)} \|\gamma_+ f_s\|^2_{L^2(\Sigma_+^\varepsilon;\mathrm{d}\xi_1)} \mathrm{d}s \leq e^{2\kappa t}\|f_0\|^2_{\mathcal{H}} + \alpha \int_0^t e^{2\kappa(t-s)} \|\gamma_+ h_s\|^2_{L^2(\Sigma_-^\varepsilon;\mathrm{d}\xi_1)} \mathrm{d}s, \quad (3.6.19)$$

for every $t \in [0,T]$. This implies that the mapping $h \mapsto f$ is α-Lipschitz for the norm defined by

$$\sup_{t\in[0,T]} \left\{ \|f_t\|^2_{\mathcal{H}} e^{-2\kappa t} + \int_0^t \|\gamma f_s\|^2_{L^2(\Sigma_+^\varepsilon;\mathrm{d}\xi_1)}\, e^{-2\kappa s}\,\mathrm{d}s \right\}.$$

The application of the Banach fixed point theorem implies then that there is $f \in C(\mathbb{R}_+, \mathcal{H})$, with a trace function $\gamma f \in L^2(\Gamma_T^\varepsilon; \mathrm{d}\xi_1 \mathrm{d}t)$, unique renormalized solution of the following evolution equation

$$\begin{cases} \partial_t f &= -v\cdot\nabla_x f + Kf - \nu f & \text{in } \mathcal{U}_T^\varepsilon \\ \gamma_- f &= \alpha \mathscr{R} \gamma_+ f & \text{on } \Gamma_{T,-}^\varepsilon \\ f_{t=0} &= f_0 & \text{in } \mathcal{O}^\varepsilon, \end{cases}$$

in the sense that for every $\phi \in \mathcal{D}(\bar{\mathcal{U}}^\varepsilon)$, and every $\beta \in C^1_{pw,*}$ there holds

$$\int_{\mathcal{O}^\varepsilon} \beta(f)(t,\cdot)\,\varphi(t,\cdot) - \int_0^t \int_{\mathcal{O}^\varepsilon} \beta'(f)\varphi K f + \beta(f)\,(\partial_t \varphi - v\cdot\nabla_x \varphi) - \nu \beta'(f) f$$
$$- \int_0^t \int_{\Sigma^\varepsilon} \gamma_+ \beta(f)\,\varphi\,(n_x\cdot v)_+ = \int_{\mathcal{O}^\varepsilon} \beta(f_0)(\cdot)\varphi(0,\cdot), \quad (3.6.20)$$

for every $t \in [0,T]$.

Step 3. For a sequence $\alpha_k \in (0,1)$ such that $\alpha_k \nearrow 1$, we consider the sequence (f_k) obtained by using the Step 2 as the solution to the modified Maxwell reflection boundary condition problem

$$\begin{cases} \partial_t f_k &= -v \cdot \nabla_x f_k + \mathscr{L} f_k & \text{in } \mathcal{U}_T^\varepsilon \\ \gamma_- f_k &= \alpha_k \mathscr{R} \gamma_+ f_k & \text{on } \Sigma_{T,-}^\varepsilon \\ f_{k,t=0} &= f_0 & \text{in } \mathcal{O}^\varepsilon. \end{cases} \tag{3.6.21}$$

From the fact that $\mathscr{R} : L^2(\Sigma_+^\varepsilon; \mathrm{d}\xi_1) \to L^2(\Sigma_-^\varepsilon; \mathrm{d}\xi_1)$ with norm less than 1 as established in Step 2 and the energy estimate (3.6.19), we deduce that f_k satisfies

$$\|f_{kt}\|_{\mathcal{H}} + (1-\alpha_k) \int_0^t e^{2\kappa(t-s)} \|\gamma_+ f_{ks}\|^2_{L^2(\Sigma_+^\varepsilon; \mathrm{d}\xi_1)} \le e^{\kappa t} \|f_0\|_{\mathcal{H}}, \tag{3.6.22}$$

for any $t \in [0,T]$ and any $k \ge 1$.

Now, for any constant $M > 0$ we take $\beta(s) = \beta_M(s) = M \wedge s^2$, and as tests functions φ we take the ones considered during the proof of Proposition 3.5. From the renormalized formulation (3.6.20) with these choices, arguing as during the proof of Proposition 3.5, and using the integral version of the Grönwall lemma we additionally have that

$$\int_{\Gamma_T^\varepsilon} (\gamma f_k)^2 \mathrm{d}\xi_2 \, \mathrm{d}t \lesssim \|f_0\|^2_{\mathcal{H}} e^{\kappa T}.$$

From the above estimates, we deduce that, up to the extraction of a subsequence, there exist $f \in L^2(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{H})$ and $\mathfrak{f}_\pm \in L^2(\Gamma_{T,\pm}^\varepsilon; \mathrm{d}\xi_2 \mathrm{d}t)$ such that

$$f_k \rightharpoonup f \text{ weakly in } L^2(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{H}), \quad \gamma_\pm f_k \rightharpoonup \mathfrak{f}_\pm \text{ weakly in } L^2(\Gamma_{T,\pm}^\varepsilon; \mathrm{d}\xi_2 \mathrm{d}t).$$

Since $\langle v \rangle \mathcal{M} \in L^2(\mathbb{R}^d)$, we have that $L^2(\Gamma_T^\varepsilon; \mathrm{d}\xi_2 \mathrm{d}t) \subset L^1(\Gamma_T^\varepsilon; \zeta_{\mathfrak{S}^\varepsilon}(x)\,(n_x \cdot v) \mathrm{d}v \mathrm{d}\sigma_x \mathrm{d}t)$. Moreover, from the very definition of the rescaled Maxwell boundary condition (3.1.9), we have that as a map

$$\mathscr{R} : L^1(\Sigma_+^\varepsilon; \zeta_{\mathfrak{S}^\varepsilon}(x)\,(n_x \cdot v) \mathrm{d}v \mathrm{d}\sigma_x) \to L^1(\Sigma_-^\varepsilon; \zeta_{\mathfrak{S}^\varepsilon}(x)\,(n_x \cdot v) \mathrm{d}v \mathrm{d}\sigma_x),$$

there holds

$$\|\mathscr{R}\|_{L^1(\Sigma^\varepsilon; \zeta_{\mathfrak{S}^\varepsilon}(x)\,(n_x \cdot v) \mathrm{d}v \mathrm{d}\sigma_x)} \le 1.$$

Altogether this implies that $\mathscr{R}(\gamma_+ f_k) \rightharpoonup \mathscr{R}(\mathfrak{f}_+)$ weakly in $L^1(\Gamma_-; \zeta_{\mathfrak{S}^\varepsilon}(x)\,(n_x \cdot v) \mathrm{d}v \mathrm{d}\sigma_x)$.

Furthermore, from [157, Proposition 8.10], we have that $\gamma f_k \rightharpoonup \gamma f$ weakly in $L^2_{\mathrm{loc}}(\Gamma_T^\varepsilon; \mathrm{d}\xi_2 \mathrm{d}t)$. Using both convergences in the boundary condition $\gamma_- f_k = \mathscr{R}(\gamma_+ f_k)$, we obtain $\gamma_- f = \mathscr{R}(\gamma_+ f)$.

We may thus pass to the limit in the weak formulation of Equation (3.6.21), obtained from the Step 2, and we obtain that $f \in C([0,T];\mathcal{H})$ is a weak solution to Equation (3.1.13). Moreover, passing to the limit in (3.6.22), we also have that (3.6.7) holds. This and the linearity give the uniqueness of the solution to Equation (3.1.13), and repeating this argument in the time intervals $[nT, (n+1)T]$ for every $n \in \mathbb{N}$ we conclude the existence and uniqueness of a weak global solution. □

3.6.3 Proof of Theorem 3.6.2

We provide now the well-posedness of Equation (3.1.21) in a weighted L^∞ framework. This is necessary because in order to provide the well-posedness of the non-linear Boltzmann equation, we need to work in the algebraic structure given by weighted L^∞ Lebesgue spaces.

Proof of Theorem 3.6.2. We split the proof into four steps.

Step 1. (Classical solutions to the transport equation with smooth coefficients) We consider now $H_1, H_2, \mathfrak{H} \in C^1_c(\bar{\mathcal{U}}^\varepsilon)$, $h_0 \in C^1_c(\mathcal{O}^\varepsilon)$ and we study the following transport evolution equation

$$\begin{cases} \partial_t h &= \mathscr{T} h + H_1 + H_2 & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- h &= \mathfrak{H} & \text{on } \Lambda^\varepsilon_- \\ h_{t=0} &= h_0 & \text{in } \mathcal{O}^\varepsilon. \end{cases} \tag{3.6.23}$$

We classically have (see for instance the proof of [157, Lemma 8.12] and the references therein) that a solution to Equation (3.6.23) is given by the representation formula

$$h(t,x,v) = e^{-\nu(v)t} h_0(x,v)\,\mathbf{1}_{t_1\le 0} + \int_{\max(0,t_1)}^t e^{-\nu(v)(t-s)}\,(H_1+H_2)(s,x,v)\,\mathrm{d}s \\ + e^{-\nu(v)(t-t_1)}\mathfrak{H}(t_1,x_1,v)\,\mathbf{1}_{t_1>0}, \tag{3.6.24}$$

and, in a smooth framework, we further have that h also satisfies the weak formulation

$$\int_{\mathcal{O}^\varepsilon} h_t\,\varphi(t,\cdot) - \int_0^t\int_{\mathcal{O}^\varepsilon} h\,\left(v\cdot\nabla_x\varphi - \nu(v)\varphi\right) + \int_0^t\int_{\Sigma^\varepsilon_+} \gamma_+ h\,\varphi\,(n_x\cdot v)_+ \\ = \int_{\mathcal{O}^\varepsilon} h_0\,\varphi(0,\cdot) + \int_0^t\int_{\Sigma^\varepsilon_-} \mathfrak{H}\,\varphi\,(n_x\cdot v)_+ + \int_0^t\int_{\mathcal{O}^\varepsilon} (H_1+H_2)\,\varphi. \tag{3.6.25}$$

for every $\varphi \in \mathcal{D}(\bar{\mathcal{U}}^\varepsilon)$ and $t\in\mathbb{R}_+$.

Moreover, multiplying (3.6.24) by ω_1, taking the $L^\infty(\mathcal{O}^\varepsilon)$ norm, and using (K2) with $\nu_2 = \nu_0/2$, we deduce the energy estimates

$$\|h\|_{L^\infty_{\omega_1}(\bar{\mathcal{U}}^\varepsilon)} \le \|h_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \|\mathfrak{H}\|_{L^\infty_{\omega_1}(\Gamma^\varepsilon_-)} + \|H_1\|_{L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)} + 2\frac{\nu_1}{\nu_0}\|H_2\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{U}^\varepsilon)}, \tag{3.6.26}$$

Step 2. (Solutions to the transport equation with L^∞ coefficients) We assume now $H_1 \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$, $H_2 \in L^\infty_{\omega_1\nu^{-1}}(\mathcal{U}^\varepsilon)$, $\mathfrak{H} \in L^\infty_{\omega_1}(\Gamma^\varepsilon_-)$ and $h_0 \in L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)$. We take then the sequence of smooth functions $H_1^n, H_2^n \in C^1_c(\bar{\mathcal{U}}^\varepsilon)$, $\mathfrak{H}^n \in C^1_c(\Gamma^\varepsilon)$ and $h_0^n \in C^1_c(\mathcal{O}^\varepsilon)$ such that, as $n\to\infty$, there holds

$$\begin{array}{llll} H_1^n \to H_1 & \text{strongly in } L^\infty_{\omega_1}(\mathcal{U}^\varepsilon), & H_2^n \to H_2 & \text{strongly in } L^\infty_{\omega_1\nu^{-1}}(\mathcal{U}^\varepsilon), \\ \mathfrak{H}^n \to \mathfrak{H} & \text{strongly in } L^\infty_{\omega_1}(\Gamma^\varepsilon_-), & h_0^n \to g_0 & \text{strongly in } L^\infty_{\omega_1}(\mathcal{O}^\varepsilon). \end{array} \tag{3.6.27}$$

From the analysis performed during the Step 1, there is h^n unique solution to Equation (3.6.23) associated with H_i^n for $i=1,2$, $\mathfrak{H}^n$, and h_0^n. Moreover, there holds the representation formula (3.6.24), the weak formulation (3.6.25), and the energy estimate (3.6.26).

In particular, defining the trace using the representation formula (3.6.24) as detailed in [157, Definition 8.1], we deduce from (3.6.26), the linearity of Equation (3.6.23), and the convergences from (3.6.27) that h^n and γh^n are Cauchy sequences in $L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$ and $L^\infty_{\omega_1}(\Gamma^\varepsilon)$ respectively. Therefore, there are functions $h \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$ and $\gamma h \in L^\infty_{\omega_1}(\Gamma^\varepsilon)$ such that, as $n\to\infty$, there holds

$$h^n \to h \quad \text{strongly in } L^\infty_{\omega_1}(\mathcal{O}^\varepsilon) \quad \text{and} \quad \gamma h^n \to \gamma h \quad \text{strongly in } L^\infty_{\omega_1}(\Gamma^\varepsilon_-).$$

Moreover, passing to the limit in the weak formulation of Equation (3.6.23) associated with H_i^n for $i=1,2$, $\mathfrak{H}^n$, and h_0^n, we deduce that h, with the trace function γh, is a weak solution of Equation (3.6.23) associated with H_i for $i = 1, 2$, $\mathfrak{H}$, and h_0, satisfying the energy estimate

$$\|h\|_{L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)} + \|\gamma h\|_{L^\infty_{\omega_1}(\Gamma^\varepsilon)} \le \|h_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \|\mathfrak{H}\|_{L^\infty_{\omega_1}(\Gamma^\varepsilon_-)} + \|H_1\|_{L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)} + 2\frac{\nu_1}{\nu_0}\|H_2\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{U}^\varepsilon)}. \quad (3.6.28)$$

In particular, (3.6.28) and the linearity of Equation (3.6.23) imply the uniqueness of this solution.

Furthermore, we may pass to the limit in (3.6.24) a.e in $(x, v) \in \mathcal{O}^\varepsilon$ such that h satisfies the Duhamel formulation (3.6.24).

Step 3. We now set $\psi^0 = 0$, $\psi_0 \in L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)$, $\alpha \in (0, 1)$, we recall that $G \in L^\infty_{\omega_1\nu^{-1}}(\mathcal{U}^\varepsilon)$ and we consider the recurrent sequence of solutions given by the following evolution equation

$$\begin{cases} \partial_t\psi^{k+1} = \mathscr{T}\psi^{k+1} + K\psi^k + G & \text{in } \mathcal{U}^\varepsilon \\ \gamma_-\psi^{k+1} = \alpha\mathscr{R}\gamma_+\psi^k & \text{on } \Gamma^\varepsilon_- \\ \psi^{k+1}_{t=0} = \psi_0 & \text{in } \mathcal{O}^\varepsilon. \end{cases} \quad (3.6.29)$$

Indeed, we remark that if we assume that, for a certain k, $\psi^k \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$, then (K1) implies that $K\psi^k \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$. Therefore Step 2 implies the existence of $\psi^{k+1} \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$, with a trace $\gamma\psi^{k+1} \in L^\infty_{\omega_1}(\Gamma^\varepsilon)$, unique weak solution of Equation (3.6.29) in the sense provided by the Step 2.

We take now $A > 0$ to be fixed later and, inspired by the recent series of papers [45, 48, 49], we define the modified weight functions

$$\omega_1^A = \omega_1^A(v) := \mathscr{M}^{-1}\xi_A(v) + (1 - \xi_A(v))\omega_1,$$

where $\xi_A(v) := \xi(|v|/A)$, for a function $\xi \in C^2(\mathbb{R}_+, \mathbb{R})$, such that $\mathbf{1}_{[0,1]} \le \xi \le \mathbf{1}_{[0,2]}$. Moreover, we remark that there is a constant $c_A > 0$ such that

$$c_A^{-1}\omega_1 \le \omega_1^A \le c_A\omega. \quad (3.6.30)$$

Using now the representation formula given by the Step 2 for the solutions of Equation (3.6.29) we have that

$$\psi^{k+1}(t, x, v) = e^{-\nu(v)t}\psi_0(x, v)\,\mathbf{1}_{t_1\le 0} + \int_{\max(0,t_1)}^t e^{-\nu(v)(t-s)}\,K\psi^k(s, x, v)\,\mathrm{d}s + \int_{\max(0,t_1)}^t e^{-\nu(v)(t-s)}\,G(s, x, v)\,\mathrm{d}s + e^{-\nu(v)(t-t_1)}\alpha\mathscr{R}\gamma_+\psi^k(t_1, x_1, v)\,\mathbf{1}_{t_1>0}. \quad (3.6.31)$$

Multiplying (3.6.31) by ω_1^A, using (K1) together with (3.6.30), and arguing as during the proof of (K2) with $\nu_2 = \nu_0/2$, we deduce that there is a constant $C_0 > 0$ satisfying

$$|\psi^{k+1}(t, x, v)|\omega_1^A = e^{-\nu_0 t}\|\psi_0\|_{L^\infty_{\omega_1^A}(\mathcal{O}^\varepsilon)} + t\,C_0\|\psi^k\|_{L^\infty_{\omega_1^A}(\mathcal{O}^\varepsilon)} + C_0\|G\|_{L^\infty_{\omega_1^A\nu^{-1}}(\mathcal{O}^\varepsilon)} + \alpha\,\omega_1^A(v)\,|\mathscr{R}\gamma_+\psi^k(t_1, x_1, v)|. \quad (3.6.32)$$

To bound the boundary term from (3.6.32) we remark that

$$\begin{aligned} |\mathscr{R}\gamma_+\psi^k(t_1, x_1, v)| &\le (1 - \iota^\varepsilon)\|\psi^k\|_{L^\infty_{\omega_1^A}(\mathcal{O}^\varepsilon)} + \iota^\varepsilon\omega_1^A\mathcal{M}\left|\int_{\mathbb{R}^3}\gamma_+\psi^k(t_1, x_1, u)\,(n_x\cdot u)_+\,\mathrm{d}u\right| \\ &\le (1 - \iota^\varepsilon)\|\psi^k\|_{L^\infty_{\omega_1^A}(\mathcal{O}^\varepsilon)} + \iota^\varepsilon\|\psi^k\|_{L^\infty_{\omega_1^A}(\mathcal{O}^\varepsilon)}\omega_1^A\mathscr{M}\int_{\mathbb{R}^3}(n_x\cdot u)_+\,\omega_1^A(u)^{-1}\mathrm{d}u. \end{aligned}$$

On the one hand, we observe that from the very definition of ω_A there holds

$$\omega_1^A \mathscr{M} = \xi_A + (1-\xi_A)\omega_1 \mathscr{M} \leq 1.$$

On the other hand, we have that

$$1 \leq \int_{\mathbb{R}^3} (n_x \cdot u)_+ \, \omega_1^A(u)^{-1} \mathrm{d}u = \int_{\mathbb{R}^3} \mathscr{M}(u)(n_x \cdot u)_+ \left(\xi_A + (1-\xi_A)\omega \mathscr{M}\right)^{-1} \mathrm{d}u := \Theta_A \underset{A\to\infty}{\longrightarrow} 1,$$

where we have used the fact that $\xi_A + (1-\xi_A)\omega\mathscr{M} \to 1$ as $A \to \infty$ and $\widetilde{\mathscr{M}} = 1$.

Using then that $\alpha \in (0,1)$ and $\Theta_A \to 1$ as $A \to \infty$, we can choose $A > 0$ large enough such that $\Theta_A < 1/\alpha$, this implies that

$$\alpha\left(1 - \iota^\varepsilon + \iota^\varepsilon \Theta_A\right) = \alpha\left(1 + \iota^\varepsilon(\Theta_A - 1)\right) \leq \alpha\Theta_A < 1,$$

where we have used the fact that $\Theta_A \geq 1$ for every $A > 0$ and $\iota^\varepsilon(x) \leq 1$ for almost every $x \in \partial\Omega^\varepsilon$.

Using then the linearity of Equation (3.6.29) and the previous analysis we have that

$$\|\psi^{k+1} - \psi^k\|_{L^\infty_{\omega_1^A}(\mathcal{O}^\varepsilon)} \leq (t\, C_0 + \alpha\Theta_A)\, \|\psi^k - \psi^{k-1}\|_{L^\infty_{\omega_1^A}(\mathcal{O}^\varepsilon)},$$

and following exactly the same argument we further obtain that

$$\|\gamma\psi^{k+1} - \gamma\psi^k\|_{L^\infty_{\omega_1^A}(\mathcal{O}^\varepsilon)} \leq (t\, C_0 + \alpha\Theta_A)\, \|\psi^k - \psi^{k-1}\|_{L^\infty_{\omega_1^A}(\mathcal{O}^\varepsilon)}.$$

These informations imply that, for a small time $T_0 > 0$, ψ^k and $\gamma\psi^k$ are Cauchy sequences in the the Banach spaces $L^\infty_{\omega_1^A}(\mathcal{U}^\varepsilon_{T_0})$ and $L^\infty_{\omega_1^A}(\Gamma^\varepsilon_{T_0})$. Therefore there are functions $\psi \in L^\infty_{\omega_1^A}(\mathcal{U}^\varepsilon_{T_0})$ and $\gamma\psi \in L^\infty_{\omega_1^A}(\Gamma^\varepsilon_{T_0})$ such that, as $k \to \infty$, there holds

$$\psi^k \to \psi \text{ strongly in} L^\infty_{\omega_1^A}(\mathcal{U}^\varepsilon_{T_0}) \quad \text{and} \quad \gamma\psi^k \to \gamma\psi \text{ strongly in} L^\infty_{\omega_1^A}(\Gamma^\varepsilon_{T_0}).$$

Moreover, using (3.6.30) and passing to the limit in the weak formulation associated with Equation (3.6.29) we deduce that ψ solves the evolution equation

$$\begin{cases} \partial_t \psi &= \mathscr{T}\psi + K\psi + G & \text{in } \mathcal{U}^\varepsilon \\ \gamma_-\psi &= \alpha\mathscr{R}\gamma_+\psi & \text{on } \Gamma^\varepsilon_- \\ \psi_{t=0} &= \psi_0 & \text{in } \mathcal{O}^\varepsilon, \end{cases} \tag{3.6.33}$$

in the time interval $[0, T_0]$. Repeating this argument in every time interval $[nT_0, (n+1)T_0]$ for every $n \in \mathbb{N}$ we deduce the existence of a global solution ψ. Furthermore, from the representation formula associated with Equation (3.6.29), and the previous strong convergence result we have that

$$\begin{aligned} \psi(t,x,v) = e^{-\nu(v)t}\psi_0(x,v)\, \mathbf{1}_{t_1 \leq 0} + \int_{\max(0,t_1)}^t e^{-\nu(v)(t-s)}\, K\psi(s,x,v)\, \mathrm{d}s \\ + \int_{\max(0,t_1)}^t e^{-\nu(v)(t-s)}\, G(s,x,v)\, \mathrm{d}s + e^{-\nu(v)(t-t_1)}\gamma\psi(t_1,x_1,v)\, \mathbf{1}_{t_1>0}, \end{aligned} \tag{3.6.34}$$

for every $t \geq 0$ and for almost every $(x,v) \in \mathcal{O}^\varepsilon$.

We can then use the well-posedness from the Step 3 of the proof of Theorem 3.6.8 and Remark 3.9 to justify the repetition of the hypocoercivity result from Theorem 3.2.1 applied to Equation (3.6.33). We may thus repeat the computations leading to the conclusion of

Proposition 3.7. Therefore, there are $\varepsilon_{11}, \theta > 0$, given by Proposition 3.1, such that for every $\varepsilon \in (0, \varepsilon_{11})$, there holds the following energy estimate

$$\|\psi_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \|\gamma\psi_t\|_{L^\infty_{\omega_1}(\Sigma^\varepsilon)} \leq Ce^{-\theta\varepsilon^2 t}\left(\|\psi_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + \sup_{s\in[0,t]} \|G\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{U}^\varepsilon)}\right), \quad (3.6.35)$$

for every $t \geq 0$, and where $C > 0$ is an universal constant independent of ε. Finally, it is worth remarking that (3.6.35) is uniform in α from the fact that this estimate comes from Proposition 3.7.

Step 4. We now take a sequence $\alpha_k \nearrow 1$, with $k \in \mathbb{N}$, and we consider the sequence $(f_k)_{k\in\mathbb{N}}$ obtained in Step 3 as the solution to the modified Maxwell reflection boundary condition problem

$$\begin{cases} \partial_t f_k &= \mathscr{T} f_k + K f_k + G & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_k &= \alpha_k \mathscr{R}\gamma_+ f_k & \text{on } \Gamma^\varepsilon_- \\ f_{k,t=0} &= f_0 & \text{in } \mathcal{O}^\varepsilon. \end{cases} \quad (3.6.36)$$

Moreover, it is worth remarking that there holds the representation formula (3.6.34) for f_k.

Using then the energy estimate (3.6.35) we obtain that f^k and γf^k are bounded sequences in $L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$ and $L^\infty_{\omega_1}(\Gamma^\varepsilon)$, therefore there is $f \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$ and $\gamma f \in L^\infty_{\omega_1}(\Gamma^\varepsilon)$ such that, as $k \to \infty$, there holds

$$f^k \rightharpoonup f \text{ weakly-}* \text{ in } L^\infty_{\omega_1}(\mathcal{U}^\varepsilon) \quad \text{and} \quad \gamma f^k \rightharpoonup \gamma f \text{ weakly-}* \text{ in } L^\infty_{\omega_1}(\Gamma^\varepsilon).$$

We can thus pass the limit in the weak formulation associated to Equation (3.6.36) and we deduce that f, with its associated trace γf, solves Equation (3.1.21) in the sense of (3.6.1).

Furthermore, we remark that by taking the representation formula (3.6.34) for f^k, multiplying it by any arbitrary function $g \in L^1_{m_1}(\mathcal{U}^\varepsilon)$, where $m_1 := \omega_1^{-1}$, we may pass to the limit as $k \to \infty$ and by density we deduce that f also satisfies the representation formula (3.6.34).

Finally, we conclude by remarking that the conclusion of Proposition 3.1 holds from the fact that we have access to a Duhamel-type formulation and using Remark 3.9. ☐

3.6.4 Proof of Theorem 3.6.3

The proof of Theorem 3.6.3 follows by exactly the same arguments used for the proof of Theorem 3.6.2, with the only difference being using the a priori estimates from Proposition 3.1 in the place of those from Proposition 3.7. ☐

3.7 Proof of Theorem 3.1.3 for strongly confining weights

We dedicate this section to prove a first part of Theorem 3.1.3.

Theorem 3.7.1. *Consider either Assumption (RH1) or (RH2) to hold, and let $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function. For every $\varepsilon \in (0, \varepsilon_{11})$, where we recall that $\varepsilon_{11} > 0$ is given by Proposition 3.1, there is $\eta^1_0(\varepsilon) \in (0,1)$ satisfying $\eta^1_0(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that if $f_0 \in L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)$ satisfies*

$$\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq (\eta^1_0(\varepsilon))^2,$$

there is a function $f \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$ with an associated trace $\gamma f \in L^\infty_{\omega_1}(\Gamma^\varepsilon)$, unique solution to the linearized rescaled Boltzmann Equation (3.1.5)-(3.1.9)-(3.1.10) *in the distributional* *sense, i.e for every test function* $\varphi \in \mathcal{D}(\bar{\mathcal{U}}^\varepsilon)$ *there holds*

$$\int_{\mathcal{O}^\varepsilon} f(t,\cdot)\,\varphi(t,\cdot)\mathrm{d}v\mathrm{d}x - \int_0^t \int_{\mathcal{O}^\varepsilon} f\left(\partial_t\varphi + v\cdot\nabla_x\varphi + K^*\varphi - \nu\varphi\right) + \varphi\,\mathcal{Q}(f,f)\,\mathrm{d}v\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\Sigma^\varepsilon} \gamma f\,\varphi\,(n_x\cdot v)\mathrm{d}v\mathrm{d}\sigma_x = \int_{\mathcal{O}^\varepsilon} f_0\,\varphi(0,\cdot)\mathrm{d}v\mathrm{d}x, \quad (3.7.1)$$

for every $t \geq 0$*, and we recall that* K^* *is defined in* (3.6.2)*. Furthermore, there holds*

$$\|f_t\|_{L^\infty_{\omega_1}(\mathcal{O})} \leq e^{-\theta\varepsilon^2 t}\,\eta_0^1(\varepsilon) \qquad \forall t \geq 0. \quad (3.7.2)$$

3.7.1 Estimate on the bilinear Boltzmann collision operator

Before studying the non-linear Boltzmann equation we state the following control on the bilinear Boltzmann collision operator.

Lemma 3.2. *Let* ω *be an admissible weight function, and let* $g, h \in L^\infty_\omega(\mathcal{O}^\varepsilon)$*. There holds*

$$\|\mathcal{Q}(g,h)\|_{L^\infty_{\omega\nu^{-1}}(\mathcal{O}^\varepsilon)} \leq C_{\mathcal{Q}}\|g\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\|h\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}.$$

for some constant $C_{\mathcal{Q}} = C_{\mathcal{Q}}(\omega) > 0$.

Remark 3.3. This type of estimate is classical in the study of the Boltzmann equation, see for instance [56, 83, 102, 108]. Moreover, we point out that Lemma 3.2 is nothing but [102, Lemma 5.16] for polynomial and stretched exponential weight functions. Therefore we will only prove this result for inverse gaussian weight functions.

Proof. We recall that, due to Remark 3.3, we assume $\omega(v) = e^{\zeta|v|^2}$, with $\zeta \in (0, 1/2)$. We also recall the Boltzmann collision operator

$$\begin{aligned}\mathcal{Q}(g,h) &= \frac{1}{2}\Big[\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-v_*)\cdot\sigma|g(v_*')h(v')d\sigma dv_* + \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-v_*)\cdot\sigma|h(v_*')g(v')\mathrm{d}\sigma\mathrm{d}v_* \\ &\quad - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-v_*)\cdot\sigma|g(v')h(v)f(v_*)\mathrm{d}\sigma\mathrm{d}v_* - \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|(v-v_*)\cdot\sigma|h(v')g(v)f(v_*)\mathrm{d}\sigma\mathrm{d}v_*\Big] \\ &=: \frac{1}{2}\left(\mathcal{Q}_1^+ + \mathcal{Q}_2^+ - \mathcal{Q}_1^- - \mathcal{Q}_2^-\right),\end{aligned}$$

where

$$v' = v - ((v-v_*)\cdot\sigma)\sigma, \qquad v_*' = v_* + ((v-v_*)\cdot\sigma)\sigma.$$

In particular, we observe that

$$|v_*'|^2 + |v'|^2 = |v|^2 + |v|^2, \quad \text{therefore} \quad \omega(v)\omega(v_*) = \omega(v')\omega(v_*'). \quad (3.7.3)$$

We then compute

$$\begin{aligned}\omega(v)\nu(v)^{-1}|\mathcal{Q}_1^+| &\leq \|g\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\|h\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\omega(v)\nu(v)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\frac{|(v-v_*)\cdot\sigma|}{\omega(v_*')\omega(v')}\mathrm{d}\sigma\mathrm{d}v_* \\ &\leq \|g\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\|h\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\nu(v)^{-1}\int_{R^3}\int_{\mathbb{S}^2}|(v-v_*)\cdot\sigma|\,\omega(v_*)^{-1}\mathrm{d}\sigma\mathrm{d}v_* \\ &\lesssim \|g\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\|h\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\nu(v)^{-1}\left(\int_{\mathbb{R}^3}|v_*|\,(\omega(v_*))^{-1}dv_* + |v|\int_{\mathbb{R}^3}(\omega(v_*))^{-1}\mathrm{d}v_*\right) \\ &\lesssim \|g\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\|h\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\nu(v)^{-1}\langle v\rangle,\end{aligned}$$

where we have used (3.7.3) to obtain the second line and we have used the convexity type inequality $1+|v| \le \sqrt{2}\langle v\rangle$ to obtain the fourth. We remark that, from (3.1.6) we have that $\nu(v)^{-1}\langle v\rangle \le \nu_0^{-1}$, therefore there holds

$$\omega(v)\nu(v)^{-1}|\mathcal{Q}_1^+| \lesssim \|g\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}\|h\|_{L^\infty_\omega(\mathcal{O}^\varepsilon)}.$$

We conclude the proof by repeating these same steps for the remaining operators $\mathcal{Q}_2^+$, $\mathcal{Q}_1^-$ and $\mathcal{Q}_2^-$. □

3.7.2 Proof of Theorem 3.7.1

We define the norm

$$[[f]]_1 := \sup_{s\in[0,\infty)} \left[e^{\theta\varepsilon^2 s}\|f_s\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)}\right],$$

where we recall that $\theta > 0$ is given by Proposition 3.1. We consider a $\lambda > 0$ to be fixed later, and we define the Banach space

$$\mathcal{Z}_1 := \{g \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon),\ [[g]]_1 \le \lambda\},$$

equipped with the strong topology in $L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$, which makes $\mathcal{Z}_1$ a bounded, convex, closed subset of $L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$.

We denote Ψ_1 as the map that to $g \in \mathcal{Z}_1$ assigns f the solution of

$$\left\{\begin{array}{lcll} \partial_t f &=& \mathscr{L} f + \mathcal{Q}(g,g) & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f &=& \mathscr{R}\gamma_+ f, & \text{on } \Gamma^\varepsilon_- \\ f_{t=0} &=& f_0, & \text{in } \mathcal{O}^\varepsilon, \end{array}\right. \tag{3.7.4}$$

which is given by Theorem 3.6.2. Indeed, we observe that Theorem 3.6.2 holds by using Lemma 3.2 and the fact that $g \in \mathcal{Z}_1$. We remark then that Proposition 3.1 implies that for every $\varepsilon \in (0,\varepsilon_{11})$ there is $C_0 = C_0(\varepsilon) > 0$ satisfying $C_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$, such that

$$\begin{aligned} [[f]]_1 &\le C_0\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_0 \sup_{s\in[0,t]} \left[e^{\theta\varepsilon^2 s}\|\mathcal{Q}(g_s,g_s)\|_{L^\infty_{\omega_1\nu^{-1}}(\mathcal{O}^\varepsilon)}\right] \\ &\le C_0\|f_0\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} + C_0 C_{\mathcal{Q}}[[g]]_1^2 \end{aligned} \tag{3.7.5}$$

where we have used Lemma 3.2 to obtain the second line and the constant $C_{\mathcal{Q}} = C_{\mathcal{Q}}(\omega_1) > 0$ is given by Lemma 3.2.

We then choose $\eta_0^1(\varepsilon) = \lambda(\varepsilon)$ and

$$\lambda = \lambda(\varepsilon) := \min\left(\frac{1}{C_0(\varepsilon)(1+C_{\mathcal{Q}})}, \frac{1}{4C_0(\varepsilon)C_{\mathcal{Q}}}, 1\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

In particular, this choice of λ together with (3.7.5) implies that $f \in \mathcal{Z}_1$, thus $\Psi_1 : \mathcal{Z}_1 \to \mathcal{Z}_1$.

Furthermore, if we take $g_1, g_2 \in \mathcal{Z}_1$ and denoting $f_i = \Psi_1(g_i)$ for $i = 1,2$, we have that $\psi = f_1 - f_2$ is the weak solution of the evolution equation

$$\left\{\begin{array}{lcll} \partial_t \psi &=& \mathscr{L}\psi + \mathcal{Q}(g_1+g_2, g_1-g_2) & \text{in } \mathcal{U}^\varepsilon \\ \gamma_-\psi &=& \mathscr{R}\gamma_+\psi & \text{on } \Gamma^\varepsilon_- \\ \psi_{t=0} &=& 0 & \text{in } \mathcal{O}^\varepsilon, \end{array}\right.$$

in the sense of Theorem 3.6.2. Using then Proposition 3.1 and Lemma 3.2 as before we have that

$$[[\psi]]_1 \le C_0 C_{\mathcal{Q}}[[g_1+g_2]]_1[[g_1-g_2]]_1 \ \le 2C_0C_{\mathcal{Q}}\lambda[[g_1-g_2]]_1 \le \frac{1}{2}[[g_1-g_2]]_1.$$

This implies that Ψ_1 is a contraction in $\mathcal{Z}_1$, thus there is a unique fixed point $f \in \mathcal{Z}_1$ for this map. We deduce that f is a weak solution of the nonlinear Boltzmann Equation (3.1.5)-(3.1.9)-(3.1.10) in the sense of (3.7.1). Furthermore, (3.7.2) comes from the very fact that $f \in \mathcal{Z}_1$. □

3.8 Proof of Theorem 3.1.3 for weakly confining weights

We dedicate this section to prove the remaining part of Theorem 3.1.3 on the well-posedness of the Boltzmann equation for weakly confining admissible weight functions.

Theorem 3.8.1. *Consider either Assumption (RH1) or (RH2) to hold, and let $\omega_0 \in \mathfrak{W}_0$ a weakly confining admissible weight function. Define furthermore $\varepsilon_{12} := \min(\varepsilon_8, \varepsilon_{11})$, where we recall that $\varepsilon_8 > 0$ is given by Proposition 3.1 and $\varepsilon_{11} > 0$ is given by Proposition 3.1.*

For every $\varepsilon \in (0, \varepsilon_{12})$ there is $\eta_0^0(\varepsilon) \in (0,1)$ such that $\eta_0^0(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for every $f_0 \in L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)$ satisfying

$$\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \le (\eta_0^0(\varepsilon))^2,$$

there is $f \in L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)$ with an associated trace $\gamma f \in L^\infty_{\omega_0}(\Gamma^\varepsilon)$, unique solution to the linearized rescaled Boltzmann Equation (3.1.5)-(3.1.9)-(3.1.10) *in the distribution sense, i.e for every test function $\varphi \in \mathcal{D}(\bar{\mathcal{U}}^\varepsilon)$ there holds*

$$\int_{\mathcal{O}^\varepsilon} f(t,\cdot)\,\varphi(t,\cdot)\mathrm{d}v\mathrm{d}x - \int_0^t \int_{\mathcal{O}^\varepsilon} f\,(\partial_t\varphi + v\cdot\nabla_x\varphi + K^*\varphi - \nu\varphi) + \varphi\,\mathcal{Q}(f,f)\,\mathrm{d}v\mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_{\Sigma^\varepsilon} \gamma f\,\varphi\,(n_x\cdot v)\mathrm{d}v\mathrm{d}\sigma_x = \int_{\mathcal{O}^\varepsilon} f_0\,\varphi(0,\cdot)\mathrm{d}v\mathrm{d}x, \quad (3.8.1)$$

for every $t \ge 0$, and we recall that K^* is defined in (3.6.2). *Furthermore, there is $\theta > 0$ such that there holds*

$$\|f_t\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \le e^{-\theta\varepsilon^2 t}\,\eta_0^0(\varepsilon) \qquad \forall t \ge 0, \quad (3.8.2)$$

Remark 3.2. The proof follows the main ideas of [36, Section 6].

3.8.1 Dissipative equation

Let $h : \mathcal{U}^\varepsilon \to \mathbb{R}$, we study the following equation

$$\begin{cases} \partial_t f_1 &= \mathscr{T} f_1 + \mathcal{A}_\delta f_1 + \mathcal{Q}(f_1 + h, f_1 + h) & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_1 &= \mathscr{R}\gamma_+ f_1 & \text{on } \Gamma_-^\varepsilon \\ f_{1,t=0} &= f_0 & \text{in } \mathcal{O}^\varepsilon. \end{cases} \quad (3.8.3)$$

We dedicate this subsection to prove the following well-posedness result.

Proposition 3.3. *Consider either Assumption (RH1) or (RH2) to hold, let $\omega_0 \in \mathfrak{W}_0$ a weakly confining admissible weight function, and let $f_0 \in L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)$ and such that*

$$\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \le \eta_1,$$

some $\eta_1 > 0$ small enough. Assume furthermore that for every $\varepsilon \in (0, \varepsilon_8)$ and $\delta \in (0, \delta_0)$ there holds

$$\sup_{s\in[0,\infty)} \left[e^{\theta\varepsilon^2 s} \|h_s\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \right] \le \eta_1, \quad (3.8.4)$$

where $\varepsilon_8, \delta_0 > 0$ are given by Proposition 3.1, and $\theta \in (0, \nu_0/2)$ is given by Proposition 3.1. For every $\varepsilon \in (0, \varepsilon_8)$ and $\delta \in (0, \delta_0)$ there is $f_1 \in L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)$ with an associated trace $\gamma f_1 \in L^\infty_{\omega_0}(\Gamma^\varepsilon)$, unique solution to Equation (3.8.3) *in the weak sense, i.e for any $\varphi \in \mathcal{D}(\bar{\mathcal{U}}^\varepsilon)$ there holds*

$$\int_{\mathcal{O}^\varepsilon} f_1(t,\cdot)\,\varphi(t,\cdot)\,\mathrm{d}x\mathrm{d}v - \int_0^t\int_{\mathcal{O}^\varepsilon} f_1\,\mathcal{A}_\delta^*\,\varphi + f\,(\partial_t\varphi + v\cdot\nabla_x\varphi - \nu\,\varphi)\,\mathrm{d}x\mathrm{d}v\mathrm{d}s$$
$$-\int_0^t\int_{\mathcal{O}^\varepsilon} \mathcal{Q}(f_1+h, f_1+h)\,\varphi\,\mathrm{d}x\mathrm{d}v\mathrm{d}s + \int_0^t\int_{\Sigma^\varepsilon} \gamma f_1\,\varphi\,(n_x\cdot v)\mathrm{d}\sigma_x\mathrm{d}v = \int_{\mathcal{O}^\varepsilon} f_0(\cdot)\varphi(0,\cdot)\mathrm{d}x\mathrm{d}v, \tag{3.8.5}$$

for every $t \geq 0$, and where we recall that $\mathcal{A}_\delta^*$ has been defined in (3.6.5)*. Furthermore, there holds*

$$\|f_{1,t}\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq Ce^{-\theta\varepsilon^2 t}\left(\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + \sup_{s\in[0,\infty)}\left[e^{\theta\varepsilon^2 s}\|h_s\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right]^2\right) \qquad \forall t \geq 0, \tag{3.8.6}$$

for some constant $C > 0$, independent of ε.

Proof. The proof is mainly a repetition of the arguments from the proof of Theorem 3.7.1.

We define the norm

$$[[g]]_0 := \sup_{s\in[0,\infty)}\left[e^{\theta\varepsilon^2 s}\|g_s\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)}\right], \tag{3.8.7}$$

we consider a $\lambda > 0$ to be fixed later, and we define the Banach space

$$\mathcal{Z}_0 := \{g \in L^\infty_{\omega_0}(\mathcal{U}^\varepsilon),\ [[g]]_0 \leq \lambda\}, \tag{3.8.8}$$

equipped with the strong topology in $L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)$, which makes $\mathcal{Z}_0$ a bounded, convex, closed subset of $L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)$.

Moreover, we denote Ψ_0 as the map that to $g \in \mathcal{Z}_0$ assigns f_1 the solution of

$$\begin{cases} \partial_t f_1 &= \mathscr{T} f_1 + \mathcal{A}_\delta f_1 + \mathcal{Q}(g+h, g+h) & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_1 &= \mathscr{R}\gamma_+ f_1, & \text{on } \Gamma_-^\varepsilon \\ f_{1,t=0} &= f_0, & \text{in } \mathcal{O}^\varepsilon, \end{cases}$$

which is given by Theorem 3.6.3. Indeed, we observe that Theorem 3.6.3 holds by using Lemma 3.2, (3.8.4), and the fact that $g \in \mathcal{Z}_0$.

We recall now that Proposition 3.1 implies that there is $C_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_8)$ and every $\delta \in (0, \delta_0)$, where $\varepsilon_8, \delta_0 > 0$ are given by Proposition 3.1, there holds

$$\begin{aligned} [[f_1]]_0 &\leq C_0\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + C_0 \sup_{s\in[0,t]}\left[e^{\theta\varepsilon^2 s}\|\mathcal{Q}(g_s+h_s, g_s+h_s)\|_{L^\infty_{\omega_0\langle v\rangle^{-1}}(\mathcal{O}^\varepsilon)}\right] \\ &\leq C_0\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + C_0 C_{\mathcal{Q}}[[g+h]]_0^2 \\ &\leq C_0\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + 2C_0C_{\mathcal{Q}}[[g]]_0^2 + 2C_0C_{\mathcal{Q}}[[h]]_0^2, \end{aligned}$$

where we have used Lemma 3.2 to obtain the second line, we remark that $C_{\mathcal{Q}} = C_{\mathcal{Q}}(\omega_0) > 0$ is given by Lemma 3.2, and we have used the Young inequality to obtain the last line.

We then choose $\eta_1 = \lambda^2$ and

$$\lambda = \min\left(\frac{1}{C_0(1+4C_{\mathcal{Q}})}, \frac{1}{8C_0C_{\mathcal{Q}}}\right),$$

and we observe that, in particular, this implies that $f_1 \in \mathcal{Z}_0$, thus $\Psi_0 : \mathcal{Z}_0 \to \mathcal{Z}_0$.

Furthermore, we consider $g_1, g_2 \in \mathcal{Z}_0$ and $f_{1,i} = \Psi_0(g_i)$ for $i = 1, 2$, and we have that $\psi = f_{1,1} - f_{1,2}$ is the weak solution of the evolution equation

$$\begin{cases} \partial_t \psi &= \mathscr{L}\psi + \mathcal{Q}(g_1 + g_2 + 2h, g_1 - g_2) & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- \psi &= \mathscr{R}\gamma_+\psi, & \text{on } \Gamma_-^\varepsilon \\ \psi_{t=0} &= 0 & \text{in } \mathcal{O}^\varepsilon, \end{cases}$$

in the sense of Theorem 3.6.2. Using again Proposition 3.1 and Lemma 3.2 we have that

$$[[\psi]]_0 \le C_0 C_{\mathcal{Q}} [[g_1 + g_2 + 2h]]_0 [[g_1 - g_2]]_0 \ \le 4 C_0 C_{\mathcal{Q}} \lambda [[g_1 - g_2]]_0 \le \frac{1}{2}[[g_1 - g_2]]_0.$$

This implies that Ψ_0 is a contraction in $\mathcal{Z}_0$, thus using the Banach fixed point theorem we deduce that there is a unique fixed point $f_1 \in \mathcal{Z}_0$ for this map. Moreover, from the very definition of the map Ψ_0 we deduce that f_1 is a weak solution of Equation (3.8.3) in the sense of (3.8.5). Furthermore, (3.8.6) comes from using Proposition 3.1 and Lemma 3.2, the fact that $f_1 \in \mathcal{Z}_0$, and the very definition of $\lambda > 0$. □

3.8.2 Functional inequality on $\mathcal{K}_\delta$

We recall the definition of $\mathcal{K}_\delta$ given by (3.1.26) and we have the following result.

Lemma 3.4. *Let ω be an admissible weight function. For every $\delta > 0$ there holds*

$$\|\mathcal{K}_\delta f\|_{L^\infty_\varsigma(\mathcal{O}^\varepsilon)} \le C_\delta \|f\|_{L^\infty(\mathcal{O}^\varepsilon)}, \tag{3.8.9}$$

for some constructive constant $C_\delta > 0$.

Proof. We note first that

$$\{|v| \le 2\delta^{-1},\, \delta \le |v - v_*| \le 2\delta^{-1}\} \subset \{|v| \le 2\delta^{-1},\, |v_*| \le 4\delta^{-1},\, \delta \le |v - v_*| \le 2\delta^{-1}\} =: \mathcal{I}$$

and there holds

$$\varsigma(v) \int_{\mathbb{R}^3} \mathbf{1}_{\mathcal{I}}(v, v_*) k(v, v_*) dv_* \lesssim \delta.$$

This concludes the proof. □

3.8.3 Residual linear equation

We consider $g : \mathcal{U}^\varepsilon \to \mathbb{R}$, and we study the following linear evolution equation

$$\begin{cases} \partial_t f_2 &= \mathscr{L} f_2 + \mathcal{K}_\delta g & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_2 &= \mathscr{R}\gamma_+ f_2 & \text{on } \Gamma_-^\varepsilon \\ f_{2,t=0} &= 0 & \text{in } \mathcal{O}^\varepsilon, \end{cases} \tag{3.8.10}$$

where we recall that $\mathcal{K}_\delta$ is defined in (3.1.26). We define the projection operator

$$\mathscr{P} f(t, x, v) := \mathcal{M}(v) \int_{\mathcal{O}^\varepsilon} f(t, x, u) \mathrm{d}u,$$

and we dedicate this subsection to prove the following well-posedness and long-time behavior result

Proposition 3.5. *Consider either Assumption (RH1) or (RH2) to be satisfied, let* $g \in L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)$*, for some weakly confining admissible weight function* $\omega_0 \in \mathfrak{W}_0$.

For every strongly confining admissible weight function $\omega_1 \in \mathfrak{W}_1$*, every* $\delta > 0$*, and every* $\varepsilon \in (0, \varepsilon_{12})$*, where we recall that* $\varepsilon_{12} > 0$ *is given in Theorem 3.8.1, there is* $f_2 \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$ *unique weak solution of Equation* (3.8.10) *in the sense of the Theorem 3.6.2.*

Moreover, if $\mathscr{P}(f_2 + g) = 0$*, and for every* $\varepsilon \in (0, \varepsilon_{12})$ *there holds*

$$\|g_t\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \leq C_g\, e^{-\theta\varepsilon^2 t} \qquad \forall t \geq 0, \tag{3.8.11}$$

for some constant $C_g > 0$*, and where* $\theta > 0$ *is given by Proposition 3.1, then there is a constant* $C = C(\varepsilon) > 0$ *satisfying* $C(\varepsilon) \to \infty$ *as* $\varepsilon \to 0$*, such that*

$$\|f_{2,t}\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq C C_g e^{-\theta\varepsilon^2 t}, \tag{3.8.12}$$

for every $t \geq 0$.

Remark 3.6. The proof of the above proposition follows the main ideas of [36, Proposition 6.8].

Proof. The existence of $f_2 \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$ is immediate by using Lemma 3.4 and Theorem 3.6.2.

Moreover, from the fact that $\mathscr{P}(f_2 + g) = 0$ we deduce that there is $C_1 > 0$ such that

$$\|\mathscr{P} f_{2,t}\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} = \|\mathscr{P}\mathcal{K}_\delta g_t\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq C_1\, C_g e^{-\theta\varepsilon^2 t} \qquad \forall t \geq 0, \tag{3.8.13}$$

where we have used, the very definition of $\mathscr{P}$, Lemma 3.4 and (3.8.11) to obtain the final inequality.

On the other hand, we define $\mathscr{P}^\perp := Id - \mathscr{P}$, we observe that $\langle\!\langle \mathscr{P}^\perp(\mathcal{K}_\delta g_t)\rangle\!\rangle_{\mathcal{O}^\varepsilon} = 0$ for all $t \geq 0$, and there holds

$$\partial_t \mathscr{P}^\perp f_{2,t} = \mathscr{L} f_2 + \mathscr{P}^\perp(\mathcal{K}_\delta g_t) = \mathscr{L}(\mathscr{P}^\perp f_{2,t}) + \mathscr{P}^\perp(\mathcal{K}_\delta g_t).$$

Therefore there holds in the distributional sense

$$\begin{cases} \partial_t(\mathscr{P}^\perp f_{2,t}) &=& \mathscr{L}(\mathscr{P}^\perp f_{2,t}) + \mathscr{P}^\perp(\mathcal{K}_\delta g) & \text{in } \mathcal{U}^\varepsilon \\ \gamma_-(\mathscr{P}^\perp f_{2,t}) &=& \mathscr{R}\gamma_+(\mathscr{P}^\perp f_{2,t}) & \text{on } \Gamma^\varepsilon_- \\ \mathscr{P}^\perp f_{2,t=0} &=& 0 & \text{in } \mathcal{O}^\varepsilon, \end{cases}$$

and Proposition 3.1 implies then that

$$\|\mathscr{P}^\perp f_{2,t}\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq C_2 e^{-\theta\varepsilon^2 t} \sup_{s\in[0,t]} \left[e^{\theta\varepsilon^2 s} \|\mathscr{P}^\perp(\mathcal{K}_\delta g_s)\|_{L^\infty_{\omega_1 \nu^{-1}}(\mathcal{O}^\varepsilon)} \right] \qquad \forall t \geq 0, \tag{3.8.14}$$

for some constant $C_2 = C_2(\varepsilon) > 0$ satisfying $C_2(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Using then (3.8.14) together with Lemma 3.4 and (3.8.11) we deduce that there is a constant $C_3 = C_3(\varepsilon) > 0$ satisfying $C_3(\varepsilon) \to \infty$ as $\varepsilon \to 0$, such that

$$\|\mathscr{P}^\perp f_{2,t}\|_{L^\infty_{\omega_1}(\mathcal{O}^\varepsilon)} \leq C_3\, C_g e^{-\theta\varepsilon^2 t} \qquad \forall t \geq 0. \tag{3.8.15}$$

We conclude by putting together (3.8.13) and (3.8.15). □

3.8.4 Proof of Theorem 3.8.1

We consider $h \in L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)$ such that that for every $\varepsilon \in (0, \varepsilon_{12})$ and $\delta \in (0, \delta_0)$ there holds

$$\sup_{s\in[0,\infty)} \left[e^{\theta\varepsilon^2 s} \|h_s\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} \right] \leq \eta_1, \tag{3.8.16}$$

where $\delta_0 > 0$ is given by Proposition 3.1, $\theta \in (0, \nu_0/2)$ is given by Proposition 3.1, and $\eta_1 > 0$ is given by Proposition 3.3. We define $f_1 \in L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)$ as the solution of the evolution equation

$$\left\{ \begin{array}{rcll} \partial_t f_1 &=& \mathscr{T} f_1 + \mathcal{A}_\delta f_1 + \mathcal{Q}(f_1 + h, f_1 + h) & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_1 &=& \mathscr{R}\gamma_+ f_1, & \text{on } \Gamma^\varepsilon_- \\ f_{1,t=0} &=& f_0, & \text{in } \mathcal{O}^\varepsilon, \end{array} \right. \tag{3.8.17}$$

which is given by Proposition 3.3. Moreover, we fix $\omega_1 \in \mathfrak{W}_1$ a strongly confining admissible weight function, and we define $f_2 \in L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)$ as the solution of the evolution equation

$$\left\{ \begin{array}{rcll} \partial_t f_2 &=& \mathscr{L} f_2 + \mathcal{K}_\delta f_1 & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- f_2 &=& \mathscr{R}\gamma_+ f_2 & \text{on } \Gamma^\varepsilon_- \\ f_{2,t=0} &=& 0 & \text{in } \mathcal{O}^\varepsilon, \end{array} \right. \tag{3.8.18}$$

which is given by Proposition 3.5. We emphasize that, defined this way, f_2 depends of h.

We recall the norm $[[\cdot]]_0$ defined in (3.8.7) and, for $\lambda \in (0, \eta_1)$ to be fixed later, we define the Banach space

$$\mathcal{Z}_0 := \{ g \in L^\infty_{\omega_0}(\mathcal{U}^\varepsilon),\ [[g]]_0 \leq \lambda \},$$

as in (3.8.8), equipped with the strong topology in $L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)$, which makes $\mathcal{Z}_0$ a bounded, convex, closed subset of $L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)$. Furthermore, we denote Ψ as the map that to $h \in \mathcal{Z}_0$ assigns f_2 as defined above.

We remark that $f_1 + f_2$ solves Equation (3.1.21) with $G = \mathcal{Q}(f_1 + h, f_1 + h)$, which together with (3.1.15) implies that $\mathscr{P}(f_1 + f_2) = 0$. We set then the constant $C_{\omega_1} > 0$ such that

$$\|g\|_{L^\infty_{\omega_0}(\mathcal{U}^\varepsilon)} \leq C_{\omega_1} \|g\|_{L^\infty_{\omega_1}(\mathcal{U}^\varepsilon)},$$

and using (3.8.16), (3.8.12) and (3.8.6) we obtain that

$$[[f_1]]_0 \leq C_1 \left(\|f_0\|_{L^\infty_{\omega_0}(\mathcal{O}^\varepsilon)} + [[h]]_0^2 \right) \leq C_1 \left((\eta_0^0)^2 + \lambda^2 \right), \quad \text{and} \quad [[f_2]]_0 \leq C_{\omega_1} C_2 C_1 \left((\eta_0^0)^2 + \lambda^2 \right),$$

where $C_1 > 0$ is given by (3.8.6) in Proposition 3.3, and $C_2 = C_2(\varepsilon) > 0$, satisfying $C_2(\varepsilon) \to \infty$ as $\varepsilon \to 0$, is given by (3.8.12) in Proposition 3.5.

We define $\eta_0^0 = \lambda$ and

$$\lambda = \lambda(\varepsilon) := \min\left(\frac{1}{8C_{\omega_1} C_1 C_2(\varepsilon)}, \frac{1}{6C_1}, \frac{1}{8C_{\mathcal{Q}} C_3}, \frac{1}{16 C_3 C_{\mathcal{Q}}}, \frac{\eta_1}{2}, 1 \right) \xrightarrow[\varepsilon\to 0]{} 0$$

where $C_{\mathcal{Q}} > 0$ is given by Lemma 3.2 and $C_3 > 0$ is the constant given by Proposition 3.1. This choices of parameters implies in particular that $[[f_1]] \leq \lambda$ and $[[f_2]]_0 \leq \lambda$, thus $f_1, f_2 \in \mathcal{Z}_0$, therefore $\Psi : \mathcal{Z}_0 \to \mathcal{Z}_0$.

Furthermore, we consider $h_1, h_2 \in \mathcal{Z}_0$ and we define $f_1, g_1 \in \mathcal{Z}_0$ as the solutions of Equation (3.8.17) associated with h_1 and h_2 respectively. We also define $f_2 = \Psi(h_1)$ and $g_2 = \Psi(h_2)$, i.e the solutions of Equation (3.8.18) associated with f_1 and g_1 respectively. We denote $\phi = f_1 - g_1$, $\psi := f_2 - g_2$, and we observe that ϕ and ψ are the weak solutions of

$$\left\{ \begin{array}{rcll} \partial_t \phi &=& \mathscr{T}\phi + \mathcal{A}_\delta \phi + \mathcal{Q}(f_1 + g_1 + h_1 + h_2, \phi + h_1 - h_2) & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- \phi &=& \mathscr{R}\gamma_+ \phi, & \text{on } \Gamma^\varepsilon_- \\ \phi_{t=0} &=& 0, & \text{in } \mathcal{O}^\varepsilon, \end{array} \right.$$

and

$$\begin{cases} \partial_t \psi &= \mathscr{L}\psi + \mathcal{K}_\delta \phi & \text{in } \mathcal{U}^\varepsilon \\ \gamma_- \psi &= \mathscr{R}\gamma_+ \psi & \text{on } \Gamma_-^\varepsilon \\ \psi_{t=0} &= 0 & \text{in } \mathcal{O}^\varepsilon, \end{cases}$$

respectively. On the one hand using Proposition 3.1 and Lemma 3.2 we have that

$$[[\phi]]_0 \leq C_3 C_{\mathcal{Q}} [[f_1 + g_1 + h_1 + h_2]]_0 [[\phi + h_1 - h_2]]_0 \leq \frac{1}{2}[[\phi]]_0 + 4\lambda C_3 C_{\mathcal{Q}} [[h_1 - h_2]]_0 \quad (3.8.19)$$

where we have used the triangular inequality and the definition of λ to obtain the last inequality. In particular, (3.8.19) and our very definition of λ implies that

$$[[\phi]]_0 \leq 8\lambda C_3 C_{\mathcal{Q}} [[h_1 - h_2]]_0 \leq \frac{1}{2}[[h_1 - h_2]]_0.$$

This implies that Ψ is a contraction in $\mathcal{Z}_0$, thus using the Banach fixed point theorem we deduce that there is a unique fixed point $f_2 \in \mathcal{Z}_0$ for this map. Moreover, from the very definition of the map Ψ we deduce that f_2 is a weak solution of Equation (3.8.18), and Proposition 3.3 further gives the uniqueness of f_1, weak solution of Equation (3.8.17) with $h = f_2$.

Putting together the aforementioned weak formulations we deduce that $f = f_1 + f_2$ is a weak solution of the Boltzmann equation (3.1.5)-(3.1.9)-(3.1.10) in the sense of (3.8.1). Finally, (3.8.2) comes from the very fact that $f = f_1 + f_2$ with $f_1, f_2 \in \mathcal{Z}_0$. This concludes the proof. □

3.9 Appendix. Control of the Jacobian for the specular reflection

In this section we prove Lemma 3.12. This is a fundamental result in order to control the gain of regularity of the term $S_{\mathscr{T}} * K$ after specular collisions. The computations follow the ideas leading to [110, Equations (2.141) and (2.142)], but we will repeat them for two main reasons.

On the one hand, the lack of a better reference to give, thus in the interest of making the chapter as self contained as possible

On the other hand, during the proof we will emphasize the local character of the arguments which will justify their use for the computations performed during the proof of Proposition 3.9 in Section 3.4.

Proof of Lemma 3.12. We define

$$X(v_*) = x_1 - (s_1 - r)\mathcal{V}_{x_1} v_*, \qquad \text{where} \qquad x_1 := x_s - (s - s_1)v_*,$$

and we recall that x_s does not depends on $v_* \in \mathbb{R}^3$ (see for instance the Step 1 of the proof of Proposition 3.10. We also recall that

$$\mathcal{V}_{x_1} v_* = v_* - 2[v_* \cdot n(x_1)]n(x_1), \qquad \text{and} \qquad n(x_1) = -\frac{\nabla_x \delta^\varepsilon(x_1)}{|\nabla_x \delta^\varepsilon(x_1)|} = -\frac{\nabla_y \delta(\varepsilon x_1)}{|\nabla_y \delta(\varepsilon x_1)|},$$

where the second inequality of the second expression comes from (3.1.8). Putting together the expressions above we obtain that

$$X(v_*) = x_s - (s - r)v_* + (s_1 - r)\frac{2}{|\nabla_y \delta(\varepsilon x_1)|^2}[v_* \cdot \nabla_y \delta(\varepsilon x_1)]\nabla_y \delta(\varepsilon x_1).$$

We now remark that using [108, Lemma 2-(2)], $s_1 = s_1(v_*)$ is a smooth function, and we can then compute the derivatives component-wise with respect to the components of v_* of the previous expression. Indeed, for every $i, j \in \{1,2,3\}$ there yields

$$\partial_{v_*^j} X_i(v_*) = -\delta_{ij}(s-r) + 2\partial_{v_*^j}\left(\frac{[v_* \cdot \nabla_y \delta(\varepsilon x_1)]\partial_{y_i}\delta(\varepsilon x_1)}{|\nabla_y \delta(\varepsilon x_1)|^2}\right)(s_1 - r) + 2\left(\frac{[v_* \cdot \nabla_y \delta(\varepsilon x_1)]\partial_{y_i}\delta(\varepsilon x_1)}{|\nabla_y \delta(\varepsilon x_1)|^2}\right)(\partial_{v_*^j} s_1)$$

where we have considered $v_* = (v_*^1, v_*^2, v_*^3)$, and δ_{ij} as the Kronecker delta function. Moreover, a direct computations gives that

$$\partial_{v_*^j}\left(\frac{[v_* \cdot \nabla_y \delta(\varepsilon x_1)]\partial_{y_i}\delta(\varepsilon x_1)}{|\nabla_y \delta(\varepsilon x_1)|^2}\right) = \frac{\partial_{y_j}\delta(\varepsilon x_1)\partial_{y_i}\delta(\varepsilon x_1)}{|\nabla_y \delta(\varepsilon x_1)|^2} + \mathscr{O}(\varepsilon),$$

where we remark that the term $\mathscr{O}(\varepsilon)$ comes from the fact that $|v_*| \lesssim 1$ and $\delta \in C^2(\mathbb{R}^3, \mathbb{R})$.

Therefore we can write

$$\partial_{v_*^j} X_i(v_*) = -\delta_{ij}(s-r) + 2a_i b_j + \mathscr{O}(\varepsilon),$$

with

$$a_i := \frac{\partial_{y_i}\delta(\varepsilon x_1)}{|\nabla_y \delta(\varepsilon x_1)|^2} \quad \text{and} \quad b_j = \partial_{y_j}\delta(\varepsilon x_1)(s_1 - r) + [v_* \cdot \nabla_y \delta(\varepsilon x_1)](\partial_{v_*^j} s_1),$$

thus a direct computation gives that

$$\det \nabla_{v_*} X(v_*) = -(s-r)^3 + 2(s-r)^2 \sum_{k=1}^3 a_k b_k + \mathscr{O}(\varepsilon).$$

We compute then

$$\begin{aligned}
\sum_{k=1}^3 a_k b_k &= \sum_{k=1}^3 \frac{\partial_{y_k}\delta(\varepsilon x_1)}{|\nabla_y \delta(\varepsilon x_1)|^2}\left[\partial_{y_k}\delta(\varepsilon x_1)(s_1 - r) + (\nabla_y \delta(\varepsilon x_1) \cdot v_*)\partial_{v_*^k} s_1\right] \\
&= \frac{(s_1 - r)}{|\nabla_y \delta(\varepsilon x_1)|^2} \sum_{k=1}^3 |\partial_{y_k}\delta(\varepsilon x_1)|^2 + \frac{[\nabla_y \delta(\varepsilon x_1) \cdot v_*]}{|\nabla_y \delta(\varepsilon x_1)|^2} \sum_{k=1}^3 \partial_{y_k}\delta(\varepsilon x_1)\partial_{v_*^k} s_1 \\
&= (s_1 - r) + \frac{[\nabla_y \delta(\varepsilon x_1) \cdot v_*][\nabla_y \delta(\varepsilon x_1) \cdot \nabla_{v_*} s_1]}{|\nabla_y \delta(\varepsilon x_1)|^2} \\
&= (s_1 - r) + [n(x_1) \cdot v_*][n(x_1) \cdot \nabla_{v_*} s_1] = s - r,
\end{aligned}$$

where we have used again [108, Lemma 2-(2)] on the last line to deduce that

$$[n(x_1) \cdot v_*][n(x_1) \cdot \nabla_{v_*} s_1] = (s - s_1).$$

It is worth remarking that the use of [108, Lemma 2-(2)] holds due to the fact that the proof is based on the Implicit Function Theorem, which holds for our case.

Altogether we have obtained that

$$\det \nabla_{v_*} X(v_*) = -(s-r)^3 + 2(s-r)^2\left((s_1 - r) + (s - s_1)\right) = (s-r)^3 + \mathscr{O}(\varepsilon).$$

The conclusion follows by putting together the previous informations. □

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RÉSUMÉ

Cette thèse est consacrée à l'étude de plusieurs équations cinétiques posées dans des domaines bornés et munies de conditions aux bords de type Maxwell. Nous nous intéressons en particulier aux problèmes où la frontière est non isotherme, c'est-à-dire qu'il existe une fonction dépendante de la variable spatiale représentant une température de paroi prescrite, constante dans le temps mais pouvant varier selon la position.

Cette étude est motivée par le problème physiquement pertinent de la description de l'évolution de particules à l'intérieur d'un domaine cylindrique, dont chacune des bases est munie de réservoirs de chaleur maintenus à des températures différentes.

Nous considérons deux cadres différents :

Premièrement, nous étudions des équations cinétiques de type Fokker–Planck pour lesquelles la température au bord peut varier de manière arbitraire, à condition qu'elle soit bornée et strictement positive. Dans ce cadre, nous utilisons des théorèmes de type Krein–Rutmann et Doblin–Harris, pour construire une solution stationnaire à ce problème et nous en décrivons sa stabilité.

Deuxièmement, nous étudions une équation de Boltzmann dans le régime de la limite hydrodynamique, dans des domaines régulières ou cylindriques, mais nous ne considérons que le problème isotherme. Dans ce cadre, nous montrons l'existence de solutions perturbatives près de l'équilibre et la stabilité asymptotique de la Maxwellienne. Cette étude est un premier pas vers l'étude de ce problème dans des conditions de Maxwell non isothermes.

MOTS CLÉS

comportement asymptotique en temps long, théorie cinétique, conditions aux bords de type Maxwell, solutions stationnaires hors équilibre, équation de Boltzmann, équations cinétiques de Fokker–Planck, réservoirs de chaleur de type BGK, théorie d'hypocoercivité, inégalité de Harnack, théorème de Krein–Rutman, ultracontractivité.

ABSTRACT

This thesis is devoted to the study of several kinetic equations set in bounded spatial domains and complemented with Maxwell boundary conditions. We are particularly interested in problems where the boundary is non-isothermal, meaning that there is a space dependent function representing a prescribed wall-temperature which is constant in time, but may vary in space.

This study is motivated by the physically relevant problem of describing the evolution of particles within a cylindrical domain, where each of its bases is equipped with heat thermostats held at different temperatures.

We consider two different setting:

First, we study kinetic Fokker-Planck equations where the wall temperature at the boundary may vary arbitrarily as long as it is bounded away from zero and infinity. In this framework, we use Krein-Rutmann and Doblin-Harris theorems to construct a stationary solution to this problem and we describe its stability.

Secondly, we study a Boltzmann equation in the regime of the hydrodynamic limit, set in smooth or cylindrical domains, but we only consider the isothermal problem. In this setting, we prove the existence of perturbative solutions near the equilibrium and the asymptotic stability of the Maxwellian. This study is a first step towards the investigation of this problem under non-isothermal Maxwell boundary conditions.

KEYWORDS

long-time asymptotic behavior, kinetic theory, Maxwell reflection boundary conditions, Non-equilibrium steady states, Boltzmann equation, kinetic Fokker-Planck equations, BGK heat thermostats, Hypocoercivity theory, Harnack inequality, Krein-Rutman theorem, ultracontractivity.