1 Dual space, strong, weak and weak* convergences

In what follows we consider X a (real or complex) vector space endowed with a norm denoted by $\|\cdot\|$ or $\|\cdot\|_X$. We assume that $(X, \|\cdot\|)$ is a Banach[1] space, that is any Cauchy sequence has a unique limit in X. The algebraic dual of X, denoted by X^*, is the space of linear mappings from X into $\mathbb{R}$ (if X is a real vector space) or $\mathbb{C}$, in the complex case. We denote by X' the topological dual space of X, that is the subspace defined by those elements of X^* which are continuous from $(X, \|\cdot\|)$ into $\mathbb{R}$ or $\mathbb{C}$. If $x \in X$ and $\ell \in X'$, the value of ℓ at x, or the action of ℓ on x, is denoted by $\langle \ell, x\rangle$ or sometimes $\ell(x)$. When $X \neq \{0\}$ is an infinite dimensional abstract vector space, it is not obvious to decide whether $X \neq \{0\}$...

This result, is a consequence of Kuratowski[2]-Zorn[3] lemma (also called Zorn's lemma), or equivalently a consequence of the Axiom of Choice.

First recall that in an ordered set $(Z, \leq)$, an element $m \in Z$ is said to be *maximal* if

$$x \in Z \quad \text{and} \quad m \leq x \Longrightarrow x = m.$$

Note that a maximal element need not to be the largest element. Also, if $(Z, \leq)$ is an ordered (non empty) set, one says that $(Z, \leq)$ is *inductive* if for any $A \subset Z$ which is totally ordered, there exists $a^* \in Z$ such that for all $x \in A$ one has $x \leq a^*$ (in short : $(Z, \leq)$ is inductive when any totally ordered subset, also called a *chain*, has an upper bound in Z. Note that it is not required that a^* belongs to A).

1.1 Lemma. *(Kuratowski-Zorn Lemma). If $(Z, \leq)$ is an ordered, non empty and inductive set, then there exists at least one maximal element in Z.*

[1] Stefan Banach, Polish mathematician, 1892–1945.
[2] Kazimierz Kuratowski, Polish mathematician, 1896–1980.
[3] Max August Zorn, German mathematician, 1906–1993.

This lemma has many applications in Algebra, for instance in the proof of existence of bases for general vector spaces, as well as in Analysis, for instance in the proof of the Hahn[4]-Banach theorem :

1.2 Theorem. *(Hahn-Banach Theorem, real case). Let X be a real vector space, and $j : X \longrightarrow \mathbb{R}$ a function such that :*
(i) *for any $\lambda > 0$ and $x \in X$, one has $j(\lambda x) = \lambda j(x)$;*
(ii) *for any $x, y \in X$ one has $j(x + y) \le j(x) + j(y)$;*
Let $E_0 \subset X$ be a vector subspace and $f_0 : E_0 \longrightarrow \mathbb{R}$ a linear mapping such that for all $x \in E_0$ one has $f_0(x) \le j(x)$.
Then there exists an extension $f : X \longrightarrow \mathbb{R}$ such that $f_{|E_0} = f_0$ and $f(x) \le j(x)$ for all $x \in X$.

There is another version of the Hahn-Banach theorem for the case of complex vector spaces :

1.3 Theorem. *(Hahn-Banach Theorem, complex case). Let X be a real or complex vector space, and $p : X \longrightarrow [0, \infty)$ a semi-norm on X. Let $E_0 \subset X$ be a vector subspace and $f_0 : E_0 \longrightarrow \mathbb{R}$ a linear mapping such that for all $x \in E_0$ one has $|f_0(x)| \le p(x)$.*

Then there exists an extension $f : X \longrightarrow \mathbb{R}$ such that $f_{|E_0} = f_0$ and $|f(x)| \le p(x)$ for all $x \in X$.

A geometric version of Hahn-Banach theorem is extremely useful and important : recall that a set K a subset of a vector space X is said to be *convex* if for all $t \in [0, 1]$ and $a, b \in K$ one has $(1 - t)a + tb \in K$.

1.4 Theorem. *(Hahn-Banach Theorem, geometrical version 1). Let $(X, \|\cdot\|)$ be a normed real vector space, A, B two non empty convex subsets of X. Assume that A is open and that $A \cap B = \emptyset$. then there exists $f \in X'$, $\alpha \in \mathbb{R}$ such that for all $a \in A$ and $b \in B$ one has $f(a) \le \alpha \le f(b)$.*

This result is interpreted in a geometrical sense by saying that the closed hyperplane $[f = \alpha]$, that is the set of $x \in X$ such that $f(x) = \alpha$, separates the convex sets A and B, that is $A \subset [f \le \alpha]$ while $B \subset [f \ge \alpha]$. Another version of the theorem concerns the separation of a closed subset from a compact one :

[4] Hans Hahan, Austrian mathematician, 1879–1934.

1.5 Theorem. *(Hahn-Banach Theorem, geometrical version 2). Let $(X, \|\cdot\|)$ be a normed real vector space, A, B two non empty convex subsets of X. Assume that A is closed, B is compact and that $A \cap B = \emptyset$. then there exists $f \in X'$, $\alpha \in \mathbb{R}$ such that for all $a \in A$ and $b \in B$ one has $f(a) < \alpha < f(b)$.*

The various forms of Hahn-Banach theorems are essential in Functional Analysis : for instance they are used in showing that for a non trivial normed vector space X, the topological dual is not reduced to $\{0\}$.

Also, denoting by $\mathscr{P}(X')$ the set of all subsets of X', one uses the Hahn-Banach theorem in order to show that there exists a *duality map* $\Psi : X \longrightarrow \mathscr{P}(X') \setminus \{\emptyset\}$ such that for any $x \in X$ one has

$$\Psi(x) = \left\{f \in X' \,;\, \|f\|_{X'} = \|x\|, \quad \text{and} \quad \langle f, x\rangle = \|x\|^2\right\}.$$

By the way, it is noteworthy that for each $x \in X$ the set $\Psi(x)$ is a convex subset of X', and when the norm of X' is *strictly convex* it is reduced to a single element[5].

Also, Hahn-Banach theorems are crucial in the following characterization of dense subsets in a Banach space : first, for a nonempty subset $A \subset X$, let us denote by $A^{\perp}$ the subspace of X' defined as being

$$A^{\perp} := \left\{f \in X' \,;\, \forall\, x \in A, \quad \langle f, x\rangle = 0\right\}.$$

1.6 *Density Lemma. Let $(X, \|\cdot\|)$ be a Banach space, and $A \subset X$ a nonempty subset. Then A is dense in X if and only if $A^{\perp} = \{0\}$.*

---

[5] A norm on a vector space $(X, \|\cdot\|)$ is said to be strictly convex if one has the following property

$$\forall x, y \in X, \quad \|x\| = \|y\| = 1, \quad \forall t \in (0,1), \quad \|(1-t)x + ty\| < 1.$$

The norm is said to be uniformly convex if one has the following property:

$$\begin{aligned}&\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x, y \in X,\\ &\|x\| \le 1,\ \|y\| \le 1,\ \|x - y\| \ge \varepsilon \quad \Longrightarrow \quad \left\|\frac{x+y}{2}\right\| \le 1 - \delta.\end{aligned}$$

A space $(X, \|\cdot\|)$ in which the norm is uniformly convex is reflexive. It is easy to see that any Hilbert space $(H, (\cdot|\cdot))$ is uniformly convex. Also it is important to know that the spaces $L^p(\Omega)$ are uniformly convex for $1 < p < \infty$.

When $(x_n)_n$ is a sequence of X one says that $(x_n)_n$ (or x_n) *converges weakly* to $x^* \in X$, if and only if for any $f \in X'$ we have

$$\langle f, x_n \rangle = f(x_n) \to f(x^*) = \langle f, x^* \rangle .$$

We shall write $x_n \rightharpoonup x^*$ or $x_n \to x^*$ in X_w. When we talk of $(x_n)_n$ or x_n converging (or converging strongly) to x^*, we mean that the convergence takes place in the sense of the norm of X, i.e. $\|x^* - x_n\| \to 0$ as $n \to \infty$.

If X is a Banach space and X' is its dual space, we say that a sequence $(f_n)_n$ in X' converges to f in X'-weak* (and we write $f_n \rightharpoonup^* f$ in X'_{w^*}), if for any $x \in X$ we have $\langle f_n, x \rangle \to \langle f, x \rangle$. Note that if $(f_n)_n$ is a sequence in X' and $(x_n)_n$ a sequence in X, then if $f_n \to f$ in X' and $x_n \rightharpoonup x$ in X_w, then one has $\langle f_n, x_n \rangle \to \langle f, x \rangle$. The same is true if $f_n \rightharpoonup^* f$ in X'_{w^*} while $x_n \to x$ dans X. In general one cannot say anything about the limit of $\langle f_n, x_n \rangle$ when $f_n \rightharpoonup^* f$ in X'_{w^*} and $x_n \rightharpoonup x$ in X_w. It is useful to recall that when $x_n \rightharpoonup x$ and $f_n \rightharpoonup^* f$ in X'_{w^*} then we have :

$$\|x\| \le \liminf_{n \to \infty} \|x_n\|, \qquad \|f\|_{X'} \le \liminf_{n \to \infty} \|f_n\|_{X'}. \tag{1.1}$$

Let X be a Banach space ; then X' is a Banach space and so one may consider its dual $X'' := (X')'$. It is easy to see that there is a canonical embedding Φ from X into X'' : if $x \in X$ denote $\ell := \Phi(x)$ as being that linear functional on X' defined by $\ell(f) := \langle f, x \rangle$. Upon identifying X and $\Phi(X)$ one may consider X as a subspace of X''. We say that X is a reflexive Banach space when this embedding Φ is also onto, that is $\Phi(X) = X''$. If this is the case then we shall identify X and X''. For instance a Hilbert space H is a reflexive Banach space and, as a matter of fact as far as only one Hilbert space is involved, thanks to F. Riesz[6] representation theorem, oe may identify also H and H'.

However it is important to have in mind that in general it is impossible to identify *simultaneously* the dual spaces of two Hilbert spaces, one being embedded into the second. To be more specific, let H_0 and H_1 be two distinct Hilbert spaces such that their norms, denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$, satisfy (for some fixed positive constant c)

$$H_1 \subset H_0, \qquad \text{and} \qquad \forall u \in H_1, \quad \|u\|_0 \le c\,\|u\|_1,$$

[6] Frigyes (also written Frédéric in French) Riesz, Hungarian mathematician, 1880–1956.

and moreover let us assume that H_1 is dense in H_0. Hence one may identify the dual H_0' of H_0 with a subsapce of H_1', the dual space of H_1, so that we can write $H_0' \subset H_1'$ with a continuous embedding. Now it is quite clear that if, for instance, we decide to identify, through F. Riesz' representation theorem, H_0 with its dual H_0', then we cannot consider at the same time H_1 as a subspace of H_0 and also identify H_1 and H_1'. Quite often, when H_0 is the Lebesgue space $L^2(\Omega)$, one identifies H_0 and H_0', hence prohibiting the identification of a *smaller* Hilbert space with its dual.

Reflexive Banach spaces play a crucial rôle in the study of variational problems and nonlinear equations, due to the following interesting property (see for instance K. Yosida, *Functional Analysis*, Appendix to chapter V, § 4) :

1.7 *Theorem (Eberlein[7]-Shmulyan[8]). A Banach space X is reflexive if and only if any bounded sequence $(x_n)_n$ in X contains a subsequence $(x_{n_i})_i$ which converges weakly in X.*

In practice, this result is used when one has to show that a certain sequence in a reflexive Banach space is relatively compact (at least for a reasonable topology) : one begins by showing that the sequence is bounded, then one considers a weakly convergent sequence (which exists thanks to the above theorem) and finally one tries to show that this weakly convergent subsequence is strongly convergent… Clearly, in general this last step is not an easy one, but using different strong topologies (for instance by considering a larger Banach space) one hopes to achieve this goal. Most difficulties in the study of nonlinear problems stem from the passage from a weak convergence to a strong one.

1.8. Remark

When X is a Banach space, one can show that any bounded sequence $(f_n)_n$ in X' contains a subsequence $(f_{n_j})_j$ which converges in X'-weak*. This is particularly useful in the case in which X is the Lebesgue space $L^1(\Omega)$, and thus X' is the Lebesgue space $L^\infty(\Omega)$. As a matter of fact, the only *usable* weak convergence in $L^\infty(\Omega)$ is the weak*-convergence of sequences. Another situation is when, $K \subset \mathbb{R}^N$ being a compact set, we have $X = C(K)$ endowed with the norm $\|\cdot\|_\infty$, and $X' = M(K)$ is the space of bounded measures on K : in this situation very

[7] William Frederick Eberlein, American mathematician, ??–.
[8] Vitold Lvovich Shmulyan, Russian mathematician, 1915–1945?

often one is led to consider a bounded sequence of measures $(\mu_n)_n$ in $M(K)$. Then the fact that $M(K)$ is the dual of the Banach space $C(K)$, allows us to state that there exists a subsequence $(\mu_{n_j})_j$ which converges weakly to some bounded measure μ in $M(K)$. $\square$

2 Exercises

Exercise 1

Let the space $L^2(\mathbb{R})$ be equipped with the scalar product and its associated norm

$$(f|g) := \int_{\mathbb{R}} f(x)\,g(x)\,dx, \qquad \|f\| := \sqrt{(f|f)}.$$

Take $f_0 \in L^2(\mathbb{R})$ be given, and for $n \geq 1$ integer define

$$f_n(x) := f_0(x+n).$$

Study the sequence $(f_n)_{n\geq 1}$ in $L^2(\mathbb{R})$ regarding its weak or strong convergence.

What can be said if we consider $L^p(\mathbb{R})$ for $1 \leq p < \infty$ instead of $L^2(\mathbb{R})$.

Exercise 2

Let the space $L^p(\mathbb{R})$ be equipped with the scalar product and its associated norm

$$(f|g) := \int_{\mathbb{R}} f(x)\,g(x)\,dx, \qquad \|f\| := \sqrt{(f|f)}.$$

Take $f_0 \in L^2(\mathbb{R})$ be given, and for $n \geq 1$ integer define

$$f_n(x) := n^{1/2} f_0(nx).$$

Study the sequence $(f_n)_{n\geq 1}$ in $L^2(\mathbb{R})$ regarding its weak or strong convergence.

Construct an analogous sequence in the space $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Exercise 3

Let H be the Hilbert spaces of functions which are square integrable for the Guassian measure $d\mu \coloneqq \exp(-|x|^2/2)\,dx$, i.e.

$$H \coloneqq \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^N)\ ;\ \int_{\mathbb{R}^N} |u(x)|^2 \exp(-|x|^2/2)\,dx < \infty \right\}.$$

We shall denote by $(\cdot|\cdot)$ and $\|\cdot\|$ the scalar product and the norm of H. Let $u \in \mathscr{S}(\mathbb{R}^N)$ a function. For $\lambda > 0$ et $b \in \mathbb{R}^N$ we denote $u_{\lambda,b}(x) \coloneqq u(\lambda x + b)$, and we denote by E_0 the vector space generated by the family $(u_{\lambda,b})_{\lambda>0,b\in\mathbb{R}^N}$. Recall that if $\alpha \in \mathbb{N}^N$ we denote by $|\alpha| \coloneqq \sum_{k=1}^N \alpha_k$, the length of α.

1) Prove that the space $C_c^\infty(\mathbb{R}^N)$ is dense in H. (Recall that it is assumed that we know already that $C_c^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$).

2) We denote by Π the vector space of polynomials generated by the functions $\varphi_\alpha(x) \coloneqq x^\alpha \coloneqq \prod_{1\le j\le N} x_j^{\alpha_j}$ for $\alpha \in \mathbb{N}^N$. For $f \in H$ fixed, justify the equality

$$\sum_{k\ge 0} \int_{\mathbb{R}^N} \frac{(-\mathrm{i}x\cdot\xi)^k}{k!} f(x)\exp(-|x|^2/2)\,dx = \int_{\mathbb{R}^N} \sum_{k\ge 0} \frac{(-\mathrm{i}x\cdot\xi)^k}{k!} f(x)\exp(-|x|^2/2)\,dx.$$

Conclude that if $f \in \Pi^\perp$, then $f \equiv 0$ and thus Π is dense in H.

3) For $n \in \mathbb{N}$ given, prove that there exist integers $a_\alpha \ge 1$ such that

$$\frac{\partial^n}{\partial\lambda^n} u_{\lambda,b}(x) = \sum_{|\alpha|=n} a_\alpha \varphi_\alpha(x) \frac{\partial^n u}{\partial x^\alpha}(\lambda x + b),$$

where as a matter of fact

$$a_\alpha \coloneqq \frac{n!}{\alpha!} \quad \text{with} \quad \alpha! \coloneqq \alpha_1!\,\alpha_2!\cdots\alpha_N!.$$

4) Let $f \in H$. Prove that if we set $h(\lambda) \coloneqq (f|u_{\lambda,b})$, then

$$h^{(n)}(0) = \frac{d^n}{d\lambda^n} h(0) = \sum_{|\alpha|=n} a_\alpha (f|\varphi_\alpha)\,\partial^\alpha u(b).$$

5) Here we assume that the function u is such that the space E_0 is not dense in H. Prove that there exists an integer $n \geq 0$, and some constants c_α, for $\alpha \in \mathbb{N}^N$ and $|\alpha| = n$, not all equal to zero, such that u satisfies the the partial differential equation :

$$\forall x \in \mathbb{R}^N, \qquad \sum_{|\alpha|=n} c_\alpha \partial^\alpha u(x) = 0. \tag{2.1}$$

6) Let $\varepsilon > 0$ be fixed, and denote by E_1 the vector space generated by the family $u_{\lambda,b}$ when λ vary in the interval $(-\varepsilon, \varepsilon)$, that is $|\lambda| < \varepsilon$, and $b \in \mathbb{R}^N$. Can one characterize the function u with a homogeneous partial differential equation such as (2.1) if E_1 is not dense in H ?

7) Let $u(x) \coloneqq \exp(-|x|^2/2)$. With the above notations for the spaces E_0 and E_1, are either of these spaces dense in H ? With this choice of u, can one restrict also the parameter b to vary under the condition $b \in \Omega$ for a non empty open set $\Omega \subset \mathbb{R}^N$?

Exercise 4

For $1 \leq p < \infty$ we denote by $\ell^p \coloneqq \ell^p(\mathbb{N}^*, \mathbb{C})$ the set of complex sequences $x \coloneqq (x_n)_{n\geq 1}$ such that $\sum_{n\geq 1} |x_n|^p < \infty$. The set of ***bounded*** complex sequences will be denoted by $\ell^\infty \coloneqq \ell^\infty(\mathbb{N}^*, \mathbb{C})$. We shall also denote by $\mathbf{c}_0 \coloneqq \mathbf{c}_0(\mathbb{N}^*, \mathbb{C})$ the st of complex sequences which converge to zero, and by $\mathbf{c} \coloneqq \mathbf{c}(\mathbb{N}^*, \mathbb{C})$ the set of complex sequences which converge to some limit. We introduce the following notation which will be proved to be a norm in this exercise :

$$\|x\|_p \coloneqq \Big(\sum_{n\geq 1} |x_n|^p\Big)^{1/p}, \quad \text{et} \ \ \|x\|_\infty \coloneqq \sup_{n\geq 1} |x_n|.$$

For $p \in [1, \infty]$ on note $p' \coloneqq p/(p-1)$ the (Hölder) conjugate of of p.

1) Let $\alpha, \beta \in \mathbb{C}$ and $1 < p < \infty$. Prove the Young inequality :

$$|\alpha\beta| \leq \frac{|\alpha|^p}{p} + \frac{|\beta|^{p'}}{p'} .$$

(One can start by showing this result for $\alpha, \beta \in \mathbb{R}_+$). Conclude that for $x \in \ell^p$ and $y \in \ell^{p'}$ we have

$$\sum_{n\geq 1} |x_n y_n| \leq \frac{1}{p}\|x\|_p^p + \frac{1}{p'}\|y\|_{p'}^{p'}. \tag{2.2}$$

2) Let $p \in [1, \infty]$. Using the previous question, show the following inequality (called the Hölder inequality) : for $x \in \ell^p$ and $y \in \ell^{p'}$:

$$\sum_{n\geq 1} |x_n y_n| \leq \|x\|_p \, \|y\|_{p'}. \tag{2.3}$$

(In (2.2) replace x by λx and y by $\lambda^{-1} y$ for a parameter $\lambda > 0$ and then choose λ appropriately).

3) Let $1 < p < \infty$ and $x, y \in \ell^p$. Upon writing

$$(|x_n| + |y_n|)^p = |x_n|(|x_n| + |y_n|)^{p-1} + |y_n|(|x_n| + |y_n|)^{p-1},$$

and using the Hölder's inequality (2.3) on each of these terms, prove the Minkowski inequality

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p. \tag{2.4}$$

4) Prove that the spaces que $(\ell^p, \|\cdot\|_p)$, for $1 \leq p \leq \infty$, as well as the spaces $(\mathbf{c}_0, \|\cdot\|_\infty)$ and $(\mathbf{c}, \|\cdot\|_\infty)$ are normed vector spaces.

5) Prove that all these spaces are Banach spaces.

6) Prove that for $1 \leq p < q \leq \infty$, if $x \in \ell^p$ then $\|x\|_q \leq \|x\|_p$, that is we have $\ell^p \subset \ell^q$ and that the imbedding is continuous. Show also that these spaces are distinct.

7) Let $p_0 < \infty$ and $x \in \ell^{p_0}$. Prove that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

8) Denote by $\mathbb{E}$ the vector space consisting of complex sequences $(x_n)_{n\geq 1}$ for which there exists $n_0 \geq 1$ such that $x_n = 0$ for $n \geq n_0$. Prove that for $1 \leq p < q \leq \infty$ we have the following strict inclusions :

$$\mathbb{E} \subset \ell^p \subset \ell^q \subset \mathbf{c}_0 \subset \mathbf{c} \subset \ell^\infty.$$

9) Prove that $\mathbf{c}_0$ and $\mathbf{c}$ are closed subspaces of ℓ^∞.

10) Prove that for $1 \leq p < \infty$ the closure of $\mathbb{E}$ in ℓ^p is equal to ℓ^p, and is equal to $\mathbf{c}_0$ if $p = \infty$.

11) For $x \in \mathbf{c}$ denote by $L(x) := \lim_{n\to\infty} x_n$. Prove that L is a continuous linear form on $\mathbf{c}$. Conclude that $\mathbf{c}$ is isomorphic to $\mathbb{C} \oplus \mathbf{c}_0$.

12) Prove that ℓ^p (for $1 \le p < \infty$) as well as $\mathbf{c}_0$ and $\mathbf{c}$ are separable Banach spaces.

13) Let $B := \{x \in \ell^\infty \; ; \; \forall\, n \in \mathbb{N}^*, \; x_n \in \mathbb{N}\}$. Verify that if $x, y \in B$ are distinct, then $\|x - y\|_\infty \ge 1$.

14) Let $A \subset \ell^\infty$ be a dense subset in ℓ^∞. If $x \in B$, we denote by a_x an element of A such that $\|x - a_x\|_\infty < 1/4$.

Prove that the mapping $x \mapsto a_x$ from B into A is injective, and conclude that ℓ^∞ is not separable.

15) Let $1 < p < \infty$. For $n \ge 1$ we denote by $\boldsymbol{e}_n$ the sequence such that $(\boldsymbol{e}_n)_k = \delta_n^k$ for $k \ge 1$ (the Kronecker symbol, $\delta_n^n = 1$ and $\delta_n^k = 0$ if $n \ne k$). We denote by $\langle \cdot, \cdot \rangle$ the duality between ℓ^p and $(\ell^p)'$: for $x \in \ell^p$ and $\xi \in (\ell^p)'$ we denote by $\langle \cdot, \cdot \rangle$ the duality between ℓ^p and $(\ell^p)'$. Also, for convenience we set $q := p' = p/(p-1)$, in order to avoid, for the time being, confusion between $\ell^{p'}$ and $(\ell^p)'$…

For $\xi \in (\ell^p)'$ we set $\alpha_n := \langle \xi, \boldsymbol{e}_n \rangle$ for $n \ge 1$. Prove that $\alpha := (\alpha_n)_{n\ge1} \in \ell^q$ and that $\|\alpha\|_q \le \|\xi\|_{(\ell^p)'}$. (Observe that setting

$$y_j := |\alpha_j|^{q-2}\overline{\alpha_j}$$

we have $y^{(n)} := \sum_{j=1}^n y_j e_j \in \ell^p$, and $|\langle \xi, y^{(n)} \rangle| \le \|\xi\|_{(\ell^p)'} \, \|y^{(n)}\|_p$).

16) With the above notations, consider the mapping $\Phi : (\ell^p)' \longrightarrow \ell^q$ defined by $\Phi(\xi) := \alpha$ (recall that $q := p' = p/(p-1)$).

Prove that $\|\xi\|_{(\ell^p)'} \le \|\Phi(\xi)\|_q$ for $\xi \in (\ell^p)'$. (If $x \in \ell^p$ consider $x^{(n)} := \sum_{j=1}^n x_j e_j$ and then find an upper bound for $\left|\langle \xi, x^{(n)} \rangle\right|$ using the Hölder inequality, and finally let $n \to \infty$).

17) Prove that Φ is surjective and is an isometry from $(\ell^p)'$ into ℓ^q with $q = p'$. From now on, we shall make the identification $(\ell^p)' = \ell^{p'}$, for $1 < p < \infty$.

18) Using an analogous approach show that the dual of ℓ^1 can be identified to ℓ^∞.

19) Prove that $(\mathbf{c}_0)' = \ell^1$.

20) Determine the dual of $\mathbf{c}$.

21) Prove that if $1 < p < \infty$ the Banach space ℓ^p is reflexive.

Exercise 5

Let $(H, (\cdot | \cdot))$ be a Hilbert space. Prove that the norm of H, defined by $\|x\| : \sqrt{(x|x)}$ for $x \in H$, is uniformly convex, that is

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall x, y \in X,$$
$$\|x\| \leq 1, \ \|y\| \leq 1, \ \|x - y\| \geq \varepsilon \quad \Longrightarrow \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

(Remember the parallelogram identity for a Hilbertian norm, that is the fact that $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for $x, y \in H$).

3 Convergence theorems

When studying various properties of functions or operators acting on functions, or manipulating series and sequences of functions, one is led quite often to handle integrability properties of functions and at the same time pass to the limit in different senses. In this note we gather a number of results which are useful in such situations.

In what follows the notion of integral is that of Lebesgue[1]. For a thorough study of this notion, the reader is strongly encouraged to refer to the excellent book by Walter Rudin[2] *Real and Complex Analysis*. However, as long as the application of the following results is concerned, one may admit that all the functions and sets involved are *measurable*, and at this stage it is not essential to get into the intricacies of the precise definition of this term… (please, be kind and don't tell anyone that I said that this aspect is not essential, or else I am going to be in big trouble…). The most important is to know how to check the assumptions or conditions of the convergence theorems below, and be aware that there might be counter-examples when one or several of conditions are not satisfied exactly as it is stated.

First of all, let us recall what is a *null set*:

3.1 Definition. *A set $A \subset \mathbb{R}^N$ is a null set (for the Lebesgue measure) if for any $\varepsilon > 0$ one may find a countable sequence of balls $(B(x_j, r_j))_{j\geq 1}$ such that*

$$A \subset \bigcup_{j\geq 1} B(x_j, r_j), \quad \textit{and} \quad \sum_{j\geq 1} r_j^N \leq \varepsilon.$$

Thinking of the fact that the *measure* of the ball $B(x_j, r_j)$ corresponds to its *volume*, that is to r_j^N up to a fixed multiplicative constant (equal to the volume of the unit ball of $\mathbb{R}^N$), a null set would correspond to a set of *measure* zero. However, some care is due here since there is a serious problem, far from being

[1] Henri Lebesgue, French mathematician, 1875–1941.
[2] Walter Rudin, American mathematician, born in 1921.

a metaphysical question : if we have two sets $B_0 \subset B_1$, and if B_1 is measurable and the measure of the set B_1 is known, can one also define the measure of the set B_0 and say that the *measure* of B_0 is not greater than that of B_1 ?…

The reader may verify that when A is reduced to a single point, i.e. $A := \left\{a\right\}$ for some $a \in \mathbb{R}^N$, then A is a null set. It is an interesting exercise to show that if $(A_n)_n$ is a countable family of null sets, and $A := \cup_{n\geq 1} A_n$, then A is also a null set. In particular one sees that if $A := \left\{a_n\ ;\ n \geq 1\right\}$ for a sequence of $a_n \in \mathbb{R}^N$, then A is a null set.

Once the notion of null set is at hand, we introduce the notion of *almost everywhere convergence* :

3.2 Definition. *A sequence of integrable functions $(f_n)_n$ (for instance defined on $\Omega \subset \mathbb{R}^N$ with values in $\mathbb{C}$) converges almost everywhere to f, if there exists a null set $A \subset \mathbb{R}^N$ such that $f_n(x) \to f(x)$ for all $x \in A^c := \mathbb{R}^N \setminus A$.*

We shall write $f_n \to f$ a.e. in Ω, and in general we shall say that an inequality, or a property, holds almost everywhere, in short a.e., if it holds outside a certain null set. In particular the functions we consider are defined a.e. and we identify two functions which are equal a.e.

When Ω in an open subset of $\mathbb{R}^N$, we denote by $L^1(\Omega)$ the Lebesgue space of integrable functions defined on Ω taking their values in $\mathbb{R}$ or $\mathbb{C}$. We shall note

$$\|f\|_1 := \|f\|_{L^1(\Omega)} := \int_\Omega |f(x)|dx.$$

One of the great advantages of Lebesgue measure is its *robustness* regarding the integrability of limits of integrable functions. The first result, which as a matter of fact is the crux in the definition of Lebesgue integral, is the the following :

3.3 Theorem. *(Monotone Convergence Theorem). Let $(f_n)_{n\geq 1}$ be a sequence of measurable functions defined on Ω such that $0 \leq f_n \leq f_{n+1}$ a.e. Then upon setting $f(x) := \sup_{n\geq 1} f_n(x)$ (meaning that $f(x) := \lim_{n\to\infty} f_n(x) \in [0, +\infty]$) we have*

$$\int_\Omega f(x)\,dx = \sup_{n\geq 1} \int_\Omega f_n(x)\,dx = \lim_{n\to\infty} \int_\Omega f_n(x)\,dx.$$

A slightly different version is :

3.4 Theorem. *(Beppo-Levi's[3] Convergence Theorem). Let $(f_n)_{n\geq 1}$ be a sequence of integrable functions defined on Ω such that $f_{n+1} \geq f_n$ a.e. Then upon setting $f(x) := \sup_{n\geq 1} f_n(x)$ (meaning that $f(x) := \lim_{n\to\infty} f_n(x) \in [0, +\infty]$) we have*

$$f \text{ is integrable } \iff \sup_{n\geq 1} \int_\Omega f_n(x)\,dx < \infty,$$

and if f is indeed integrable we have

$$\int_\Omega f(x)\,dx = \lim_{n\geq 1} \int_\Omega f_n(x)\,dx = \sup_{n\geq 1} \int_\Omega f_n(x)\,dx,$$

$$\text{and} \quad \lim_{n\to\infty} \int_\Omega |f(x) - f_n(x)|\,dx = 0.$$

Quite often it is not known whether a sequence $(f_n)_n$ of integrable functions converges, or not, almost everywhere, and nothing is known about the monotonicity of the sequence : in these situations the following lemma is of great help :

3.5 Theorem. *(Fatou's[4] Lemma). Let $f_n \geq 0$ for $n \geq 1$ be a sequence of integrable functions. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n(x)\,dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x)\,dx.$$

The next important convergence theorem is

3.6 Theorem. *(Lebesgue's Dominated Convergence Theorem). If $(f_n)_n$ converges to f a.e. and if there exists an integrable function g such that for all $n \geq 1$ one has $|f_n| \leq g$ a.e., then f is integrable and moreover*

$$\int_{\mathbb{R}^N} f(x)\,dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} f_n(x)\,dx, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} |f(x) - f_n(x)|\,dx = 0.$$

In other terms, we may pass to the limit under the integral sign, provided there is a.e. convergence and *domination* of the sequence $(f_n)_n$ by a fixed integrable function g.

---

[3] Beppo Levi, Italian mathematician, 1875–1961.
[4] Pierre Joseph Louis Fatou, French mathematician, 1878–1929.

A typical use of this dominated convergence theorem is in the proof of the following result. Let $f : [a,b] \times \mathbb{R}^N \longrightarrow \mathbb{C}$ such that for all $t \in [a,b]$ the function $x \mapsto f(t,x)$ is integrable and, a.e. in $x \in \mathbb{R}^N$, the function $t \mapsto f(t,x)$ is continuous at $t_0 \in [a,b]$. Then, upon setting

$$F(t) := \int_{\mathbb{R}^N} f(t,x)\,dx,$$

can one say that F is continuous at t_0 ? In general the answer is negative (it is a good exercise to exhibit a counter-example…), but if we assume moreover that there exists an integrable function $g : \mathbb{R}^N \longrightarrow [0,\infty)$ such that for all $t \in [a,b]$ and a.e. in $x \in \mathbb{R}^N$ one has $|f(t,x)| \leq g(x)$, then F is continuous at t_0. Indeed, if $t_n \to t_0$ is a sequence of points in $[a,b]$ and $f_n(x) := f(t_n,x)$, then we know that $f_n(x) \to f_0(x) := f(t_0,x)$, and that $|f_n(x)| \leq g(x)$ a.e. in $x \in \mathbb{R}^N$. Therefore, according to theorem 3.6, we may write :

$$\lim_{n\to\infty} |F(t_n) - F(t_0)| \leq \lim_{n\to\infty} \int_{\mathbb{R}^N} |f_n(x) - f_0(x)|\,dx = 0.$$

Another noteworthy result concerns the *derivation under the integral sign* :

3.7 Theorem. . *Let $f : [a,b] \times \Omega \longrightarrow \mathbb{C}$ be such that for all $t \in [a,b]$ the function $x \mapsto f(t,x)$ is integrable and, a.e. in $x \in \Omega$, the function $t \mapsto f(t,x)$ is of class C^1 on $[a,b]$. Set*

$$F(t) := \int_\Omega f(t,x)\,dx,$$

and assume that there exists an integrable function $g : \Omega \longrightarrow [0,\infty)$ such that

$$\left|\frac{\partial f(t,x)}{\partial t}\right| \leq g(x) \quad \text{a.e. in } x \in \Omega$$

Then $F \in C^1([a,b])$ and

$$\frac{F'(t) = \int_\Omega \partial f(t,x)}{\partial t\,dx.} \tag{3.1}$$

The proof of this result is quite easy : for $t_0 \in [a,b]$ and a sequence $h_n \to 0$, according to the mean value theorem there exists $\theta \in (0,1)$ depending on x, t_0 and h_n, such that :

$$(3.2)\qquad \frac{F(t_0+h_n)-F(t_0)}{h_n} = \int_\Omega \frac{f(t_0+h_n,x)-f(t_0,x)}{h_n}\,dx = \int_\Omega \frac{\partial f(t_0+\theta h_n,x)}{\partial t}\,dx.$$

Then setting

$$f_n(x) := \frac{\partial f(t_0+\theta h_n,x)}{\partial t}$$
$$f_0(x) := \frac{\partial f(t_0,x)}{\partial t},$$

we know that $f_n \to f_0$ and $|f_n| \le g$ a.e. Therefore, according to the dominated convergence theorem 3.6, in (3.2) we may pass to the limit under the integral sign and obtain (3.1).

Another important result in the handling of integrals is when one has to integrate a series of functions. Let $(f_n)_{n\ge0}$ be a sequence of positive functions. Then using the monotone convergence theorem 3.3 we see that

$$\int_{\mathbb{R}^N} \sum_{n=0}^{\infty} f_n(x)\,dx = \sum_{n=0}^{\infty} \int_{\mathbb{R}^N} f_n(x)\,dx,$$

where, by convention, we assme that the equality holds in $[0,+\infty]$. Actualy this is a particular case of Fubini[5]-Tonelli[6] theorem, of which the version we give below is particularly useful. Before stating this result let us make the following convention : if $m,k \ge 1$ are given integers and $N := k+m$, a generic element $x \in \mathbb{R}^N$ is written as $x = (x_1,x_2)$ with $x_1 \in \mathbb{R}^k$ and $x_2 \in \mathbb{R}^m$.

3.8 Theorem. *(Fubini-Tonelli Theorem). If $f : \mathbb{R}^N \longrightarrow \mathbb{C}$ and either of the following three integrals*

$$\int_{\mathbb{R}^N} |f(x)|dx,$$
$$\int_{\mathbb{R}^k}\left(\int_{\mathbb{R}^m} |f(x_1,x_2)|dx_2\right)dx_1,$$
$$\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^k} |f(x_1,x_2)|dx_1\right)dx_2$$

is finite, then the two others are also finite, and the three integrals are equal. Moreover we have

[5] Guido Fubini, Italian mathematician, 1879–1943.
[6] Leonida Tonelli, Italian mathematician, 1885–1946.

$$\int_{\mathbb{R}^N} f(x)\,dx = \int_{\mathbb{R}^k}\Big(\int_{\mathbb{R}^m} f(x_1,x_2)\,dx_2\Big)\,dx_1 = \int_{\mathbb{R}^m}\Big(\int_{\mathbb{R}^k} f(x_1,x_2)\,dx_1\Big)\,dx_2.$$

Practically this result is used as follows : in order to compute a *double* integral by inverting the order of the integrations, one shows first that one of the above three integrals is finite.

A useful version of the Fubini-Tonelli theorem is the following useful result concerning the integral of the sum of a series of functions :

3.9 Theorem. *If $f_n : \mathbb{R}^N \longrightarrow \mathbb{C}$, and if either of the two expressions*

$$\int_{\mathbb{R}^N} \sum_{n=0}^{\infty} |f_n(x)|dx, \qquad \sum_{n=0}^{\infty} \int_{\mathbb{R}^N} |f_n(x)|dx$$

is finite, then the other one is also finite and they are both equal. Moreover when these quantities are finite we have

$$\int_{\mathbb{R}^N} \sum_{n=0}^{\infty} f_n(x)\,dx = \sum_{n=0}^{\infty} \int_{\mathbb{R}^N} f_n(x)\,dx.$$

Recall that a sequence of continuous functions $(\varphi_n)_n)$ defined for instance on a compact metric space (K,d) is said to be *equicontinuous* when for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $x, y \in K$ and $d(x,y) < \delta$, for all $n \geq 1$ we have $|\varphi_n(x) - \varphi_n(y)| < \varepsilon$: in other terms the modulus of continuity is the same for all φ_n. Now, an analogous notion exists regarding the integrability of a sequence of functions :

3.10 Definition. *A sequence of integrable functions (f_n) defined on Ω is said to be equi-integrable if for any $\varepsilon > 0$ there exist $\delta > 0$ and a subset $A \subset \Omega$ with* $\mathrm{meas}(\Omega) < \infty$*, such that for all $n \geq 1$ and all measurable sets $E \subset \Omega$ with* $\mathrm{meas}(E) < \delta$ *one has :*

$$\int_{A^c} |f_n(x)|dx < \varepsilon, \qquad \int_E |f_n(x)|dx < \varepsilon.$$

This notion is particularly useful in proving the convergence of sequences of integrable functions of which one knows they converge a.e. :

3.11 Theorem. *(Vitali's[7] Lemma). Let $(f_n)_n$ be a sequence of integrable functions defined on Ω, such that $f_n \to f$ a.e. Then*

$$\lim_{n\to\infty} \|f - f_n\|_{L^1(\Omega)} = 0 \iff (f_n)_n \text{ is equi-integrable on } \Omega.$$

The reader is strongly encouraged to prove the following corollary which is a slight improvement of the Lebesgue's dominated convergence theorem :

3.12 Corollary. *Assume that $(f_n)_n$ and $(g_n)_n$ are two sequences of integrable functions such that $f_n \to f$ and $|f_n| \le g_n$ a.e. Moreover assume that $g_n \to g$ in $L^1(\Omega)$ as well as a.e. Then $f_n \to f$ in $L^1(\Omega)$.*

Finally the following result, which is *almost* the converse of Lebesgue's dominated convergence theorem, is of some importance in the study of nonlinear equations : it establishes a certain relationship between convergence in the sense of the norm of $L^1(\Omega)$ and almost everywhere convergence.

3.13 Theorem. *Let $(f_n)_n$ be a sequence of integrable functions such that $f_n \to f$ in $L^1(\Omega)$. Then there exists a subsequence $(f_{n_k})_k$ and $g \in L^1(\Omega)$ such that*

$$f_{n_k} \to f \quad \text{a.e. in } \Omega, \qquad \text{and} \qquad |f_{n_k}| \le g \quad \text{a.e. in } \Omega.$$

When using this result, one should be aware that we may only assert that a subsequence $(f_{n_k})_k$ is converging to f, and not all the sequence $(f_n)_n$.

4 Exercises

Exercise 1

Exercise 2

Give an example of a nonnegative sequences of integrable functions $(f_n)_n$ on a domain Ω such that

[7] Giovanni Vitali, Italian mathematician, 1863–1909.

$$\int_{\Omega} \liminf_{n\to\infty} f_n(x)\,dx < \liminf_{n\to\infty} \int_{\Omega} f_n(x)\,dx,$$

and another (non trivial…) example where

$$\int_{\Omega} \liminf_{n\to\infty} f_n(x)\,dx = \liminf_{n\to\infty} \int_{\Omega} f_n(x)\,dx.$$

5 Change of variables in integrals

In this section we show the theorem concerning the change of variables in the Lebesgue integral. Here m denotes the Lebesgue measure on $\mathbb{R}^N$. We begin with an algebraic result regarding a ceratin type of matrix decomposition (E. Cartan's[1] decomposition). Recall that an orthogonal matrix is a matrix R such that

$$R^*R = RR^* = I,$$

where I is the identity matrix.

5.1 Lemma. *If T is an invertible matrix, there exist two orthogonal matrices R_1 and R_2, and an invertble diagonal matrix D such that $T = R_1DR_2$.*

Proof. Indeed the matrix T^*T, being self-adjoint and positive definite, is diagonalisable and can be written in the form $T^*T = U^*\Delta U$ where U is an orthogonal matrix and Δ is diagonal with positive entries. Setting

$$D \coloneqq \Delta^{1/2}, \qquad R_2 \coloneqq U, \qquad R_1 \coloneqq TR_2^*D^{-1},$$

one checks that $R_1^*R_1 = I$ et $T = R_1DR_2$. □

In a first step, we consider linear changes of variables.

5.2 Lemma. *Let T be an invertible matrix of order N. If $\varphi(x) \coloneqq Tx$ and $\Omega \subset \mathbb{R}^N$ is a borelian set, we have*

$$m(\varphi(\Omega)) = |\det(T)|\, m(\Omega).$$

Proof. Let $m_1(\Omega) \coloneqq m(\varphi(\Omega))$. One verifies that m_1 is a measure, invariant under translation, and therefore m_1 is a multiple of m i.e.

$$\exists\, c(T) > 0, \quad \forall\, \Omega \text{ measurable}, \qquad m_1(\Omega) = c(T)\, m(\Omega). \tag{5.1}$$

[1] Elie Cartan, French mathematician, 1869–1951.

One can see easily that $c(T_1T_2) = c(T_1)c(T_2)$, for two invertible matrices T_1 and T_2. If R is an orthogonal matrix, the unit ball of $\mathbb{R}^N$ is invariant under R, and thus $c(R) = 1$. Taking Ω the set $]0,1[^N$ and for T the diagonal matrix D such that $D_{ii} = \lambda_i$, one sees that

$$c(D) = \prod_{i=1}^{N} |\lambda_i| = |\det(D)|.$$

However, according to Lemma 5.1, any invertible matrix T has a decomposition $T = R_1DR_2$, where R_1 and R_2 are orthogonal matrices and D is a diagonal matrix. In (5.1) we have thus $c(T) = c(D) = |\det(D)| = |\det(T)|$ and the lemma is proved. □

Next we show the theorem regarding the change of variables in the regular case, more precisely when the change of variabes is of class C^2.

Recall that when ω and Ω are two open subsets of $\mathbb{R}^N$ and $\varphi : \omega \longrightarrow \Omega$ is a C^1 function, a matrix representing $\varphi'(x)$ is called the *jacobian matrix* of φ at x, and one denotes by $J_\varphi(x)$ the *jacobian* of φ, that is the determinant of $\varphi'(x)$:

$$J_\varphi(x) := \det \varphi'(x) = \det\left[\left(\frac{\partial \varphi_i}{\partial x_j}\right)_{1\le i,j\le N}\right].$$

When ω, Ω are two open sets, we shall write $\omega \subset\subset \Omega$ to mean that $\overline{\omega}$ is compact $\overline{\omega} \subset \Omega$.

5.3 Proposition. *Let Ω and ω be two open subsets of $\mathbb{R}^N$ and $\varphi : \omega \longrightarrow \Omega$ a diffeomorphism of class C^2. Then upon setting $J_\varphi(x) := \det(\varphi'(x))$, for any measurable nonnegative function $f : \Omega \longrightarrow \mathbb{R}$ we have :*

$$\int_\Omega f(y)\,dy = \int_\omega f(\varphi(x))\,|J_\varphi(x)|\,dx. \tag{5.2}$$

Proof. It is sufficient to prove (5.2) for functions of the type $f := 1_A$ where A is a borelian set of Ω. More specifically, since the σ-algebra of the borelian sets of ω is generated by sets of the type

$$P := \prod_{i=1}^{i=N} [a_i, b_i) \subset\subset \omega,$$

and since φ is a diffeomorphism of ω on Ω, one sees that the borelian σ-algebra of Ω is generated by sets of the type $\varphi(P)$, with $P \subset\subset \omega$. Finally, it is sufficient to show (5.2) when f is the characteristic function of a set such as $\varphi(P)$, with

$$P \coloneqq \prod_{i=1}^{N} [a_i, b_i[\subset\subset \omega.$$

Let $n \geq 1$ be an integer and $h \coloneqq 1/n$. Upon dividing each interval $[a_i, b_i[$ in n disjoint sub-intervals of length $(b_i - a_i)h$, we obtain a decomposition of P into a union of disjoint sets P_j :

$$P = \bigcup_{1 \leq j \leq M} P_j, \qquad P_i \cap P_j = \varnothing \quad \text{if} \quad 1 \leq j \neq i \leq M, \qquad M \coloneqq n^N.$$

Since φ is one-to-one, we have

$$m(\varphi(P)) = \sum_{j=1}^{M} m(\varphi(P_j)). \tag{5.3}$$

Let x_j be the center of P_j and $T_j \coloneqq \varphi'(x_j)$. For $x \in P_j$, there exists $\xi \in P_j$ such that :

$$\varphi(x) = \varphi(x_j) + T_j(x - x_j) + \frac{1}{2}\,\varphi''(\xi)\,(x - x_j, x - x_j). \tag{5.4}$$

Let us introduce the following notations : if $\delta > 0$ and Q is a subset of $\mathbb{R}^N$, we shall denote

$$\begin{aligned} Q_\delta^+ &= \Big\{x \in \mathbb{R}^N \,;\ \operatorname{dist}(x, Q) \leq \delta\Big\}, \\ Q_\delta^- &= \Big\{x \in \mathbb{R}^N \,;\ \operatorname{dist}(x, Q^c) \geq \delta\Big\}. \end{aligned} \tag{5.5}$$

Next set $\lambda \coloneqq \sup_{\xi \in P} \|\varphi''(\xi)\|$, and

$$\begin{aligned} \delta &\coloneqq \frac{\lambda}{2}\left[\max_{1 \leq j \leq M} \operatorname{diam}(P_j)\right]^2, \\ Q_j &\coloneqq \varphi(x_j) + T_j(P_j - x_j). \end{aligned}$$

One sees that for h small enough (or equivalently for n large enough), we have

$$(Q_j)_\delta^- \subset \varphi(P_j) \subset (Q_j)_\delta^+,$$

and since there exists a constant C depending only on P such that $\delta \le C\,h^2$, this implies that, since $m(Q_j) = m(T_jP_j)$:

$$\begin{aligned}|m(\varphi(P_j)) - m(T_jP_j)| &\le m\Big((Q_j)^+_\delta\Big) - m\Big((Q_j)^-_\delta\Big)\\ &\le C\cdot h^{N-1}\cdot\delta \le C\,h^{N+1}.\end{aligned}$$

(Here and in what follows C denotes a constant of which we state the dependence or independence on various parameters). Setting

$$J_\varphi(x_j) := \det\varphi'(x_j) = \det T_j,$$

and using Lemma 5.2 which says $m(T_jP_j) = |\det(T_j)|$, one gets, for some constant C independent of h :

$$-C\,h^{N+1} + |J_\varphi(x_j)|m(P_j) \le m(\varphi(P_j)) \le |J_\varphi(x_j)|m(P_j) + C\,h^{N+1}.$$

Therefore, upon summation on j we have :

$$-MCh^{N+1} \le m(\varphi(P)) - \sum_{j=1}^{M} |J_\varphi(x_j)|m(P_j) \le MCh^{N+1}. \tag{5.6}$$

On the other hand, using the dominated convergence theorem and letting $h \to 0$, or $n \to \infty$, we have :

$$\begin{aligned}\lim_{n\to+\infty}\sum_{j=1}^{M} |J_\varphi(x_j)|m(P_j) &= \lim_{n\to+\infty}\int_\omega \sum_{j=1}^{M} 1_{P_j}(x)\,|J_\varphi(x_j)|\,dx\\ &= \int_\omega 1_P(x)\,|J_\varphi(x)|\,dx.\end{aligned}$$

Finally, since $M\,h^{N+1} = h$, we conclude from (5.6) that

$$m(\varphi(P)) = \int_\omega 1_P(x)\,|J_\varphi(x)|\,dx\,,$$

and the theorem is proved when φ is of class C^2. □

For the general case, if the function φ is only of class C^1, one may approximate it by a sequence of C^2 functions and obtain the following :

5.4 Theorem. *Let Ω and ω be two open subsets of $\mathbb{R}^N$ and $\varphi : \omega \longrightarrow \Omega$ a C^1 diffeomorphism. Then setting $J_\varphi(x) := \det(\varphi'(x))$ (the jacobian of φ), for any measurable nonnegative function $f : \Omega \longrightarrow \mathbb{R}_+$, we have*

$$\int_{\Omega} f(y)\,dy = \int_{\omega} f(\varphi(x))\,|J_{\varphi}(x)|\,dx.$$

Proof. Clearly it is enough to show the result for $f = 1_{\varphi(P)}$ where $P \subset\subset \omega$ is as above. If $\varepsilon > 0$ is given and $Q := P_{\varepsilon}^{+}$ with above notations introduced in (5.5), then for $\varepsilon > 0$ small enough we have $Q \subset \omega$ and since there exists a sequence $(\varphi_n)_n$ of C^2 diffeomorphisms such that

$$\varphi_n \to \varphi \qquad \text{dans } C^1(Q).$$

we have that, according to Proposition 5.3,

$$m(\varphi_n(P)) = \int_{\omega} 1_P(x)\,|J_{\varphi_n}(x)|\,dx.$$

However, it is clear that $m(\varphi_n(P)) \to m(\varphi(P))$ when $n \to \infty$ and (using the dominated convergence theorem) we have

$$\int_{\omega} 1_P(x)\,|J_{\varphi_n}(x)|\,dx \to \int_{\omega} 1_P(x)\,|J_{\varphi}(x)|\,dx.$$

Consequently we have $m(\varphi(P)) = \int_{\omega} 1_P(x)\,|J_{\varphi}(x)|\,dx$ and the proof of the theorem is complete. □

6 The surface measure

Let ω be an open subset of $\mathbb{R}^{N-1}$ and $F : \omega \longrightarrow \mathbb{R}^N$ a C^1 function. We shall say that F is an *immersion* if F and F' are injective. Observe that saying $F'(x)$ is injective means that for all $x \in \omega$ the matrix $F'(x)$ has rank $N-1$. Also note that if F is an immersion, then $F'(x)^* F'(x)$ is a square $(N-1) \times (N-1)$ matrix, which is positive definite and its generic entries are $(\partial_i F(x)\,|\,\partial_j F(x))$, where $(\cdot|\cdot)$ denotes the scalar product of $\mathbb{R}^N$. The surface measure is defined as follows :

6.1 Definition. *Let $\omega \subset \mathbb{R}^{N-1}$ be an open set, F a C^1 function from ω into $\mathbb{R}^N$. Suppose that F is an immersion and define the surface*

$$S := \left\{F(x)\,;\, x \in \omega\right\}.$$

If $u : S \longrightarrow \mathbb{R}$ is a continuous function with compact support, we define the integral of u on S as being :

(6.1) $$\int_S u(\sigma)\, d\sigma := \int_\omega u(F(x))\, (\det(F'(x)^* F'(x)))^{1/2}\, dx.$$

The measure $d\sigma$ is called the surface measure of S.

In order for this definition to be of any interest, one has to show that it depends only on the surface S and is independent of the particular immersion (or *parametrization*) F of S. In other terms, we have to show that if $\Omega \subset \mathbb{R}^{N-1}$ is another open set and $G : \Omega \longrightarrow \mathbb{R}^N$ is another immersion such that

$$S = \left\{F(x)\,;\ x \in \omega\right\} = \left\{G(x)\,;\ x \in \Omega\right\},$$

then the two surface measures defined through F and G coincide.

6.2 Proposition. *The surface measure defined on S in Definition 6.1 is independent of the immersion F.*

Proof. Indeed, let Ω be an(other) open subset of $\mathbb{R}^{N-1}$ and $G : \Omega \longrightarrow \mathbb{R}^N$ an immersion such that $S = G(\Omega)$. Then $h := F^{-1} \circ G$ is a C^1 diffeomorphism of Ω on ω. Now let $u : S \longrightarrow \mathbb{R}$ be a continuous function with compact support. Denoting by $d^*\sigma$ the surface measure defined through G, we have :

$$\int_S u(\sigma)\, d^*\sigma = \int_\Omega u(G(y))\, [\det G'(y)^* G'(y)]^{1/2}\, dy.$$

However, since $G = F \circ h$, and $G'(y) = F'(h(y))\, h'(y)$, using the change of variables Theorem 5.3, we can write

$$\begin{aligned}
\int_S u(\sigma)\, d^*\sigma &= \int_\Omega u(F \circ h(y))\, [\det F'(h(y))^* F'(h(y))]^{1/2}\, |\det h'(y)|\, dy \\
&= \int_\omega u(F(x))\, [\det F'(x)^* F'(x)]^{1/2}\, dx \\
&= \int_S u(\sigma)\, d\sigma,
\end{aligned}$$

where we have naturally used the change of variables $x := h(y)$ and $dy = |\det h'(y)|^{-1} dx$. □

7 Exercises

Exercise 1

Exercise 2

Give an example of a nonnegative sequences of integrable functions $(f_n)_n$ on a domain Ω such that

$$\int_\Omega \liminf_{n\to\infty} f_n(x)\,dx < \liminf_{n\to\infty} \int_\Omega f_n(x)\,dx,$$

and another (non trivial...) example where

$$\int_\Omega \liminf_{n\to\infty} f_n(x)\,dx = \liminf_{n\to\infty} \int_\Omega f_n(x)\,dx.$$

7 The space $\mathscr{D}(\Omega)$

In this section and the followings we introduce the functional spaces which will let us have a beautiful environment of functional spaces useful in the study of partial differential equations. We follow here the points of views introduced by L. Schwartz[1].

Let $\Omega \subset \mathbb{R}^N$ an open domain (that is non empty, open and connected). If a continuous function $\varphi : \Omega \longrightarrow \mathbb{C}$ is given we denote by $\mathrm{supp}(\varphi)$, the support of φ, the set

$$\mathrm{supp}(\varphi) := \overline{\left\{x \in \Omega\,;\ \varphi(x) \neq 0\right\}}. \tag{7.1}$$

For any integer $k \geq 0$, we denote by $C_c^k(\Omega)$ the space of functions defined on Ω which have a compact support and are of class C^k, that is which have continuous derivatives up to order k, and then we set

$$C_c^\infty(\Omega) := \bigcap_{k\geq 0} C_c^k(\Omega).$$

Indeed it is not obvious that this space contains any non trivial function. We shall see later on that in fact this space is dense in any space $L^p(\Omega)$ for $1 \leq p < \infty$. But before arriving to that point, we give just an example of such functions.

The reader may joyfully show that the following function ρ_0 defined on $\mathbb{R}^N$ belongs to $C_c^\infty(\mathbb{R}^N)$ and that its support is the closed ball $[\,|x| \leq 1\,]$. Recall that for $x \in \mathbb{R}^N$ we denote $|x|^2 := \sum_{k=1}^N |x_k|^2$. Define

$$\rho_0(x) := \begin{cases} \exp\left(\dfrac{-1}{1-|x|^2}\right) & \text{for } \ |x| < 1 \\ \qquad 0 & \text{for } \ |x| \geq 1. \end{cases}$$

The support of ρ_0 is clearly the closed unit ball $[\,|x| \leq 1\,]$. Now if Ω is any non empty domain and $x_0 \in \Omega$, let $n_0 \in \mathbb{N}^*$ be such that $B(x_0, 2/n_0) \subset \Omega$. Then for

[1] Laurent Schwartz, French mathematician, 1915–2002.

any integer $n \geq n_0$, setting $\rho_n(x) \coloneqq \rho_0(n(x - x_0))$, we have $\rho_n \in C_c^\infty(\Omega)$ with $\mathrm{supp}(\rho_n) = [|x - x_0| \leq 1/n]$.

Now, we are going to define a notion of convergence for sequences of functions in $C_c^\infty(\Omega)$. First we set $\mathscr{D}(\Omega) \coloneqq C_c^\infty(\Omega)$, and for $\varphi \in \mathscr{D}(\Omega)$ and a sequence $(\varphi_n)_{n\geq 1}$ of functions in $\mathscr{D}(\Omega)$, we shall say that $(\varphi_n)_n$ *converges to* φ *in* $\mathscr{D}(\Omega)$ if the following two conditions are satisfied :

(i) there exists a compact $K \subset \Omega$ such that for all $n \geq 1$ we have $\mathrm{supp}(\varphi_n) \subset K$;
(ii) for any $\alpha \in \mathbb{N}^N$ we have $\|\partial^\alpha \varphi_n - \partial^\alpha \varphi\|_\infty \to 0$ as $n \to \infty$.

In general the knowledge of all convergent sequences in a topological space is not enough in order to define its topology, but in this case on the one hand we shall only be concerned with sequences indexed by integers, and on the other hand one can show that one may extend the above convergence notion to generalized families (that is indexed by a general set) and thus prove that the space $\mathscr{D}(\Omega)$ is endowed with a topology (called the inductive limit of $C_c^k(\Omega)$ topologies).

A linear operator $L : \mathscr{D}(\Omega) \longrightarrow \mathscr{D}(\Omega)$ will be continuous whenever we have

$$(\varphi_n)_n \to \varphi \quad \text{in} \quad \mathscr{D}(\Omega) \qquad \Longrightarrow \qquad (L\varphi_n)_n \to L\varphi \quad \text{in} \quad \mathscr{D}(\Omega).$$

For instance if for $\alpha \in \mathbb{N}^N$ and $|\alpha| \leq m$ we are given a family of functions $c_\alpha \in C^\infty(\Omega)$, then the differential operator of order m defined by

$$L\varphi \coloneqq \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \varphi,$$

is a linear continuous operator from $\mathscr{D}(\Omega)$ into itself.

8 The space of distributions $\mathscr{D}'(\Omega)$

If $\Omega \subset \mathbb{R}^N$ is an open non empty domain, we shall say that a linear continuous operator (or a continuous linear form) $T : \mathscr{D}(\Omega) \longrightarrow \mathbb{C}$, or $\mathbb{R}$, is a *distribution on* Ω. Thus T is a distribution on Ω when the mapping $\varphi \mapsto T(\varphi) = T\varphi$ is linear from $\mathscr{D}(\Omega)$ into $\mathbb{C}$ (or $\mathbb{R}$) and for any $\varphi \in \mathscr{D}(\Omega)$, and any sequence $(\varphi_n)_n$ in $\mathscr{D}(\Omega)$, we have

$$\varphi_n \to \varphi \quad \text{in} \quad \mathscr{D}(\Omega) \qquad \Longrightarrow \qquad T(\varphi_n) \to T(\varphi) \quad \text{in} \quad \mathbb{C}.$$

The set of all distributions on Ω, that is the dual space of $\mathscr{D}(\Omega)$, will be denoted by $\mathscr{D}'(\Omega)$.

As we have already done with the dual spaces of Banach spaces, we shall usually use the duality brackets and note

$$\langle T, \varphi \rangle \coloneqq T(\varphi) \qquad \text{for all } T \in \mathscr{D}'(\Omega), \quad \varphi \in \mathscr{D}(\Omega).$$

It may happen that when we are using several vectorial topological spaces containing $\mathscr{D}(\Omega)$, and their duals, in order to be more precise we use the notations

$$\langle T, \varphi \rangle_{\mathscr{D}'(\Omega), \mathscr{D}(\Omega)}$$

to signify that the duality is taken in the sense of distributions.

As a topological vector space, $\mathscr{D}'(\Omega)$ is endowed with the weak topology associated to $\mathscr{D}(\Omega)$. More precisely, we shall say that a sequence $(T_n)_n$ of $\mathscr{D}'(\Omega)$ converges to $T \in \mathscr{D}'(\Omega)$ when we have

$$\forall \varphi \in \mathscr{D}(\Omega), \qquad \langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \quad \text{as } \ n \to \infty.$$

Let us recall the following result :

8.1 Lemma. *Let $f \in L^1_{\text{loc}}(\Omega)$ be a function such that for any compact $K \subset \Omega$ we have*

$$\int_K f(x)\, dx = 0.$$

Then we have $f = 0$ a.e. in Ω.

Now, for any $f \in L^1_{\text{loc}}(\Omega)$, one can easily check that the linear form T_f defined by

$$\forall f \in \mathscr{D}(\Omega), \qquad \left\langle T_f, \varphi \right\rangle \coloneqq \int_\Omega \varphi(x) f(x)\, dx,$$

is a distribution, that is $T_f \in \mathscr{D}'(\Omega)$. On the other hand, if we have a function $g \in L^1_{\text{loc}}(\Omega)$ such that

$$T_f = T_g, \qquad \text{that is } \ \forall \varphi \in \mathscr{D}(\Omega), \qquad \int_\Omega \varphi(x) f(x)\, dx = \int_\Omega \varphi(x) g(x)\, dx,$$

one can first infer that for any compact $K \subset \Omega$ we have

$$\int_K (f(x) - g(x))\, dx = 0,$$

and thus, thanks to Lemma 8.1, we have $f = g$ a.e. in Ω. This allows us to conclude that the mapping $f \mapsto T_f$ is an injection, and thus upon identifying T_f with f, we have the inclusion

$$L^1_{\mathrm{loc}}(\Omega) \subset \mathscr{D}'(\Omega).$$

In particular all the spaces $L^p(\Omega)$ for $1 \le p \le \infty$ may be identified with a subspace of $\mathscr{D}'(\Omega)$. Whenever $f \in L^1_{\mathrm{loc}}(\Omega)$, we shall write $\langle f, \varphi \rangle$ instead of $\langle T_f, \varphi \rangle$.

9 The space $\mathscr{S}(\mathbb{R}^N)$

For $\alpha, \beta \in \mathbb{N}^N$ and $\varphi \in C^\infty(\mathbb{R}^N)$, let us introduce the semi-norms

$$p_{\alpha,\beta}(\varphi) := \sup_{x \in \mathbb{R}^N} \left| x^\alpha \partial^\beta \varphi(x) \right|. \tag{9.1}$$

The Laurent Schwartz' space of rapidly decreasing smooth functions is defined as being

$$\mathscr{S}(\mathbb{R}^N) := \left\{ \varphi \in C^\infty(\mathbb{R}^N) \; ; \; \forall \alpha, \beta \in \mathbb{N}^N, \; p_{\alpha,\beta}(\varphi) < \infty \right\}. \tag{9.2}$$

Essentially, this is the space of smooth functions which decay to zero at infinity, as well as all their derivatives, faster than $(1 + |x|^2)^{-m}$ for any $m \ge 0$.

A sequence $(\varphi_k)_{k \ge 1}$ of $\mathscr{S}(\mathbb{R}^N)$ is said to converges to $\varphi \in \mathscr{S}(\mathbb{R}^N)$ if and only if we have

$$\forall \alpha, \beta \in \mathbb{N}^N, \qquad \lim_{k \to \infty} p_{\alpha,\beta}(\varphi_k - \varphi) = 0. \tag{9.3}$$

One may define a family of norms $\|\cdot\|_{m,n}$ on $\mathscr{S}(\mathbb{R}^N)$ by setting

$$\|\varphi\|_{m,n} := \sum_{|\alpha| \le m, |\beta| \le n} p_{\alpha,\beta}(\varphi),$$

so that the convergence of a sequence $(\varphi_k)_k$ to φ is equivalent to

$$\forall m, n \in \mathbb{N}, \qquad \lim_{k \to \infty} \|\varphi_k - \varphi\|_{m,n} = 0.$$

As a matter of fact one can show that upon setting

$$d(\varphi, \psi) := \sum_{m,n \ge 0} 2^{-(m+n)} \frac{\|\varphi - \psi\|_{m,n}}{1 + \|\varphi - \psi\|_{m,n}} \tag{9.4}$$

the space $(\mathscr{S}(\mathbb{R}^N), d)$ is a complete metric space which is also a locally convex topological vector space (such a topological vector space is called a *Fréchet*[2] *space*).

One has obviously $\mathscr{D}(\mathbb{R}^N) \subset \mathscr{S}(\mathbb{R}^N)$, and any function $\varphi \in \mathscr{S}(\mathbb{R}^N)$ can be approximated by a sequence of functions $(\varphi_n)_{n\geq 1}$ of elements of $\mathscr{D}(\mathbb{R}^N)$. In order to see that these functional spaces are distinct, one may examine the functions

$$\varphi_{\lambda,\alpha}(x) := \exp\Big(-\lambda\Big(1+|x|^2\Big)^{\alpha}\Big),$$

and see that for any $\lambda > 0$ and $\alpha > 0$ we have $\varphi_{\lambda,\alpha} \in \mathscr{S}(\mathbb{R}^N)$, while clearly $\varphi_{\lambda,\alpha} \notin \mathscr{D}(\mathbb{R}^N)$.

10 The space $\mathscr{S}'(\mathbb{R}^N)$ of tempered distributions

A continuous linear from $T : \mathscr{S}(\mathbb{R}^N) \longrightarrow \mathbb{C}$, or $\mathbb{R}$, is called a *tempered distribution*: thus if $(\varphi_k)_k$ is a sequence in $\mathscr{S}(\mathbb{R})^N$ and $\varphi \in \mathscr{S}(\mathbb{R}^N)$ we have

$$\forall m, n \in \mathbb{N}, \qquad \lim_{k\to\infty} \|\varphi_k - \varphi\|_{m,n} = 0 \qquad \Longrightarrow \qquad \lim_{k\to\infty} \langle T, \varphi_k \rangle = \langle T, \varphi \rangle .$$

The space of tempered distributions on $\mathbb{R}^N$ is denoted by $\mathscr{S}'(\mathbb{R}^N)$, and it is endowed with the weak topology associated to $\mathscr{S}(\mathbb{R}^N)$: thus a sequence $(T_k)_k$ of distributions in $\mathscr{S}'(\mathbb{R}^N)$ converges to T if, and only if,

$$\forall \varphi \in \mathscr{S}(\mathbb{R}^N), \qquad \lim_{k\to\infty} \langle T_k, \varphi \rangle = \langle T, \varphi \rangle .$$

Since a function $\varphi \in \mathscr{S}(\mathbb{R}^N)$ satisfies

$$\forall m \in \mathbb{N}, \qquad |\varphi(x)| \leq \frac{\|\varphi\|_{m,0}}{(1+|x|^2)^m},$$

it is clear that for $f \in L^1(\mathbb{R}^N)$ the mapping

$$\varphi \mapsto \int_{\mathbb{R}^N} \varphi(x) f(x)\, dx$$

is continuous on $\mathscr{S}(\mathbb{R}^N)$. On the other hand, for $1 < p \leq \infty$ and $f \in L^p(\mathbb{R}^N)$ we have

[2] René Maurice Fréchet, French mathematician, 1878–1973.

$$\int_{\mathbb{R}^N} |\varphi(x) f(x)| \, dx \le \|\varphi\|_{m,0} \|f\|_p \left(\int_{\mathbb{R}^N} \left(1 + |x|^2\right)^{-mp'} dx \right)^{1/p'}.$$

Clearly when m is large enough so that $2mp' > N$, the integral on the right hand side is finite and therefore the mapping

$$\varphi \mapsto \int_{\mathbb{R}^N} \varphi(x) f(x) \, dx$$

is also continuous on $\mathscr{S}(\mathbb{R}^N)$. Thus we may indentify the spaces $L^p(\mathbb{R}^N)$, for $1 \le p \le \infty$, to a subspace of $\mathscr{S}'(\mathbb{R}^N)$.

The space $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is not included in $\mathscr{S}'(\mathbb{R}^N)$, since for instance if $f(x) \coloneqq \exp(|x|^2)$, then the mapping $\varphi \mapsto \langle f, \varphi \rangle$ well defined on $\mathscr{D}(\mathbb{R}^N)$ cannot be extended to $\mathscr{S}(\mathbb{R}^N)$, because $\varphi_0(x) \coloneqq \exp(-\sqrt{1 + |x|^2})$ is an element of $\mathscr{S}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \varphi_0(x) f(x) \, dx = +\infty$.

11 The Fourier transform

For $\varphi, \psi \in \mathscr{S}(\mathbb{R}^N)$ we define the Fourier[3] transform of φ by setting

$$(11.1) \qquad (\mathscr{F}(\varphi))(\xi) \coloneqq \widehat{\varphi}(\xi) \coloneqq (2\pi)^{-N/2} \int_{\mathbb{R}^N} \varphi(x) \, \mathrm{e}^{-\mathrm{i}x \cdot \xi} \, dx,$$

and its conjugate (which will be proved to be its inverse and its adjoint in $L^2(\mathbb{R}^N)$)

$$(11.2) \qquad (\overline{\mathscr{F}}(\psi))(x) \coloneqq (2\pi)^{-N/2} \int_{\mathbb{R}^N} \psi(\xi) \, \mathrm{e}^{\mathrm{i}x \cdot \xi} \, d\xi.$$

The numerical factor $(2\pi)^{-N/2}$ is of no significant importance and is chosen differently by various authors (one can even remove this factor and replace the term $\mathrm{e}^{-\mathrm{i}x \cdot \xi}$ by $\mathrm{e}^{-2\mathrm{i}\pi x \cdot \xi}$). as we shall see in a moment our choice is made in order to have

$$\mathscr{F}(g) = g, \qquad \text{when} \;\; g(x) \coloneqq \exp(-|x|^2/2).$$

Another variant of the definition of the Fourier transform is by modifying the Lebesgue measure dx and setting

[3] Jean-Baptiste Joseph Fourier, French mathematician and physicist, 1768–1830.

(11.3)
$$d^*x \coloneqq (2\pi)^{-N/2}\,dx, \ \text{and} \ (\mathscr{F}(\varphi))(\xi) = \int_{\mathbb{R}^N} \varphi(x)\,e^{-ix\cdot\xi}\,d^*x.$$

With this renormalization of the Lebesgue measure we have

$$\int_{\mathbb{R}^N} g(x)\,d^*x = 1.$$

Our first observation regarding the Fourier transform is that

$$\forall \xi \in \mathbb{R}^N, \qquad |\widehat{\varphi}(\xi)| \leq (2\pi)^{-N/2}\,\|\varphi\|_{L^1(\mathbb{R}^N)},$$

showing that the Fourier transform can be extended as a bounded linear operator from $L^1(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$. As a matter of fact using the dominated convergence theorem of Lebesgue, one can see that when $\varphi \in L^1(\mathbb{R}^N)$ we have $\mathscr{F}(\varphi) \in C(\mathbb{R}^N)$. Also (see *Exercise*) one may show easily that $\mathscr{F}(\varphi)(\xi) \to 0$ when $|\xi| \to \infty$.

Thus if $\varphi \in L^1(\mathbb{R}^N)$, we have $\mathscr{F}(\varphi) \in C_0(\mathbb{R}^N)$, the space of continuous functions converging to zero at infinity.

One checks easily that

$$\mathscr{F}(\partial^\beta\varphi)(\xi) = (i\xi)^\beta\,\mathscr{F}(\varphi)(\xi), \qquad \partial^\alpha\mathscr{F}(\varphi)(\xi) = \mathscr{F}((-ix)^\alpha\varphi).$$

Combining these, one sees that

$$(i\xi)^\beta\,\partial^\alpha\mathscr{F}(\varphi) = \mathscr{F}(\partial^\beta(-ix)^\alpha\varphi).$$

Now, when $\varphi \in \mathscr{S}(\mathbb{R}^N)$, we have $\partial^\beta(-ix)^\alpha\varphi \in \mathscr{S}(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$, and thus its Fourier transform belongs to $C_0(\mathbb{R}^N)$. Therefore for any $\alpha, \beta \in \mathbb{N}^N$ we have $(i\xi)^\beta\,\partial^\alpha\mathscr{F}(\varphi) \in C_0(\mathbb{R}^N)$.

From these observations we conclude that $\mathscr{F}(\varphi) \in \mathscr{S}(\mathbb{R}^N)$, and that the mapping $\varphi \mapsto \mathscr{F}(\varphi)$ is continuous on $\mathscr{S}(\mathbb{R}^N)$.

If, for $h \in \mathbb{R}^N$, we denote by τ_h the translation operator defined by

$$\tau_h\varphi \coloneqq \varphi(\cdot + h), \quad \text{that is} \ (\tau_h\varphi)(x) \coloneqq \varphi(x+h),$$

for $\varphi \in \mathscr{S}(\mathbb{R}^N)$, one sees that

(11.4)
$$\mathscr{F}(\tau_h\varphi)(\xi) = e^{ih\cdot\xi}\widehat{\varphi}(\xi).$$

Also if for $\lambda > 0$ we denote $\varphi_\lambda(x) := \varphi(\lambda x)$, one checks that

$$\mathscr{F}(\varphi_\lambda)(\xi) = \lambda^{-N}\widehat{\varphi}\big(\lambda^{-1}x\big). \tag{11.5}$$

In order to show that the Fourier transform $\mathscr{F}$ is a continuous isomorphism on $\mathscr{S}(\mathbb{R}^N)$, we compute first the Fourier transform of the Gaussian function $g(x) := \exp(-|x|^2/2)$. This function being an even function on $\mathbb{R}^N$, we have

$$\widehat{g}(\xi) = (2\pi)^{-N/2}\int_{\mathbb{R}^N} g(x)\cos(x\cdot\xi)\,dx$$

from which by an integration by parts (since $g(x)x = -\nabla g(x)$) we infer that

$$\begin{aligned}\nabla\widehat{g}(\xi) &= -(2\pi)^{-N/2}\int_{\mathbb{R}^N} g(x)\sin(x\cdot\xi)\,x\,dx\\ &= -\widehat{g}(\xi)\,\xi.\end{aligned}$$

This yields, since $\widehat{g}(0) = 1$,

$$\widehat{g}(\xi) = \widehat{g}(0)\exp\Big(-\frac{|\xi|^2}{2}\Big) = g(\xi). \tag{11.6}$$

Using the rescaling property (11.5), for $t > 0$ we have

(11.7)

$$\text{if } g_t(x) := \exp(-t|x|^2), \qquad G_t(\xi) := \mathscr{F}(g_t)(\xi) = (2t)^{-N/2}\exp\Big(-\frac{|x|^2}{4t}\Big).$$

Using Fubini's theorem it is easy to see that for $g, \varphi \in \mathscr{S}(\mathbb{R}^N)$ we have, for all $x \in \mathbb{R}^N$ the F. Riesz identity

$$\int_{\mathbb{R}^N} g(\xi)\,\widehat{\varphi}(\xi)\,e^{ix\cdot\xi}\,d\xi = \int_{\mathbb{R}^N}\widehat{g}(y)\,\varphi(x+y)\,dy. \tag{11.8}$$

In particular, taking $g := g_t$ given in (11.7), we have $\widehat{g} = G_t$ and thus

$$\begin{aligned}\int_{\mathbb{R}^N} g_t(\xi)\,\widehat{\varphi}(\xi)\,e^{ix\cdot\xi}\,d\xi &= \int_{\mathbb{R}^N} G_t(y)\,\varphi(x+y)\,dy\\ &= \int_{\mathbb{R}^N}\exp\Big(-\frac{|z|^2}{2}\Big)\,\varphi(x+\sqrt{2t}\,z)\,dz.\end{aligned}$$

Letting $t \to 0$ we obtain

$$\int_{\mathbb{R}^N} \widehat{\varphi}(\xi)\, e^{ix\cdot\xi}\, d\xi = \varphi(x) \int_{\mathbb{R}^N} \exp\left(-\frac{|z|^2}{2}\right) dz = (2\pi)^{N/2} \varphi(x),$$

which proves that $\overline{\mathscr{F}}$ defined by (11.2) is the left inverse of the Fourier transform $\mathscr{F}$, that is $\overline{\mathscr{F}}\mathscr{F} = I$.

Using (11.8) in the form

$$\int_{\mathbb{R}^N} g(x)\, (\overline{\mathscr{F}}(\varphi))(x)\, e^{-ix\cdot\xi}\, dx = \int_{\mathbb{R}^N} (\overline{\mathscr{F}}(g))(y)\, \varphi(\xi + y)\, dy,$$

and following the same procedure as above, we see that $\overline{\mathscr{F}}$ is also the right inverse of $\mathscr{F}$, that is $\mathscr{F}\overline{\mathscr{F}} = I$.

12 Exercises

Exercise 1

We consider here various functions in $L^1_{\mathrm{loc}}(\mathbb{R})$ of which we study their derivatives in the sense of distributions.

1) Let $Y(x) \coloneqq x 1_{(0,\infty)}$. Determine the distribution derivatives Y' and Y''.

2) Let $f(x) \coloneqq |x|$. Determine f' and f''.

3) Let $g(x) \coloneqq \exp(|x|/2)$. Determine g' and g''.

4) Let $K(x) \coloneqq \exp(-|x|)/2$. Determine K' and K'' as well $-K'' + K$.

Exercise 2

For $h \in \mathbb{R}^N$ we define the translation operator τ_h on $\mathscr{D}(\mathbb{R}^N)$ by setting $(\tau_h \varphi)(x) \coloneqq \varphi(x + h)$ for $\varphi \in \mathscr{D}(\mathbb{R}^N)$.

1) Verify that this operator can also be defined on $\mathscr{D}'(\mathbb{R}^N)$ by setting

$$\forall T \in \mathscr{D}'(\mathbb{R}^N), \quad \forall \varphi \in \mathscr{D}(\mathbb{R}^N), \qquad \langle \tau_h T, \varphi \rangle \coloneqq \langle T, \tau_{-h} \varphi \rangle,$$

and that $T \mapsto \tau_h T$ is a linear continuous operator.

2) We denote by $\boldsymbol{e}_i$ the i-th element of the canonical basis of $\mathbb{R}^N$. Verify that as $\varepsilon \to 0$

$$\frac{\tau_{\varepsilon \boldsymbol{e}_i} T - T}{\varepsilon} \to \partial_i T \qquad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

Exercise 3

Consider the function $f(x) := (1/2)\log(x^2)$ on $\mathbb{R}$.

1) Verify that $f \in L^1_{\text{loc}}(\mathbb{R})$ so that the linear form $\varphi \mapsto \langle Tf, \varphi\rangle := \int_{\mathbb{R}} \varphi(x) f(x)\, dx$ is well defined.

2) Prove that the distribution derivative T_f' is given by

$$\forall \varphi \in \mathscr{D}(\mathbb{R}), \qquad \left\langle T_f', \varphi \right\rangle = \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0)}{x}\, dx.$$

This distribution is called the *finite part of* $1/x$, and is sometimes written as

$$\left\langle \text{F.P.}\frac{1}{x}, \varphi \right\rangle = \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0)}{x}\, dx.$$

One should be aware that the integral sign on the right hand side of the above equality is not a classical Lebesgue integral, in particular one cannot separate the integral into two pieces, say

$$\int_{[|x|<1]} \frac{\varphi(x) - \varphi(0)}{x}\, dx \qquad \text{and} \qquad \int_{[|x|\geq 1]} \frac{\varphi(x) - \varphi(0)}{x}\, dx,$$

since the latter is not defined as a Lebesgue integral.

3) More generally show that for any integer $k \geq 2$ one has

$$\left\langle T_f^{(k)}, \varphi \right\rangle = (-1)^{k-1}(k-1)! \int_{\mathbb{R}} \left(\varphi(x) - \sum_{j=0}^{k-1} \varphi^{(j)}(0) \frac{x^j}{j!} \right) \frac{dx}{x^k}.$$

These distributions are sometimes called *finite part of* $1/x^k$ and one sometimes write

$$\left\langle \text{F.P.}\frac{1}{x^k}, \varphi \right\rangle = \int_{\mathbb{R}} \left(\varphi(x) - \sum_{j=0}^{k-1} \varphi^{(j)}(0) \frac{x^j}{j!} \right) \frac{dx}{x^k},$$

and thus we have that the k-th derivative of F.P. $\frac{1}{x}$ in the sense of distributions is given by

$$\left(\text{F.P.}\frac{1}{x}\right)^{(k)} = (-1)^k k!\ \text{F.P.}\frac{1}{x^{k+1}}.$$

Exercise 4

For $\varphi \in \mathscr{D}(\mathbb{R})$ consider

$$\langle T_1, \varphi\rangle := \sum_{n\geq 0} (-1)^n \varphi^{(n)}(n).$$

Does T_1 belong to the space $\mathscr{D}'(\mathbb{R})$?

Exercise 5

For $\varphi \in \mathscr{D}(\mathbb{R})$ consider

$$\langle T_2, \varphi\rangle := \sum_{n\geq 0} (-1)^n \varphi^{(n)}(0).$$

Does T_2 belong to the space $\mathscr{D}'(\mathbb{R})$?

Exercise 6

For $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ let $T_n(x) := \mathrm{e}^{inx}$.

1) Show that $\langle T_n, \varphi\rangle \to 0$ when $|n| \to \infty$, for each $\varphi \in \mathscr{S}(\mathbb{R})$. (Consider the functions $f_n := T_n/in$).

2) Conclude that for any $\varphi \in C_c(\mathbb{R})$, or $\varphi \in L^1(\mathbb{R})$, we have

$$\lim_{|n|\to\infty} \int_{\mathbb{R}} \varphi(x)\,\mathrm{e}^{inx}\,dx = 0.$$

3) Deduce the Riemann-Lebesgue Lemma which asserts that

$$\forall f \in L^1(\mathbb{R}^N), \qquad \lim_{|\xi|\to\infty} \int_{\mathbb{R}^N} f(x)\,\mathrm{e}^{ix\cdot\xi}\,dx = 0.$$

Exercise 7

In $\mathscr{D}'(\mathbb{R})$ we consider the following distributions

$$u_n := \sum_{k=1}^{n} \delta_{k^{-2}} - n\delta_0$$
$$v_n := \sum_{k=1}^{n} \frac{1}{k}\delta_{k^{-1}} - \log(n)\,\delta_0$$
$$w_n := \sum_{k=1}^{n} \frac{(-1)^k}{k}\delta_{k^{-2}}.$$

Show that these distributions are convergent and determine their limit in $\mathscr{D}'(\mathbb{R})$.

Exercise 8

A distribution $T \in \mathscr{D}'(\Omega)$ is said to be nonnegative if for any $\varphi \in \mathscr{D}(\Omega)$ such that $\varphi \geq 0$ we have $\langle T, \varphi\rangle \geq 0$. Thus if $f \in L^1_{\mathrm{loc}}(\Omega)$ is a nonnegative function, it is also a nonnegative distribution. We are going to prove that a nonnegative distribution T is a Radon measure on Ω, that is for any $\varphi \in C_c(\Omega)$ such that $\mathrm{supp}(\varphi) \subset K$ we have $|\langle T, \varphi\rangle| \leq c(K)\,\|f\|_\infty$ for a constant $c(K)$ depending on K and Ω.

1) Give an example of a nonnegative distribution which is not a function.

2) Give an example of a distribution which is not nonnegative.

3) Let T be a positive distribution and $K \subset \Omega$ a compact set. We fix a compact set $K_0 \subset \Omega$ such that $K \subset \mathrm{int}(K_0)$, and then we consider $\theta \in \mathscr{D}(\Omega)$ such that $\mathrm{supp}(\theta) \subset \mathrm{int}(K_0)$ and $0 \leq \theta \leq 1$, while $\theta \equiv 1$ on K. Prove that for any $\varphi_n \in \mathscr{D}(\Omega)$ such that $\mathrm{supp}(\varphi_n) \subset K$ we have

$$-\|\varphi_n\|_\infty \langle T, \theta\rangle \leq \langle T, \varphi_n\rangle \leq \|\varphi_n\|_\infty \langle T, \theta\rangle.$$

Conclude that if $\varphi \in C_c(\Omega)$ and $\mathrm{supp}(\varphi) \subset K$ one can define $\langle T, \varphi\rangle$ as the limit of $\langle T, \varphi_n\rangle$ for an appropriate choice of the sequence $(\varphi_n)_n$, that is T is a Radon nonnegative measure on Ω.

Exercise 9

On $\mathbb{R}^2$ we consider the functions

$$G(x) := \log(|x|^2), \qquad G_\varepsilon(x) := \log(\varepsilon^2 + |x|^2),$$

where $\varepsilon > 0$.

1) Show that $G_\varepsilon \to G$ in $\mathscr{D}'(\mathbb{R}^2)$.

2) Compute ∇G_ε and ΔG_ε, and deduce ∇G and ΔG in the sense of distributions. (One may define the function $f_\varepsilon(t) := \log(\varepsilon^2 + t)$ for $t \geq 0$ and note that $G_\varepsilon(x) = f_\varepsilon(|x|^2)$).

Exercise 10

Assume $N \geq 3$. On $\mathbb{R}^N$ we consider the functions

$$G(x) := |x|^{2-N}, \qquad G_\varepsilon(x) := (\varepsilon^2 + |x|^2)^{(2-N)/2},$$

where $\varepsilon > 0$.

1) Show that $G_\varepsilon \to G$ in $\mathscr{D}'(\mathbb{R}^N)$.

2) Compute ∇G_ε and ΔG_ε, and deduce ∇G and ΔG in the sense of distributions. (One may define the function $f_\varepsilon(t) := (\varepsilon^2 + t)^{(2-N)/2}$ for $t \geq 0$ and note that $G_\varepsilon(x) = f_\varepsilon(|x|^2)$).

Exercise 11

Recall that the Laplacian operator is defined by

$$\Delta\varphi := \sum_{j=1}^{N} \partial_{jj}\varphi,$$

whenever the above expression makes sense, for instance when $\varphi \in C^2(\mathbb{R}^N)$. We shall also denote denote by M the multiplier operator

$$M(x) := 1 + |x|^2, \quad \text{and} \quad (M\varphi)(x) := (1 + |x|^2)\,\varphi(x), \tag{12.1}$$

where $|x|^2 = \sum_{k=1}^{N} |x_k|^2$. We denote the Fourier transform of $\varphi \in \mathscr{S}(\mathbb{R}^N)$ by

$$(\mathscr{F}(\varphi))(\xi) := \widehat{\varphi}(\xi) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} \varphi(x)\, \mathrm{e}^{-\mathrm{i}x\cdot\xi}\, dx.$$

For $m, n \in \mathbb{N}$ and $\varphi \in C^\infty(\mathbb{R}^N)$ let us denote by $|\varphi|_{m,n}$ the quantity

(12.2) $$|\varphi|_{m,n}^2 := \|M^m(I-\Delta)^n\varphi\|_2^2 + \|(I-\Delta)^m M^n\varphi\|_2^2.$$

1) Verify that $|\varphi|_{m,n}$ is well-defined for $\varphi \in \mathscr{S}(\mathbb{R}^N)$, and $\varphi \mapsto |\varphi|_{m,n}$ is a norm.

2) Verify that for $\varphi \in \mathscr{S}(\mathbb{R}^N)$ we have $|\varphi|_{m,n}^2 = |\widehat{\varphi}|_{m,n}^2$.

3) Show that the space $\mathscr{S}(\mathbb{R}^N)$ can be characterized by

(12.3)
$$\mathscr{S}(\mathbb{R}^N) := \Big\{\varphi \in C^\infty(\mathbb{R}^N)\ ;\ \forall m,n \in \mathbb{N},\ |\varphi|_{m,n} < \infty\Big\},$$

and that a sequence $(\varphi_k)_k$ converges to φ in $\mathscr{S}(\mathbb{R}^N)$ if, and only if,

$$\forall m,n \in \mathbb{N}, \qquad |\varphi_k - \varphi|_{m,n} \to 0 \quad \text{as } k \to \infty.$$

4) We define a linear operator $(L, D(L))$ by setting $Lu := -\Delta u + Mu$ for

$$u \in D(L) := \Big\{u \in L^2(\mathbb{R}^N)\ ;\ -\Delta u + Mu \in L^2(\mathbb{R}^N)\Big\}.$$

Verify that $u \in D(L)$ if, and only if, one has $\widehat{u} \in D(L)$, and that

$$D(L) = \Big\{u \in H^2(\mathbb{R}^N)\ ;\ \widehat{u} \in H^2(\mathbb{R}^N)\Big\}.$$

Find an analogous characterization for $D(L^n)$ for any integer $n \geq 2$.

5) Prove that the space $\mathscr{S}(\mathbb{R}^N)$ can be characterized as

$$\mathscr{S}(\mathbb{R}^N) = \bigcap_{n\geq 0} D(L^n).$$

Show that a sequence $(\varphi_k)_{k\geq 0}$ of $\mathscr{S}(\mathbb{R}^N)$ converges to φ in $\mathscr{S}(\mathbb{R}^N)$ if, and only if, for any integer $n \geq 0$ we have $\|L^n(\varphi_k - \varphi)\|_2 \to 0$ as $k \to \infty$.

13 Exercises

Exercise 1

Consider the function $u(x) := (1 - |x|)^+$ on $\mathbb{R}$. Determine all $s \in \mathbb{R}$ such that $u \in H^s(\mathbb{R})$.

Exercise 2

Let Ω be a smooth bounded domain of $\mathbb{R}^N$, $k \in \mathbb{R}$ given and $f \in L^2(\Omega)$. We study here the equation

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = \text{constant} & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{n}} \, d\sigma = k. \end{cases} \tag{13.1}$$

1) Let $V := \{v \in H^1(\Omega)\ ;\ v \text{ is constant on } \partial\Omega\}$. Prove that V is a closed subspace of $H^1(\Omega)$. In the sequel we will denote by $m(v)$ the value of $v \in V$ on the boundary.

2) Write the variational formulation of (13.1) and prove that this equation has a unique solution.

3) Is the mapping $(k, f) \mapsto u$ continuous from $\mathbb{R} \times L^2(\Omega)$ into V ?

4) When $N = 1$ and $\Omega = (0, 1)$ write and interpret equation (13.1).

Exercise 3

Show that if $u \in H^1(\Omega)$ we have $|u| \in H^1(\Omega)$ and

$$\nabla|u| = u1_{[u>0]} - u1_{[u<0]}, \qquad \|\nabla|u|\|_{L^2} = \|\nabla u\|_{L^2}.$$

(Consider the function $j_\varepsilon(t) := \sqrt{\varepsilon^2 + t^2} - \varepsilon$, and then show that $u_\varepsilon := j_\varepsilon(u)$).

As an application show the following *weak maximum principle*. Let $A := (a_{ij})_{1\le i,j\le N}$ is a uniformly positive definite matrix with $a_{ij} \in L^\infty(\Omega)$, and $c \in L^\infty(\Omega)$ be a nonnegative function. Show that if $f \in L^2(\Omega)$ and $f \ge 0$, and $u \in H_0^1(\Omega)$ satisfies

$$-\mathrm{div}(A(x)\,\nabla u) + c(x)\,u = f, \qquad u = 0 \ \text{ on } \ \partial\Omega,$$

then one has $u \ge 0$ in Ω. Show that the same result holds if c^- is sufficiently small in $L^\infty(\Omega)$.

Exercise 4

We know that $H_0^1(\Omega)$ is closed subspace of the Hilbert space $H^1(\Omega)$. In this *Exercise* we characterize the orthogonal of the space $H_0^1(\Omega)$, depending on the norm chosen on $H^1(\Omega)$.

1) We assume first that the norm of $H^1(\Omega)$ is given by

$$|||u|||_1^2 := \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2.$$

Show that $u_1 \in H^1(\Omega)$ is orthogonal to $H_0^1(\Omega)$ (with respect to the scalar product associated to $|||\cdot|||_1$) if, and only if, there exists $\varphi \in H^{1/2}(\partial\Omega)$ such that

$$\begin{cases} -\Delta u_1 + u_1 = 0 & \text{in } \ \Omega \\ u_1 = \varphi & \text{on } \ \partial\Omega. \end{cases} \tag{13.2}$$

Denoting by $\mathbb{H}_1 := \left(H_0^1(\Omega)\right)^\perp$, show that $H^1(\Omega) = H_0^1 \oplus \mathbb{H}_1$ is isomorphic to $H_0^1(\Omega) \times H^{1/2}(\partial\Omega)$. What does this imply regarding the dual of $H^1(\Omega)$?

2) With the above notations, show that if $u \in H^1(\Omega)$ and $\gamma_0(u) = \varphi$, then $|||u_1|||_1 \le |||u|||_1$, and if this inequality is an equality we have $u = u_1$.

3) Show that the following

$$|||u|||_2^2 := \|\nabla u\|_{L^2}^2 + \int_{\partial\Omega} |u(\sigma)|^2\, d\sigma,$$

defines a Hilbertian norm on $H^1(\Omega)$, equivalent to $|||\cdot|||_1$.

4) Show that $u_2 \in H^1(\Omega)$ is orthogonal to $H_0^1(\Omega)$ (for the scalar product associated to $|||\cdot|||_2$) if, and only if, there exists $\varphi \in H^{1/2}(\partial\Omega)$ such that

$$\begin{cases} -\Delta u_2 = 0 & \text{in } \Omega \\ u_2 = \varphi & \text{on } \partial\Omega. \end{cases} \tag{13.3}$$

5) We set $\mathbb{H}_2 := \{ u_2 \in H^1(\Omega) \; ; \; u_2 \text{ satisfies } (13.3) \}$. Show that

$$H^1(\Omega) = H_0^1(\Omega) \oplus \mathbb{H}_2$$

is isomorphic to $H_0^1(\Omega) \times H^{1/2}(\partial\Omega)$.

6) With the above notations, show that if $u \in H^1(\Omega)$ and $\gamma_0(u) = \varphi$, then $|||u_2|||_1 \leq |||u|||_2$, and if this inequality is an equality we have $u = u_2$.

Exercise 5

We denote by $\operatorname{sgn}(t)$ the *sign function* that is

$$\operatorname{sgn}(t) := \begin{cases} +1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0. \end{cases}$$

Recall that if Ω is a subset of $\mathbb{R}^N$ and $T_1, T_2 \in \mathscr{D}(\Omega)$, we write $T_1 \geq T_2$ when we have $\langle T_1, \varphi \rangle \geq \langle T_2, \varphi \rangle$ for all $\varphi \in \mathscr{D}(\Omega)$ such that $\varphi \geq 0$. Our aim here is to prove the Kato[1] inequality :

$$\text{if } f \in L^1_{\mathrm{loc}}(\Omega), \qquad \Delta|u| \geq \operatorname{sgn}(u)\Delta u \quad \text{in } \mathscr{D}'(\Omega). \tag{13.4}$$

1) Let $j(t) := |t|$ and for $\varepsilon > 0$, define $j_\varepsilon(t) := (\varepsilon^2 + t^2)^{1/2} - \varepsilon$. Prove that j_ε is convexe and that $0 \leq j_\varepsilon(t) \leq j(t)$ for $t \in \mathbb{R}$. Verify also that for $t \in \mathbb{R}$ fixed

$$\lim_{\varepsilon \to 0} j_\varepsilon(t) = j(t), \qquad \lim_{\varepsilon \to 0} j'_\varepsilon(t) = \operatorname{sgn}(t).$$

Can one determine a sens to a limit for $j''_\varepsilon(t)$?

2) Let $\varphi \in \mathscr{D}(\Omega)$ fixed. Prove that if $u \in L^1_{\mathrm{loc}}(\Omega)$, then

[1] Toshio (or Tosio) Kato, Japanese mathematician, 1917–1999.

$$\lim_{\varepsilon\to 0}\int_\Omega j_\varepsilon(u(x))\,\varphi(x)\,dx = \int_\Omega j(u(x))\,\varphi(x)\,dx.$$

3) With the above assumptions on φ and u, we assume that a partial derivative ∂u belongs to $L^1_{\mathrm{loc}}(\Omega)$. Prove that

$$\lim_{\varepsilon\to 0}\int_\Omega \partial\big(j_\varepsilon(u(x))\big)\,\varphi(x)\,dx = \int_\Omega \operatorname{sgn}(u(x))\,\partial u(x)\,\varphi(x)\,dx.$$

4) Let $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $\partial u \in L^1_{\mathrm{loc}}(\Omega)$. Verify that $\operatorname{sgn}(u)\,\partial u \in \mathscr{D}'(\Omega)$.

5) Let $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $\partial u \in L^1_{\mathrm{loc}}(\Omega)$. Prove that

$$\partial|u| = \operatorname{sgn}(u)\,\partial u \quad \text{in } \mathscr{D}'(\Omega)\,.$$

(Note that $\langle \partial|u|, \varphi\rangle = -\int_\Omega |u|\,\partial\varphi\,dx = -\lim_{\varepsilon\to 0}\int_\Omega j_\varepsilon(u)\,\partial\varphi\,dx$)

6) Let $u \in L^1_{\mathrm{loc}}(\Omega)$ such that for $1 \le i \le N$, we have $\partial_i u \in L^2_{\mathrm{loc}}(\Omega)$ and $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$. Compute $\Delta j_\varepsilon(u)$ and verify that it belongs to $L^1_{\mathrm{loc}}(\Omega)$. Deduce that if $\varphi \in \mathscr{D}(\Omega)$ and $\varphi \ge 0$, then

$$\int_\Omega j_\varepsilon(u(x))\,\Delta\varphi(x)\,dx \ge \int_\Omega j'_\varepsilon(u(x))\,\Delta u(x)\,\varphi(x)\,dx.$$

7) Prove that if $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$ then $j'_\varepsilon(u)\Delta u \to \operatorname{sgn}(u)\Delta u$ in $\mathscr{D}'(\Omega)$.

8) Let $u \in L^1_{\mathrm{loc}}(\Omega)$ such that for $1 \le i \le N$ we have $\partial_i u \in L^2_{\mathrm{loc}}(\Omega)$ and $\Delta u \in L^1_{\mathrm{loc}}(\Omega)$. Prove that we have the Kato inequality (13.4).

9) More generally, show that if $u \in L^1_{\mathrm{loc}}(\Omega)$ is such that we have $\partial_{ii} u \in L^1_{\mathrm{loc}}(\Omega)$ for some integer i, then

$$\partial_{ii}|u| \ge \operatorname{sgn}(u)\,\partial_{ii}u \qquad \text{in } \mathscr{D}'(\Omega)\,.$$

Exercise 6

Let $f(r) := \exp(-r)/r$ for $r > 0$. For $x \in \mathbb{R}^3$ and $\lambda > 0$ (a fixed constant which will be determined later on) we consider

$$Y(x) := \lambda\,\frac{\exp(-|x|)}{|x|}\,.$$

1) Replacing $|x|$ by $\sqrt{\varepsilon^2 + |x|^2}$, find a sequence $Y_\varepsilon \in \mathscr{S}(\mathbb{R}^3)$ suh that $Y_\varepsilon \to Y$ in $\mathscr{S}'(\mathbb{R}^3)$ as $\varepsilon \to 0$.

2) Compute $T := -\Delta Y + Y$ in $\mathscr{S}'(\mathbb{R}^3$.

3) Determine the solution $Y_* \in \mathscr{S}'(\mathbb{R}^3)$ of the equation

$$-\Delta Y_* + Y_* = \delta_0.$$

(The function Y_* is called the *Yukawa potential*).

4) Determine all the p's in the interval $[1, \infty]$ such that $Y_* \in L^p(\mathbb{R}^3)$.

5) For a given $f \in L^2(\mathbb{R}^3)$ show that the unique solution of

$$u \in \mathscr{S}'(\mathbb{R}^3), \qquad -\Delta u + u = f$$

is given by $u = Y_* * f$. (One may also consider $u_\varepsilon = Y_\varepsilon * f$ and compute $-\Delta u_\varepsilon + u_\varepsilon$, then let $\varepsilon \to 0$).

6) Show that when $f \in L^2(\mathbb{R}^3)$ then $u \in C_0(\mathbb{R}^3)$, and that if $f \in L^1(\mathbb{R}^3)$ then $u \in L^1(\mathbb{R}^3)$.

7) Let $\omega > 0$ be given. Taking your inspiration in the above study how would you suggest to solve the equation $-\Delta u + \omega^2 u = f$ for a given $f \in L^2(\mathbb{R}^3)$?

8) Let again $\omega > 0$ be given and consider the function

$$H_{\pm\omega}(x) := \lambda_* \frac{\exp(\pm \mathrm{i}\, \omega\, |x|)}{|x|}$$

where λ_* is a constant which will be fixed later on. Does $H_{\pm\omega}$ belong to a space $L^p(\mathbb{R}^3)$ for some $p \in [1, \infty]$? Determine all the exponents $p \in [1, \infty]$ such that $H_{\pm\omega}$ belong to $L^p_{\mathrm{loc}}(\mathbb{R}^3)$.

9) Compute

$$S := -\Delta H_{\pm\omega} - \omega^2 H_{\pm\omega} \qquad \text{in } \mathscr{S}'(\mathbb{R}^3).$$

10) Let $f \in C_c(\mathbb{R}^3)$ be given. Show that the function

$$u(x) = \int_{\mathbb{R}^3} H_\omega(x-y)\, f(y)\, dy$$

is solution to the Helmholtz[2] equation

$$u \in \mathscr{S}'(\mathbb{R}^3), \qquad -\Delta u = \omega^2 u + f. \tag{13.5}$$

Does u belong to a space $L^p(\mathbb{R}^3)$ for some $p \in [1, \infty]$? Is u continuous ? Is the solution unique ?

11) Show that there exists a unique $H \in \mathscr{S}'(\mathbb{R}^3)$ such that

$$-\Delta H = \omega^2 H + \delta_0$$

and which satisfies

$$\frac{x}{|x|} \cdot \nabla H(x) - \mathrm{i}\omega H(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty. \tag{13.6}$$

(This is called the *Sommerfeld*[3] *radiation condition* for the Helmholtz equation).

12) Prove that for $f \in C_c(\mathbb{R}^3)$ given, the Helmholtz equation (13.5) has a unique solution u satsifying the Sommerfeld radiation condition

$$\frac{x}{|x|} \cdot \nabla u(x) - \mathrm{i}\omega u(x) = O\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty.$$

[2] Hermann Ludwig Ferdinand von Helmholtz, German physicist and medical doctor, 1821–1894.
[3] Arnold Johannes Wilhelm Sommerfeld, German physicist, 1868–1951.