Constructive Krein-Rutman result for kinetic Fokker-Planck equations in a domain.

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FRAMEWORK

We consider the Kinetic Fokker-Planck (KFP) equation

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f + b \cdot \nabla_v f + cf \quad \text{on} \quad (0, +\infty) \times \Omega \times \mathbb{R}^d \tag{1}$$

on the function $f := f(t, x, v)$ depending on the time variable $t \geq 0$, the position variable $x \in \Omega$, where $\Omega \subset \mathbb{R}^d$ is a suitably smooth domain, $d \geq 3$, and the velocity variable $v \in \mathbb{R}^d$. We also assume that

$$b = b(x, v) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2d}, \mathbb{R}^d), \quad c = c(x, v) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2d}, \mathbb{R}). \tag{2}$$

We complement this evolution equation with the Maxwell type reflection condition on the boundary

$$\gamma_- f = \mathscr{R}\gamma_+ f = \iota_S \mathscr{S}\gamma_+ f + \iota_D \mathscr{D}\gamma_+ f \quad \text{on} \quad \Gamma_-, \tag{3}$$

and with an initial condition

$$f(0, x, v) = f_0(x, v) \quad \text{on} \quad \mathcal{O} = \Omega \times \mathbb{R}^d. \tag{4}$$

1.1 Boundary conditions

Consider $\delta \in W^{2,\infty}(\mathbb{R}^d, \mathbb{R})$ such that $|\delta(x)| := \mathrm{dist}(x, \partial\Omega)$ on a neighborhood of the boundary and thus $n_x = n(x) := -\nabla\delta(x)$. Then we define $\Sigma^x_\pm := \{v \in \mathbb{R}^d; \pm v \cdot n_x > 0\}$ the sets of outgoing (Σ^x_+) and incoming (Σ^x_-) velocities at point $x \in \partial\Omega$, then the sets

$$\Sigma_\pm := \{(x, v);\ x \in \partial\Omega,\ v \in \Sigma^x_\pm\}, \quad \Gamma_\pm := (0, T) \times \Sigma_\pm,$$

and finally the outgoing and incoming trace functions $\gamma_\pm f := \mathbf{1}_{\Gamma_\pm}\gamma f$. The specular reflection operator $\mathscr{S}$ is defined by

$$(\mathscr{S}g)(x, v) := g(x, \mathcal{V}_x v), \quad \mathcal{V}_x v := v - 2n(x)(n(x) \cdot v), \tag{5}$$

and the diffusive operator $\mathscr{D}$ is defined by

$$(\mathscr{D}g)(x, v) := \mathscr{M}_x(v)\widetilde{g}(x), \quad \widetilde{g}(x) := \int_{\Sigma^x_+} g(x, w)\, n(x) \cdot w \, dw, \tag{6}$$

where $\mathscr{M}_x$ stands for the Maxwellian function

$$\mathscr{M}_x(v) := (2\pi\Theta_x)^{-(d-1)/2} \exp(-|v|^2/(2\Theta_x)) > 0, \tag{7}$$

associated to the wall temperature Θ_x which is assumed to satisfy

$$\Theta_x \in W^{1,\infty}(\Omega), \quad 0 < \Theta_* \leq \Theta_x \leq \Theta^* < \infty. \tag{8}$$

As for the accommodation coefficient $\iota := \iota_S + \iota_D$, we assume $\iota_S, \iota_D, \iota : \partial\Omega \to [0, 1]$.

1.2 Confinement in the velocity variable and admissible weight functions

We now assume

$$\liminf_{|v|\to\infty} \inf_\Omega b \cdot \frac{v}{|v|} = +\infty, \quad \frac{|b|}{\langle v\rangle}, \mathrm{div}_v\, b, c = \mathcal{O}\Big(b \cdot \frac{v}{|v|^2}\Big), \tag{9}$$

and more precisely there are $R_0, b_0, b_1 > 0$, $\gamma > 1$ and for any $p \in [1, \infty]$ there exists $k_p \geq 0$ such that

$$\forall\, x \in \Omega,\ v \in B^c_{R_0}, \quad b_0|v|^\gamma \leq b \cdot v \leq b_1|v|^\gamma, \quad c - \frac{1}{p}\mathrm{div}_v\, b \leq k_p\, b \cdot \frac{v}{|v|^2}. \tag{10}$$

We now introduce the class of the so-called *admissible* weight functions $\omega : \mathbb{R}^d \to (0, \infty)$ we will work with, which will be either a polynomial weight function

$$\omega = \langle v\rangle^k := (1 + |v|^2)^{k/2}, \quad k > k_* := \max(d + 1, k_1', k_\infty), \tag{11}$$

$k_1' := \max(k_1 + (d + \gamma - 1)/2, k_1 + d/2 + 1)$, and we set $s := 0$ in that case, or either an exponential weight function

$$\omega = \exp(\zeta\langle v\rangle^s), \tag{12}$$

with the restrictions

$$\begin{aligned} &s < \min(\gamma, 2),\ \zeta > 0; \quad s = \gamma < 2,\ \zeta \in (0, b_0/2); \\ &s = \gamma = 2,\ \zeta \in (0, \min(1/\Theta^*, b_0)/2); \quad s = 2 < \gamma,\ \zeta \in (0, 1/(2\Theta^*)). \end{aligned} \tag{13}$$

Furthermore we will call a strongly confining an admissible weight function ω such that

$$\limsup_{|v|\to\infty} \sup_\Omega \left(2\left(1 - \frac{1}{p}\right)\frac{|\nabla_v\omega|^2}{\omega^2} + \left(\frac{2}{p} - 1\right)\frac{\Delta_v\,\omega}{\omega} - b \cdot \frac{\nabla_v\omega}{\omega} + c - \frac{1}{p}\mathrm{div}_v\, b\right) = -\infty. \tag{14}$$

Finally for a given measure space $(E, \mathscr{E}, \mu)$, a weight function $\rho : E \to (0, \infty)$ and an exponent $p \in [1, \infty]$, we define the weighted Lebesgue space L^p_ρ associated to the norm

$$\|g\|_{L^p_\rho} = \|\rho g\|_{L^p}.$$

MAIN RESULTS

Here we present three main results, which are improvements of the recent work [2, Section 11] by slightly generalizing the framework to a position dependent wall temperature and by providing a fully constructive approach for the exponential stability of the first eigenfunction.

Theorem 2.1: Existence and uniqueness

For any admissible weight function ω and any initial datum $f_0 \in L^p_\omega(\mathcal{O})$, $p \in [1, \infty]$, there exists a unique global weak solution f to the kinetic Fokker-Planck equation (1), (3), (4). In particular, for any $(x_0, v_0) \in \mathcal{O}$, there exists a unique fundamental solution associated to the initial datum $f_0 := \delta_{(x_0, v_0)}$.

Theorem 2.2: Long time asymptotic

There exist two weight functions ω_1, m_1 and an exponent $r > 2$ such that $L^r_{\omega_1} \subset (L^2_{m_1})'$ and there exists a unique eigentriplet $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times L^r_{\omega_1}(\mathcal{O}) \times L^2_{m_1}(\mathcal{O})$. Furthermore, there exist some constructive constants $C \geq 1$ and $\lambda_2 < \lambda_1$ such that for any strongly confining admissible weight function ω, any exponent $p \in [1, \infty]$ and any initial datum $f_0 \in L^p_\omega(\mathcal{O})$, the associated solution f to the kinetic Fokker-Planck equation (1), (3), (4) satisfies

$$\|f(t) - \langle f_0, \phi_1\rangle f_1 e^{\lambda_1 t}\|_{L^p_\omega(\mathcal{O})} \leq Ce^{\lambda_2 t}\|f_0 - \langle f_0, \phi_1\rangle f_1\|_{L^p_\omega(\mathcal{O})}, \quad \forall t \geq 0, \tag{15}$$

for some constant $C > 0$.

Both the above well-posedness and the longtime behaviour results are based on the following ultracontractivity property.

Theorem 2.3: Ultracontractivity

There exist $\Theta, C > 0$ and $\kappa \geq 0$ such that for every strongly confining admissible weight function ω and any solution f to the KFP equation (1), (3), (4) satisfy

$$\|f(T, \cdot)\|_{L^\infty_\omega(\mathcal{O})} \leq C\frac{e^{\kappa T}}{T^\Theta}\|f_0\|_{L^1_\omega(\mathcal{O})}, \quad \forall T > 0, \tag{16}$$

for some constant $C > 0$.

2.1 Elements of the proof

The ultracontractivity estimate is a generalization to bounded domains of a similar local result from [1]. This is done by the use of the following key propositions.

Proposition 2.1. *For any admissible weight function ω there exists $C = C(\omega, \Omega) > 0$ such that for any solution f to the KFP equation* (1)–(3)*, any $T > 0$ and any smooth function $0 \leq \varphi \in \mathcal{D}((0, T))$, there holds*

$$\int_{(0,T]\times\mathcal{O}} f^2\omega^2 \left\{\frac{(n_x \cdot \hat{v})^2}{\delta^{1/2}} + \langle v\rangle^\varsigma\right\}\varphi^2 + \int_{(0,T]\times\mathcal{O}} |\nabla_v(f\omega)|^2\varphi^2 \leq C\int_{(0,T]\times\mathcal{O}} f^2\omega^2\left[|\partial_t\varphi^2| + \varphi^2\right],$$

with $\varsigma := \gamma + s - 2$.

Proposition 2.2. *We set $\nu := \max(2, \gamma - 1)$. There exists $p > 2$, $\alpha > p$ and $C \in (0, \infty)$ such that any solution f to the KFP equation* (1)–(3) *satisfies*

$$\left\|f\varphi\frac{\omega}{\langle v\rangle^\nu}\delta^{\alpha/p}\right\|_{L^p((0,T]\times\mathcal{O})} \leq C\|(\varphi + |\varphi'|)f\omega\|_{L^2((0,T]\times\mathcal{O})}, \tag{17}$$

for any $0 \leq \varphi \in \mathcal{D}((0, T))$, any $T > 0$ and any admissible weight function ω.

Then by combining this two estimates by using interpolation and repeating this on the solutions of the dual problem we may obtain Theorem 2.3 by using a Nash type argument.

References

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